On a development of the phenomenological renormalization group

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OUTLINE

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2. Combining phenomenological RG and CDA
3. Application to $Z(N)$ models
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I. RG in spin and gauge models

Gauge and spin models on d-dimensional lattice $\Lambda_0 = L^d$ are defined as

$$Z(\Lambda_0, \{t_k\}) \;=\; \int \prod_l dU_l \prod_p Q_0(\mathrm{Tr}U_p, \{t_k\}) \;,$$

$$Z(\Lambda_0, \{t_k\}) \;=\; \int \prod_x dV_x \prod_l Q_0(\mathrm{Tr}V_l, \{t_k\}) \;,$$

where $U_l, V_x \in Z(N), O(N), SU(N), U(N)$ and $U_p = \prod_{l \in p} U_l$, $V_l = V_x V^{\dagger}_{x+e_n}$. The most general form of the Boltzmann weight

$$Q(\mathrm{Tr}U, \{t_k\}) = \sum_{\{r\}} t_r \, \chi_r(U) \;,\; t_0 = 1 \;,\; 0 \leq t_r \leq 1$$

Write partition function on a decimated lattice Λ_1 ($L = bL_1$, $b = 2$)

$$Z(\Lambda_0, \{t_k\}) \;=\; A(\{t_k\})\, Z(\Lambda_1, \{t_k^{(1)}\})$$

with unchanged Boltzmann weight and a new set of couplings $\{t_k^{(1)}\}$. Most real-space RGs on the lattice amount to a prescription of how to (approximately) compute constant $A(\{t_k\})$ and new couplings $\{t_k^{(1)}\}$.

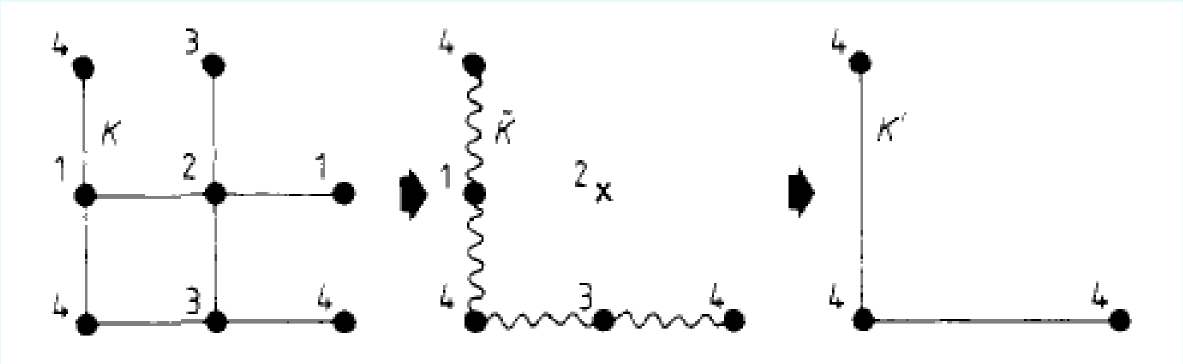
- Migdal-Kadanoff: new coupling from bond moving operation
- Cluster decimation approximation (CDA): new coupling from small clusters
- Phenomenological RG: uses finite-size scaling

CDA

➜ *R. E. Goldstein, J. S. Walker, J. Phys A: Math. Gen. 18 (1985) 1275*

$$Z(L, K) \approx Z(L/2, K_1) \; .$$

New set of couplings K_1 can be computed from the CDA.



CDA for 2×2 lattice with the periodic boundary conditions.

$$Z(2, K) = Z(1, \tilde{K}) \; .$$

For $2D$ Ising model: $K_c = 0.492$, $\nu = 1.1919$

Phenomenological RG

➜ *M.P. Nightingale, Physica* **83A** *(1976) 561.*

In the vicinity of a critical point $bm(L) \approx m(L/b)$. Phenomenological RG is designed to determine critical points and indices from this finite-size relation. Usually, one considers a system on a strip $M \times \infty$ (in $2d$). If $\lambda_i(M)$ are eigenvalues of a corresponding transfer matrix then

$$\left(\frac{\lambda_1(M)}{\lambda_0(M)}\right)^2 = \frac{\lambda_1(M/2)}{\lambda_0(M/2)} \, .$$

- Results for $2d$ Ising model

M	β_c	ν
2	0.435665	0.9873
8	0.43833	0.9768
16	0.440439	0.9948
32	0.440657	0.9988
64	0.440683	0.9997
Exact	0.440687	1

II. Combining phenomenological RG and CDA

- $bm(L) \approx m(L/b)$ is treated as equation for new coupling

- CDA: as a cluster we consider lattice strip $M^{(d-1)} \times L$, $L \rightarrow \infty$

Suppose that correlation function has the following general form

$$\Gamma_r(M, \{t_k\}; R) \; = \; D_r(M, \{t_k\}, R) \; [B_r(M, \{t_k\})]^R \; .$$

The function $B_r(M, \{t_k\})$ encodes an exponential decay

$$B_r(M, \{t_k\}) \; = \; \frac{\lambda_r(M, \{t_k\})}{\lambda_0(M, \{t_k\})}$$

of $\Gamma_r(M, \{t_k\}; R)$ in representation r.

Basic idea is to present the original correlation function via the correlation function $\Gamma_r(M/2, \{t_k^{(1)}\}; R/2)$, calculated on the strip of the width $M/2$ with new couplings $\{t_k^{(1)}\}$, in the form

$$\Gamma_r(M, \{t_k\}; R) = \frac{D_r(M, \{t_k\}, R)}{D_r(M/2, \{t_k^{(1)}\}, R/2)} \Gamma_r(M/2, \{t_k^{(1)}\}; R/2) .$$

Last equation holds if

$$B_r^2(M, \{t_k\}) = B_r(M/2, \{t_k^{(1)}\})$$

for all r. This system determines new couplings $t_k^{(1)}$ on the lattice strip $(M/2, L/2)$. Via CDA these exact relations are used to approximate the partition and the correlation functions on Λ_0 as

$$Z(\Lambda_0, \{t_k\}) = \left[\frac{\lambda_0(M, \{t_k\})}{\lambda_0^{(1/2)}(M/2, \{t_k^{(1)}\})}\right]^{L^2/M} Z(\Lambda_1, \{t_k^{(1)}\}) ,$$

$$\Gamma_r(\Lambda_0, \{t_k\}; R) = \frac{D_r(M, \{t_k\}, R)}{D_r(M/2, \{t_k^{(1)}\}, R/2)} \Gamma_r(\Lambda_1, \{t_k^{(1)}\}; R/2) .$$

III. Application to $Z(N)$ models

- $B_r(M, \{t_k\})$ is calculated with transfer matrix technics
- Transfer matrix is constructed from evolution of independent couplings
- Dual formulation is used

Exact solution for the free energy of $2d$ standard Potts models on lattice strips $M = 2, 3, 4$ and zero magnetic field $h = 0$.

➜ *J. Salas, S.-C. Chang, R. Shrock, J. Stat. Phys. 107 (2002) 1207*

This has been extended to:

- Two-point correlation function and second moment correlation length
- Non zero external field
- some vector models $Z(N = 4, 5, 6)$ with arbitrary couplings

Standard Potts models, M=2

N	β_c	ν	y_h
2	0.435657	0.987303	1.788717
	0.440687	1	1.875
3	0.655143	0.874834	1.749592
	0.670035	0.833333	1.866667
4	0.799504	0.807699	1.717679
	0.823959	0.666666	1.875
5	0.906325	-	-
	0.939487	-	-

Standard Potts models, M=L

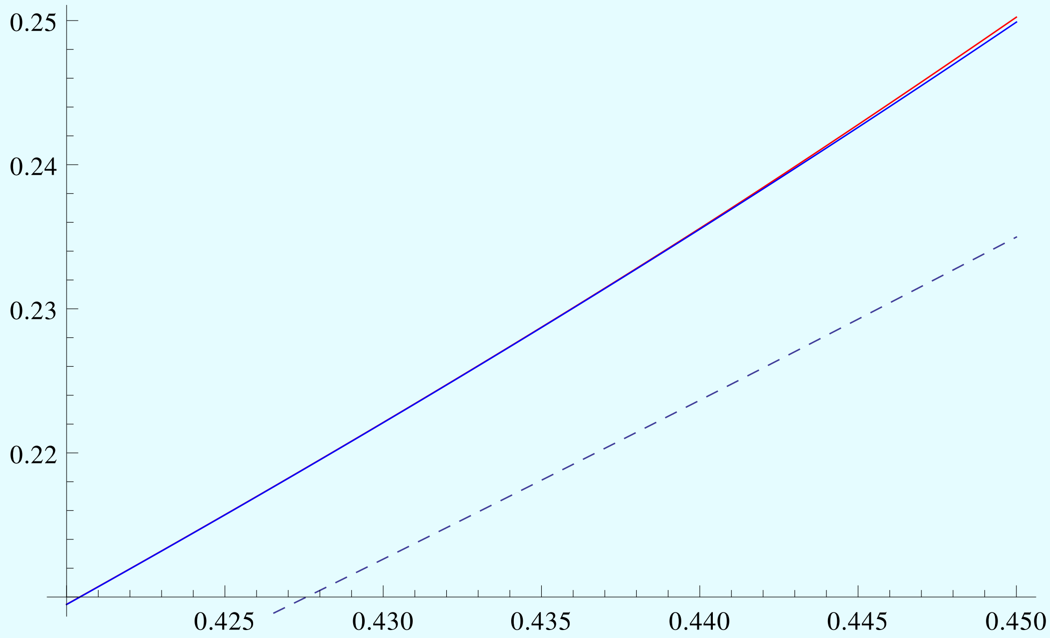
RG based on a preservation of the second moment correlation length

N	L	β_c	$\beta_{c(e)}$	ν	$\nu_{(e)}$
2	16	0.441905		1.04733	
	32	0.440965		1.01295	
	64	0.440664	0.440687	0.998986	1.0
3	8	0.33703		0.971028	
	16	0.33531		0.887692	
	32	0.33505		0.857852	
	64	0.3350186	0.335018	0.849067	5/6
5	16	0.234663		-	-
	32	0.234726		-	-
	64	0.2348156	0.234872	-	-
13	8	0.1165		-	-
	16	0.117		-	-
	32	0.1175	0.117482	-	-

Critical coupling β_c and critical exponent ν for $Z(N), N = 2, 3, 5, 13$

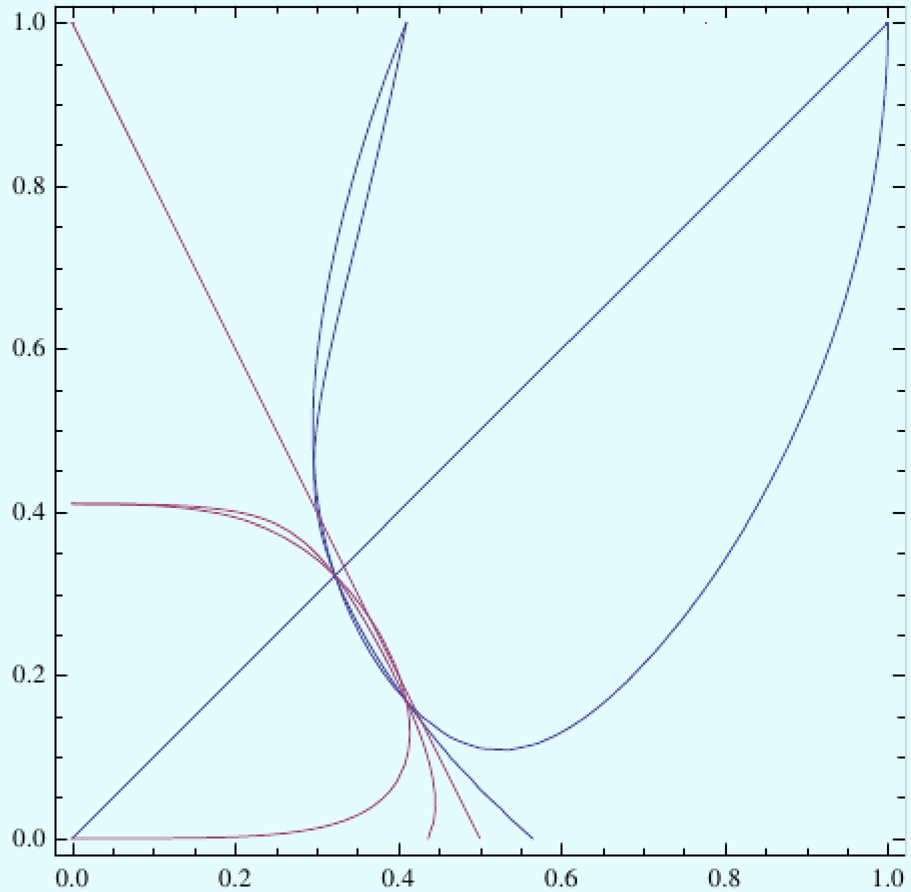
Free enrgy

$$F = \frac{2\cosh(\beta)}{M} \sum_{i \in Iter} \frac{1}{4^{i+1}} \log \frac{\lambda_0(\{t_i\}, M)}{\sqrt{\lambda_0(\{t_{i+1}\}, M/2)}} .$$



Free energy of the Ising model in the vicinity of the critical point. RG $M = 16 \rightarrow 8$. Dashed line - one iteration. Blue line - 5 iterations. Red line - exact.

Ashkin-Teller model, N=4



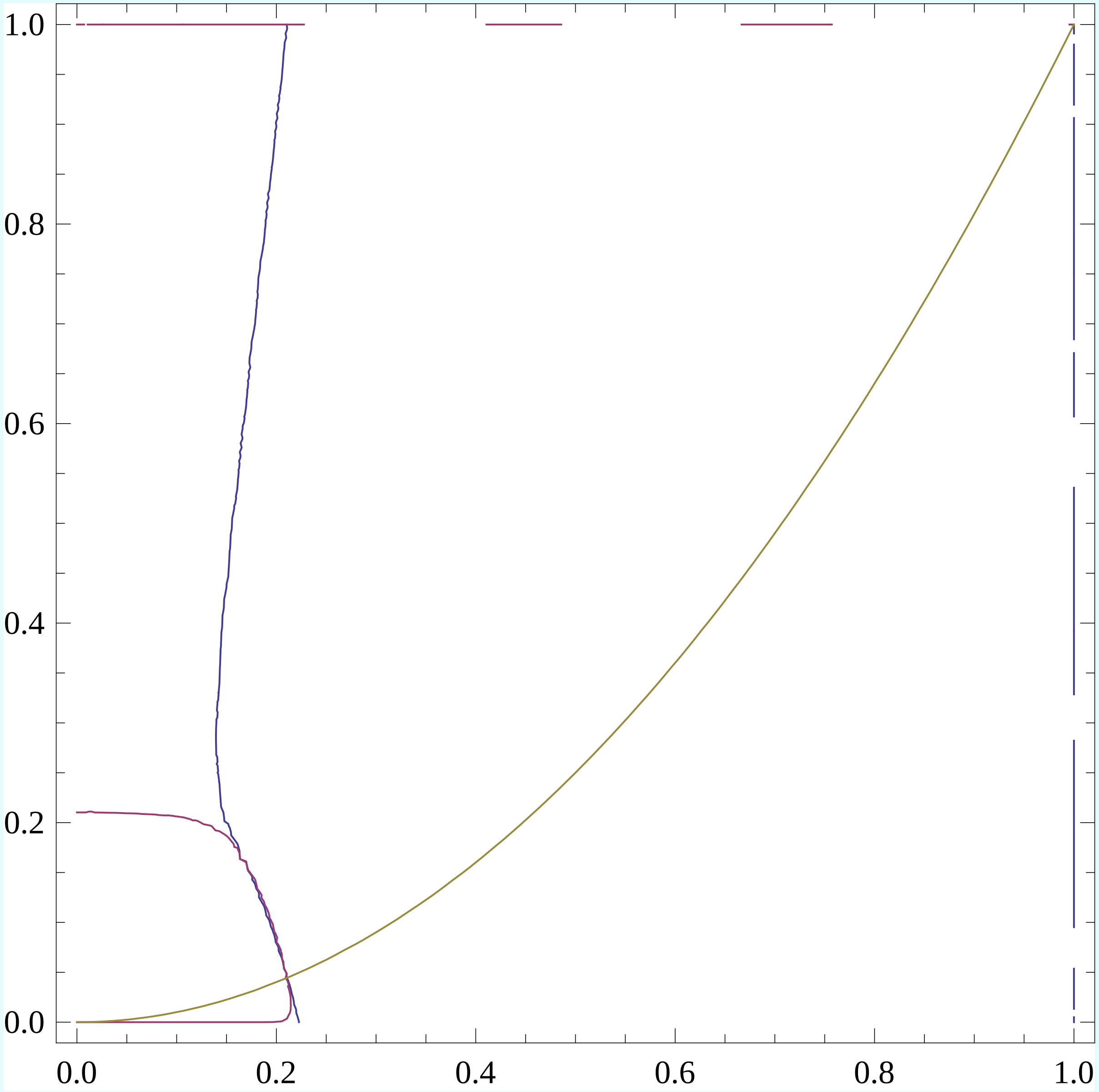
Contour plot of the fixed points after 1-st and 2-nd iterations. Shown are line of the standard Potts model ($t_1 = t_2$) and self-dual line.

Three-dimensional spin vector $Z(N)$ models

RG: $2 \times 2 \times L \to 1 \times 1 \times L/2$

N	β_c	ν
2	0.2146	0.6167
	0.22171	0.63
3	0.3237	-
	0.367	-
4	0.4292	0.6167
	0.44342	0.63

$3D$ $Z(4)$ model with arbitrary couplings



Contour plot for fixed points after 3 iterations. Yellow line corresponds to a vector $Z(4)$ model $t_2 = t_1^2$

Three-dimensional gauge vector $Z(N)$ models at zero temperature

(also talk by V. Chelnokov, Theor. Devel., Friday)

RG: $2 \times 2 \times L \to 1 \times 1 \times L/2$

N	β_c
2	0.777 0.7614
3	1.172 1.084
4	1.554 1.523
5	2.17894 2.1796

V. Summary and perspectives

- Phenomenological RG in combination with CDA

- Some new exact results for the free energy (including non-zero external field) and correlation function for $Z(N)$ models on small strips in $2d$ and $3d$

- Application to models with discrete symmetries

- Models with continuous symmetry: XY, principal chiral models, $O(N)$ non-linear sigma-models.

- Extension to gauge models