The Schrödinger Functional in Numerical Stochastic Perturbation Theory

Dirk Hesse *(Università di Parma)*

31st International Symposium on Lattice Field Theories
July-August 2013, Mainz, Germany

In collaboration with
M. Brambilla *(Parma)*, M. Dalla Brida *(TC Dublin)*,
F. Di Renzo *(Parma)*, and S. Sint *(TC Dublin)*

Overview

- Stochastic quantization and NSPT
  - NSPT in the Schrödinger functional
- The running SF coupling in NSPT

Preliminaries

We work in the Schrödinger Functional as defined in [Lüscher, Narayanan, Weisz, and Wolff 1992].

I.e. the SU(3) lattice gauge fields $U_\mu(x)$ obey the **boundary conditions**

$$U_k(x)|_{x_0=0} = e^{C(\mathbf{x})}$$
$$U_k(x)|_{x_0=T} = e^{C'(\mathbf{x})}$$
$$U_\mu(x+\hat{k}L) = U_\mu(x)$$

and may be **decomposed** (in the perturbative regime) as

$$U_\mu(x) = e^{a\, g_0\, q_\mu(x)} V_\mu(x)$$

(sometimes I'll also write $U_{x\mu}$)

Stochastic Quantization

Introduce an extra d.o.f., the *stochastic time*.
The field evolution in stochastic time is given
by the Langevin Equation.

$$\partial_t U_{x\mu}(t;\eta) = -\big\{\nabla_{x\mu} S_G[U(t;\eta)] + \eta_{x\mu}(t)\big\} U_{x\mu}(t;\eta)$$

with Gaussian noise $\eta = \eta^a T^a$

$$\langle \eta^a_{x\mu}(t) \rangle_\eta = 0\,, \quad \langle \eta^a_{x\mu}(t) \eta^b_{y\nu}(u) \rangle_\eta = 2\delta^{ab}\delta_{xy}\delta_{\mu\nu}\delta(t-u)$$

[Parisi & Wu, 1981]

Stochastic Quantization

The expectation value of an observable w.r.t. the stochastic noise may be written as

$$\langle O \rangle_\eta = \frac{1}{Z_\eta} \int \mathcal{D}[\eta]\, O[U(t;\eta)]\, e^{-\frac{1}{4}\int \mathrm{d}x\mathrm{d}t\, \eta^2(x,t)}$$

$$= \int \mathcal{D}U\, O[U]\, \underline{P[U,t]}$$

Where the time evolution of the weight is given by the Fokker-Planck equation

$$\underline{\dot{P}[U,t]} = \int \mathrm{d}x\, \mathrm{d}\mu\, \frac{\delta}{\delta U_{x\mu}} \left(\frac{\delta S[U]}{\delta U_{x\mu}} + \frac{\delta}{\delta U_{x\mu}} \right) P[U,t]$$

Stochastic Quantization

One can prove that in a well-defined sense the weight converges to the one encountered in the path integral formalism, order by order...

$$P^{(0)}[U(\eta;t)] \stackrel{t\to\infty}{\longrightarrow} P^{(0)}_{\rm PI}[U] = \frac{1}{Z^{(0)}} e^{-S^{(0)}_G[U]}\,,$$

$$P^{(1)}[U(\eta;t)] \stackrel{t\to\infty}{\longrightarrow} P^{(1)}_{\rm PI}[U]\,,\ldots$$

Hence one may obtain perturbative expansion of any observable in the large-time-limit

$$\lim_{t\to\infty}\left\langle \mathcal{O}\left[\sum_k g_0^k\, U^{(k)}(t;\eta)\right]\right\rangle_\eta = \sum_k g_0^k\, \mathcal{O}^{(k)}$$

Numerical Stochastic Perturbation Theory

Solve the Langevin equation on a lattice numerically
e.g. using an Euler Scheme

$$U_{x\mu}(t+\epsilon;\eta) = e^{-F_{x\mu}[U]}U_{x\mu}(t;\eta)\,, \quad F_{x\mu}[U] = \epsilon\nabla_{x\mu}S_G[U] - \sqrt{\epsilon}\eta_{x\mu}$$

... the perturbative expansion of which may be consistently truncated ...

$$U^{(0)} \to U^{(0)}\,, \quad U^{(1)} \to U^{(1)} - F^{(1)}$$

$$U^{(2)} \to U^{(2)} - F^{(2)} + \frac{1}{2}\left(F^{(1)}\right)^2 - F^{(1)}U^{(1)}\,, \ldots$$

[Di Renzo, Marchesini, et al. 1994]

Numerical Stochastic Perturbation Theory

Set up a computer simulation, storing all gauge fields as a series

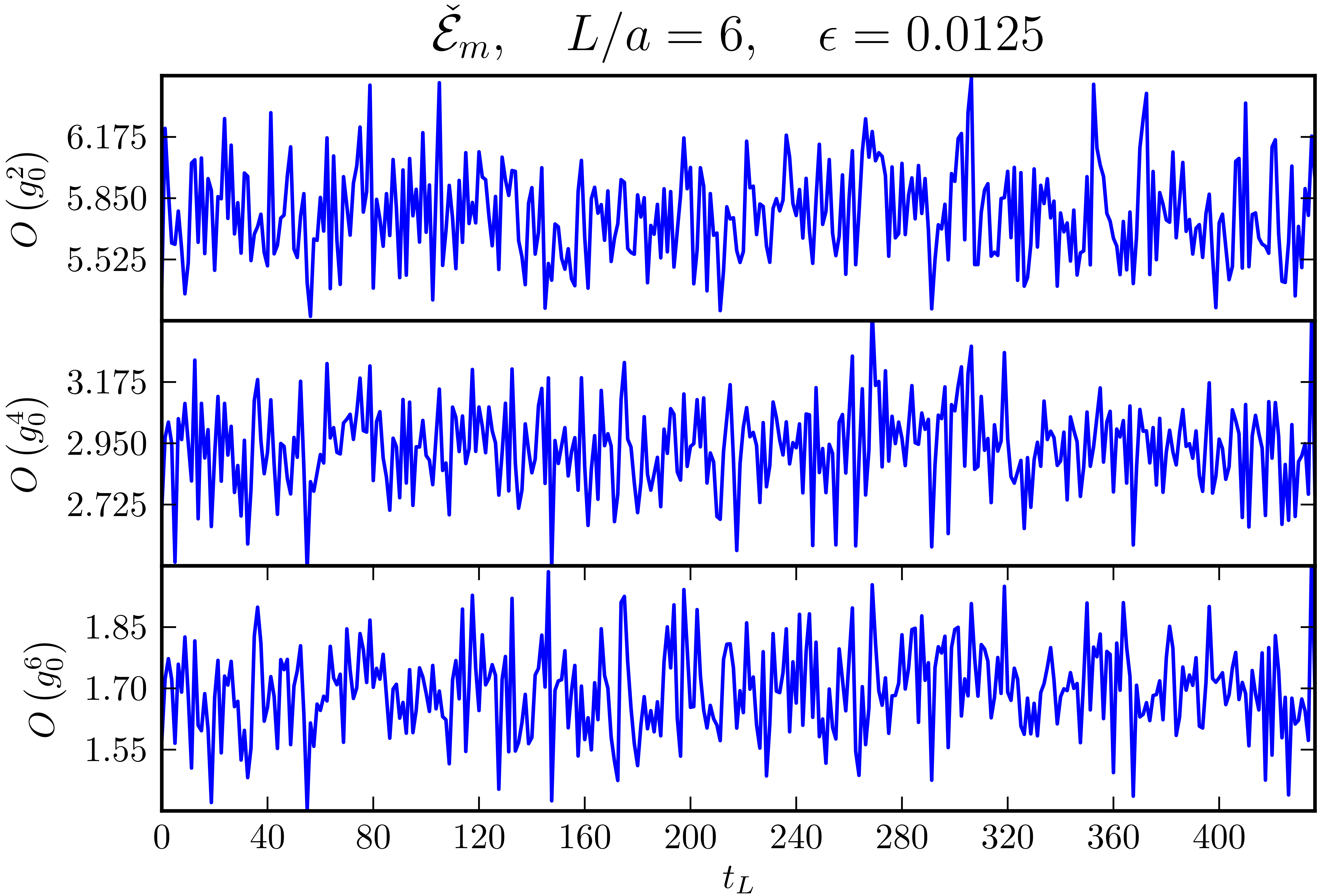
$$U = \sum_{k=0}^{N} g_0^k \, U^{(k)}$$

and perform all computations order by order, e.g.

$$(U \cdot U')^{(k)} = \sum_{l=0}^{N} U^{(l)} U'^{(k-l)}$$

- Obtain expectation value through single long history
- Perform the limit $\epsilon \to 0$

Sample History



Gauge Suppression

Historically: *Stochastic gauge fixing*.

$$F_{x\mu}[U] = \epsilon \nabla_{x\mu} S_G[U] - \sqrt{\epsilon}\eta_{x\mu}$$

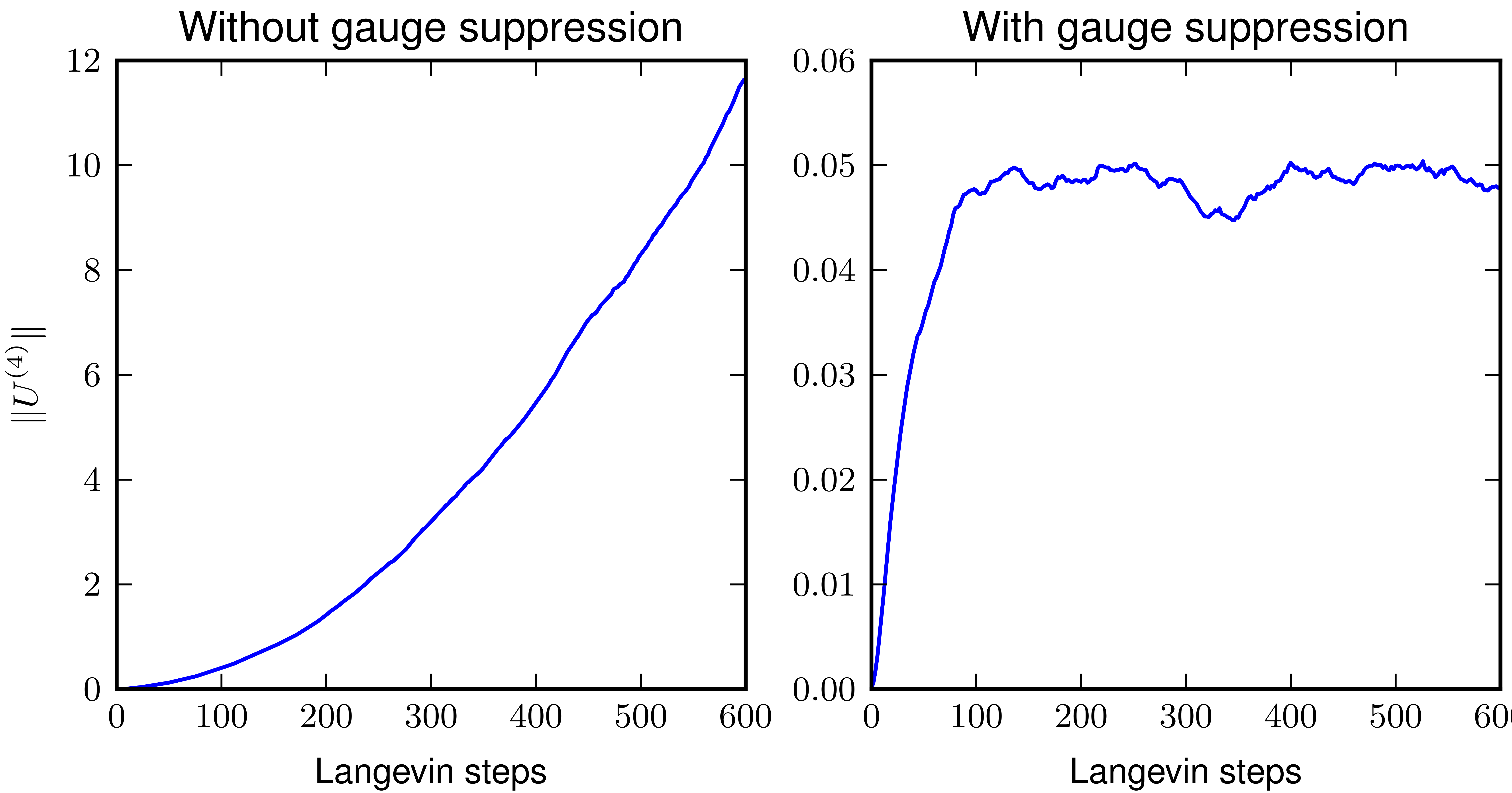
Drives *transversal* gauge modes to classical solution.

Random walk.

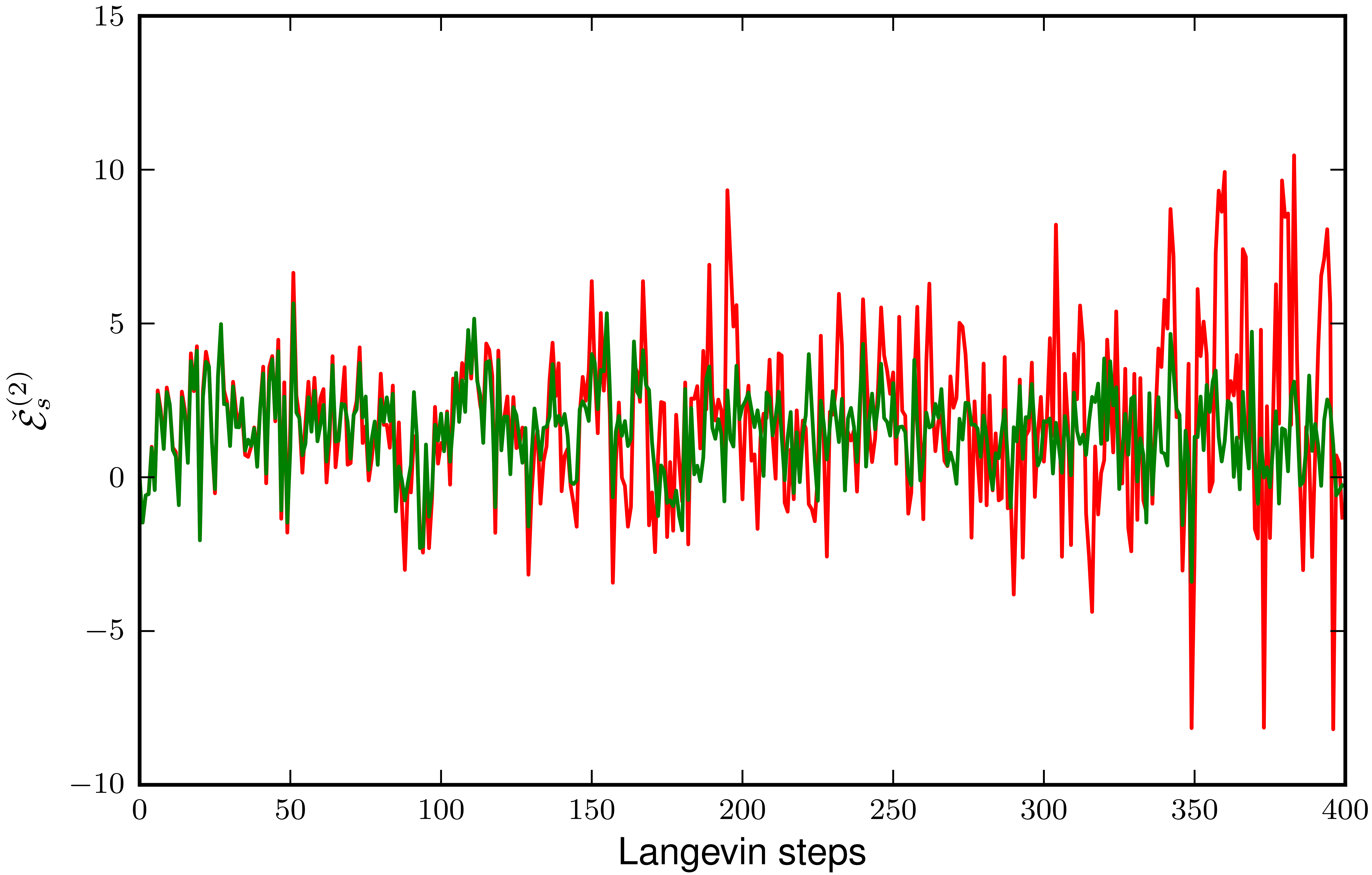
→ Perform properly chosen gauge transformation to suppress longitudinal modes and avoid growing variance.

[Zwanziger 1981]

Numerical Tests



Numerical Tests



Running Coupling

SF Coupling

In the SF, we can define a running coupling, using the external scale L, employing the free energy

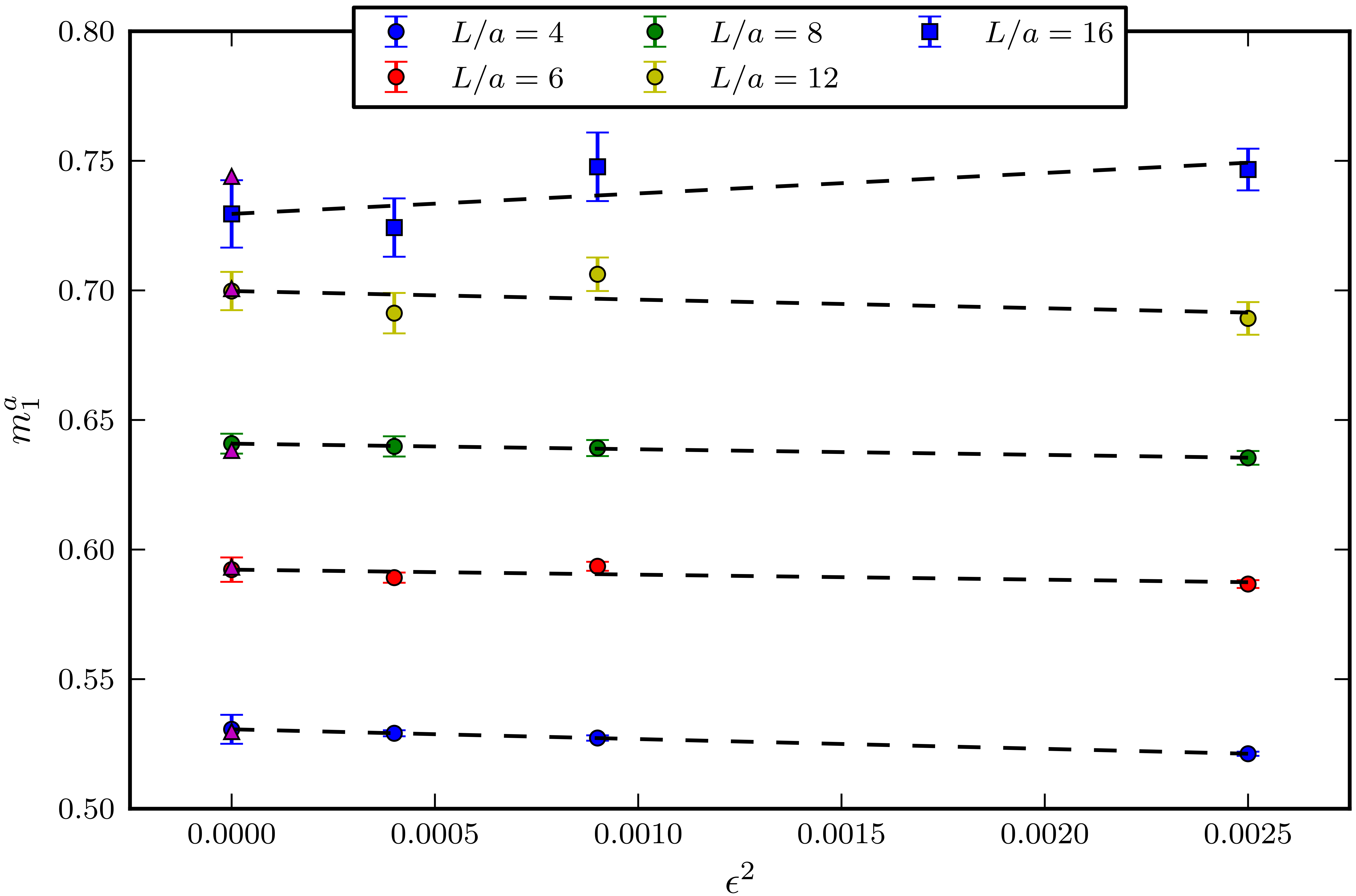
$$e^{-\Gamma} = \int \mathcal{D}[U] e^{-S_G[U]}$$

with which we define

$$\overline{g}^2_{SF}(L) = \frac{k}{\partial_\eta \Gamma|_{\eta=0}} = g_0^2 + m_1(L/a)\, g_0^4 + m_2(L/a)\, g_0^6 + \ldots$$

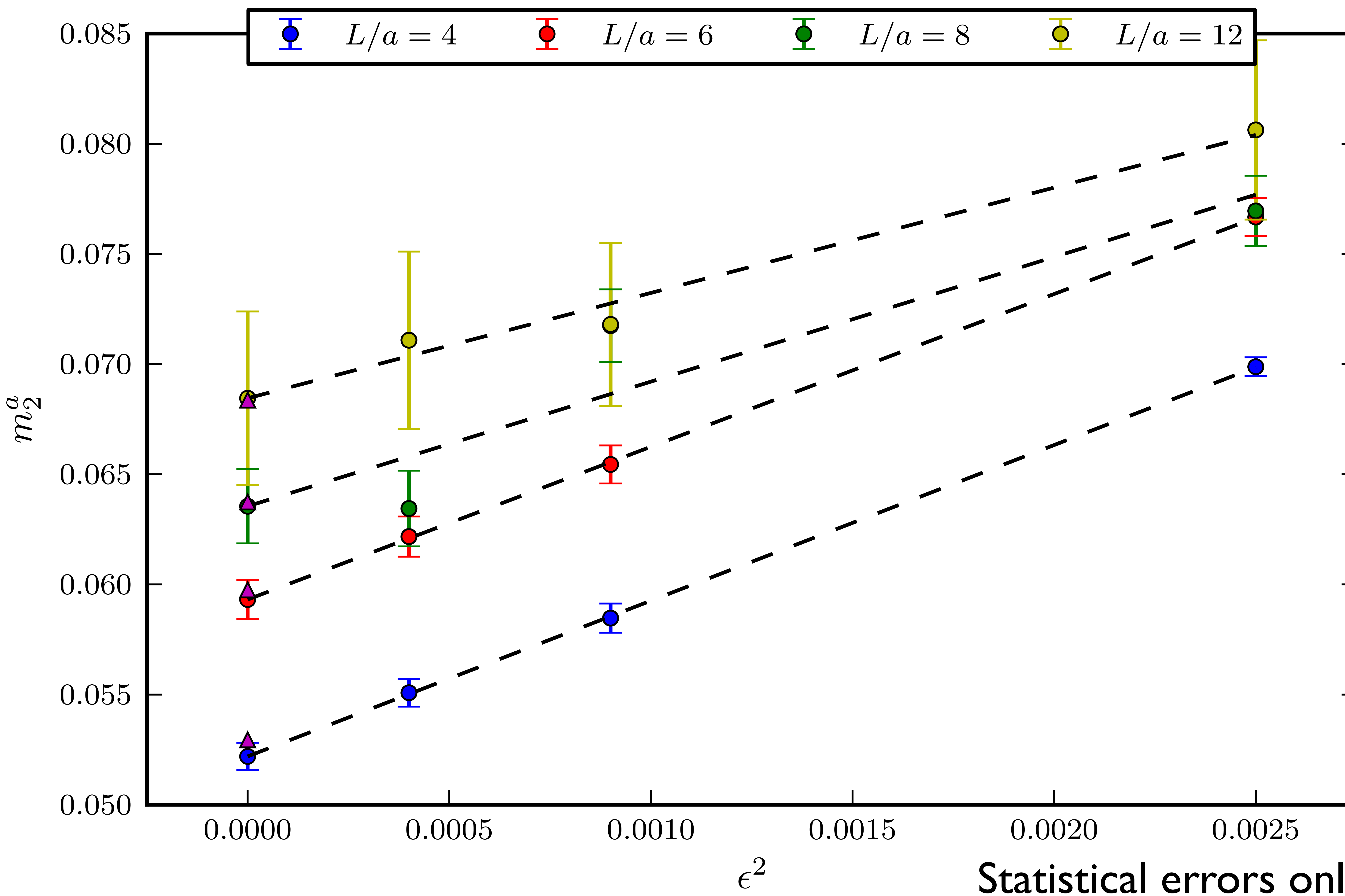
[Lüscher, Weisz, Wolff 1991]

One Loop



Known values from [ALPHA Collaboration 1998].

Two Loops



Statistical errors only.

Known values from [ALPHA Collaboration 1998].

Comparison with known results

Defining $\delta m_i^a = \dfrac{m_i^{a,\mathrm{NSPT}}}{m_i^a} - 1$, we obtain

L/a	4	6	8	12	16
δm_1^a	0.0017(21)	-0.0016(31)	0.0041(56)	-0.001(10)	-0.019(16)
δm_2^a	-0.014(12)	-0.007(15)	-0.003(26)	0.001(58)	-0.045(91)

Statistical errors only.

L/a	4	6	8	12	16
δm_1^a	0.002(11)	-0.0016(79)	0.0041(60)	-0.001(11)	-0.019(17)
δm_2^a	-0.01(19)	-0.01(17)	-0.003(69)	0.001(72)	-0.05(20)

Statistical and systematic errors.

Numerical Effort

L	ϵ	τ_{int}	N_{eff}
4	0.02	2.309(67)	24978
	0.03	1.665(43)	34668
	0.05	1.032(22)	55882
6	0.02	7.78(31)	12212
	0.03	5.58(19)	17091
	0.05	3.58(10)	26514
8	0.02	11.79(67)	5074
	0.03	8.19(40)	7308
	0.05	5.20(21)	11506
12	0.02	17.44(161)	1651
	0.03	12.19(97)	2363
	0.05	9.18(65)	3138
16	0.02	13.59(184)	655
	0.03	9.92(157)	443
	0.05	5.57(53)	1597

Final result for $L/a = 12$

$$m_2^a = 0.0684(49)$$

Corresponding to a precision of

$$\frac{\delta m_2^a}{m_2^a} \approx 7.2\%$$

With wall-clock time of 210h on a single node of TCD's Lonsdale cluster (AMD Opteron, 8 cores / node)

Conclusion & Outlook

Conclusion

- [x] NSPT is a useful tool to extract perturbative expansions with decent precision.
- [x] Valuable for *gradient flow observables* (next talk!).

Shopping list

- [x] MPI parallelization
- [ ] Fermions
- [ ] Higher order integrator(s)

Appendix

Gauge Suppression

To construct a gauge transformation that suppresses the unwanted modes, one minimizes

$$W[U] = -\, a^2 \sum_{(x,\mu)\in\Lambda} \cosh\{a g_0 q_\mu(x)\}\,.$$

This is archived by applying a gauge transformation $\Omega(x) = e^{-i\,\epsilon\,\omega(x)}$

$$\omega(x) = \begin{cases} 0 & x_0 = T\,, \\ -\alpha\left(\frac{a}{L}\right)^3 \sum_{x,x_0=0} \sinh\{a\, g_0\, q_0(x)\}_{jj}\,\big|_{\text{traceless}} & x_0 = 0\,, \\ -\alpha \sum_\mu D^*_\mu \sinh\{a\, g_0\, q_\mu(x)\}\,\big|_{\text{traceless}} & \text{else}\,. \end{cases}$$