Schrödinger functional boundary conditions and improvement of the $\mathrm{SU}(N)$ pure gauge action for $N > 3$

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Outline

Motivation

- Schrödinger functional boundary fields know only for $N = 2, 3, 4$
  - More general analysis needed
- Boundary improvement for the gauge fields know for $N = 2, 3$
  - Needed for reliable coupling constant measurements on the lattice
- Applications in beyond the standard model physics and large N limit

Theory: Lattice action

$\mathcal{O}(a)$ improved $\mathrm{SU}(N)$ gauge action in the Schrödinger functional scheme

$$S = S_G + \delta S_{G,b} + S_{gf} + S_{FP},$$

$$S_G = \frac{1}{g_0^2} \sum_p \mathrm{Tr}[1 - U(p)],$$

$$\delta S_{G,b} = \frac{1}{g_0^2} (c_t - 1) \sum_{p_t} \mathrm{Tr}[1 - U(p_t)],$$

$$c_t = 1 + \left(c_t^{(1,0)} + N_{\mathrm{F}} c_t^{(1,1)} \right) g_0^2 + \mathcal{O}(g_0^4)$$

- For the specific form of S_{gf} and S_{FP}, see[1]

[1] M. Lüscher, R. Narayanan, P. Weisz, U. Wolff, hep-lat/9207009v1

Theory: Schrödinger functional

Schrödinger functional boundary conditions

$$U_k(t=0,\vec{x}) = \exp[aC_k], \quad U_k(t=L,\vec{x}) = \exp[aC'_k]$$

$$C_k = \frac{\mathrm{i}}{L}\mathrm{diag}(\phi_1(\eta),\ldots,\phi_n(\eta)), \quad C'_k = \frac{\mathrm{i}}{L}\mathrm{diag}(\phi'_1(\eta),\ldots,\phi'_n(\eta))$$

- These boundary conditions induce a constant chromo-electric field

Effective action

$$\Gamma = -\ln\left\{\int D[\psi]D[\bar{\psi}]D[U]D[c]D[\bar{c}]e^{-S}\right\} = g_0^{-2}\Gamma_0 + \Gamma_1 + \mathcal{O}(g_0^2)$$

Theory: Step scaling

Running coupling

$$g^2 = \frac{\partial\Gamma_0}{\partial\eta} \Big/ \frac{\partial\Gamma}{\partial\eta} = g_0^2 - g_0^4 \frac{\partial\Gamma_1}{\partial\eta} \Big/ \frac{\partial\Gamma_0}{\partial\eta} + \mathcal{O}(g_0^6)$$

Lattice step scaling function and its perturbative expansion

$$\begin{aligned} \Sigma(u, s, L/a) &= g^2(g_0, sL/a)\big|_{g^2(g_0, L/a)=u} \\ &= u + \left[\Sigma_{1,0}(s, L/a) + \Sigma_{1,1}(s, L/a) N_F\right] u^2, \end{aligned}$$

Definition of δ_i

$$\delta_i = \frac{\Sigma_{1,i}(2, L/a)}{\sigma_{1,i}(2)} = \frac{\Sigma_{1,i}(2, L/a)}{2 b_{0,i} \ln 2}, \qquad i = 0, 1.$$

$$b_{0,0} = 11 N_c/(48\pi^2), \qquad b_{0,1} = N_f T_R/(12\pi^2).$$

Fundamental domain

Boundary fields ϕ and ϕ' are within the fundamental domain if

$$\phi_1 < \phi_2 < \ldots < \phi_n, \quad |\phi_i - \phi_j| < 2\pi, \quad \sum_{i=1}^{N} \phi_i = 0.$$

Vectors ϕ form a $N-1$ simplex with vertices

$$\begin{aligned}
\mathbf{X}_1 &= \frac{2\pi}{N}(-N+1, 1, 1, \ldots, 1) \\
\mathbf{X}_2 &= \frac{2\pi}{N}(-N+2, -N+2, 2, \ldots, 2) \\
\mathbf{X}_3 &= \frac{2\pi}{N}(-N+3, -N+3, -N+3, 3, \ldots, 3,) \\
&\vdots \\
\mathbf{X}_{N-1} &= \frac{2\pi}{N}(-1, -1, \ldots, -1, N-1) \\
\mathbf{X}_N &= (0, 0, \ldots, 0).
\end{aligned}$$

Fundamental domain $N = 4$

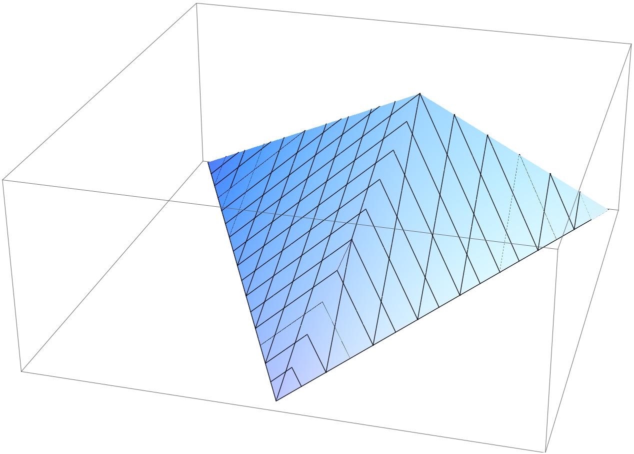

Figure : Fundamental domain of $\mathrm{SU}(4)$

Fundamental domain $N > 3$

- Define a mapping[2] $R_{i,j}(\phi)$ s.t. it reflects points in FD w.r.t. $(N-2)$ d hyperplane
  - Goes through vertices $\mathbf{X}_k$, $k \neq i, j$
  - Intersects line connecting $\mathbf{X}_i$ and $\mathbf{X}_j$ at the middle

Composite mapping from FD to itself

$$M(\phi) = \left(R_{1,N-1} \circ R_{2,N-2} \circ \cdots \circ R_{[N/2],N-[N/2]}\right)(\phi)$$

- ϕ' derived using above mapping

Transformation rule for components of ϕ' and ϕ

$$\phi'_i = \phi_{N-i+1}$$

[2] $R_{i,i}(\phi)$ is the identity mapping and $[x]$ means the integer part of x

Fundamental domain $N = 4$

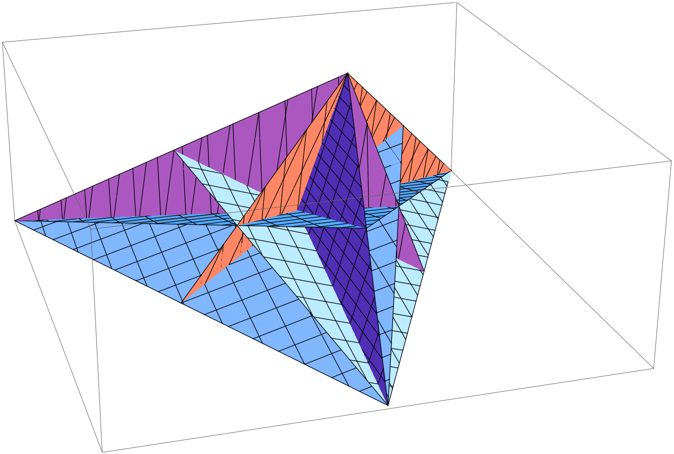

Figure : All possible $R_{i,j}(\phi)$ hyperplanes on FD of $\mathrm{SU}(4)$

Fundamental domain $N > 3$

Conjecture: Signal to noise maximized if ϕ and ϕ' chosen s.t.

- they are as far from the edges and each other as possible

- ϕ and ϕ' transformed to each other using the previous transformation
- We choose ϕ to be in the middle of a line connecting $\mathbf{X}_1$ and the centeroid of FD
- We associate flow[3] $t(\eta)$ to direction which is mirrored by $R_{1,N-1}(\phi)$ and points outside from FD

$$
\begin{aligned}
t(\eta) &= \frac{\eta N}{2\pi(N-2)} \left(\mathbf{X}_1 - \mathbf{X}_{N-1}\right) \\
&= \left(-\eta, \frac{2\eta}{N-2}, \ldots, \frac{2\eta}{N-2}, -\eta\right)
\end{aligned}
$$

[3] The normalization is chosen to match the standard case of $\mathrm{SU}(3)$

Fundamental domain $N > 3$

Example of the boundary fields

$$\mathrm{SU}(4) \quad \phi = \begin{cases} -\eta - 9\pi/8 \\ \eta + \pi/8 \\ \eta + 3\pi/8 \\ -\eta + 5\pi/8 \end{cases} \qquad \phi' = \begin{cases} \eta - 5\pi/8 \\ -\eta - 3\pi/8 \\ -\eta - \pi/8 \\ \eta + 9\pi/8 \end{cases}$$

$$\mathrm{SU}(5) \quad \phi = \begin{cases} -\eta - 6\pi/5 \\ 2\eta/3 \\ 2\eta/3 + \pi/5 \\ 2\eta/3 + 2\pi/5 \\ -\eta + 3\pi/5 \end{cases} \qquad \phi' = \begin{cases} \eta - 3\pi/5 \\ -2\eta/3 - 2\pi/5 \\ -2\eta/3 - \pi/5 \\ -2\eta/3 \\ \eta + 6\pi/5 \end{cases}$$

$$\mathrm{SU}(6) \quad \phi = \begin{cases} -\eta - 5\pi/4 \\ \eta/2 - \pi/12 \\ \eta/2 + \pi/12 \\ \eta/2 + \pi/4 \\ \eta/2 + 5\pi/12 \\ -\eta + 7\pi/12 \end{cases} \qquad \phi' = \begin{cases} \eta - 7\pi/12 \\ -\eta/2 - 5\pi/12 \\ -\eta/2 - \pi/4 \\ -\eta/2 - \pi/12 \\ -\eta/2 + \pi/12 \\ \eta + 5\pi/4 \end{cases}$$

Comparison to literature

- First approximation of signal strength is $\partial_\eta \Gamma_0 = \partial_\eta g_0^2 S[V]$
- In[4] Lucini et.al. used boundary condition

$$\phi = \begin{cases} -\eta/2 - \sqrt{2}\pi/4 \\ -\eta/2 - (2-\sqrt{2})\pi/4 \\ \eta/2 + (2-\sqrt{2})\pi/4 \\ \eta/2 + \sqrt{2}\pi/4 \end{cases} \qquad \phi' = \begin{cases} \eta/2 - (2+\sqrt{2})\pi/4 \\ \eta/2 - (4-\sqrt{2})\pi/4 \\ -\eta/2 + (4-\sqrt{2})\pi/4 \\ -\eta/2 + (2+\sqrt{2})\pi/4 \end{cases}$$

- $\partial_\eta \Gamma_0[\mathrm{Us}] = 48L^2 \sin((2\eta + \pi/2)/L^2)$ vs.
  $\partial_\eta \Gamma_0[\mathrm{Lucini}] = 24L^2 \sin((\eta - \pi/2)/L^2)$
- Enhancement by a factor of 2

[4] B. Lucini and G. Moraitis, hep-lat/0805.2913

Boundary improvement with $N > 3$

- Improvement coefficient $c_t^{(1,0)}$ previously know to one loop only for $N = 2, 3$
  - $c_t^{(1,0)}(\mathrm{SU}(2)) = -0.0543(5)$, $c_t^{(1,0)}(\mathrm{SU}(3)) = -0.08900(5)$
- Can similarly[5] be calculated for $N > 3$
- Preliminary results

N	$c_t^{(1,0)}$	$\delta c_t^{(1,0)}$
2	-0.0543	0.0002
3	-0.088	0.005
4	-0.1220	0.0002
5	-0.154	0.004
6	-0.1859	0.0008
7	-0.218	0.004
8	-0.249	0.004

[5] M. Lüscher, R. Narayanan, P. Weisz, U. Wolff, hep-lat/9207009v1

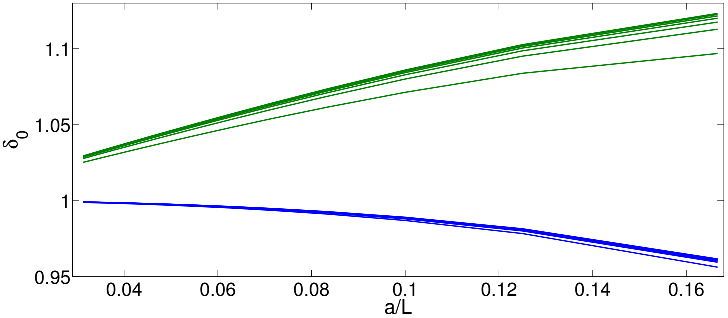

Figure : The unimproved (green) and improved (blue) one loop lattice step scaling function normalized to the continuum limit as a function of a/L for $\mathrm{SU}(N)$ pure gauge with $2 \leq N \leq 8$

Boundary improvement with $N > 3$

- We expect
  $c_t^{(1,0)} = AC_2(R) + BC_2(G) = (A/2 + B)N - A/(2N)$

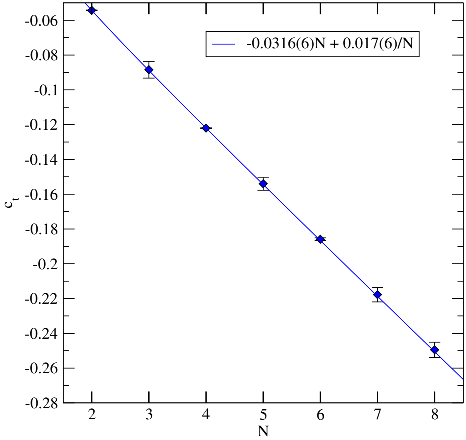

Figure : $c_t^{(1,0)}$ as a function of N and $C_1 N + C_2/N$ fit to the data

Thank you!