Renormalization of the momentum density on the lattice using shifted boundary conditions

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Why is there a need for a renormalized Energy-momentum-tensor (EMT) on the lattice?

- In the continuum we have Poincaré invariance $\Longrightarrow \partial_\mu T^{\mu\nu} = 0$ for on-shell correlation functions.
- Due to Ward Identities (WI) one can show that no renormalization constant is needed.

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Setup & Definitions: $T_{\mu\nu}$ (SU(3) YM-theory)

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\begin{aligned}
T_{\mu\nu} &= \theta_{\mu\nu} + \frac{1}{4}\delta_{\mu\nu}\theta \\
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Continuum...

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$

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The momentum density $\hat{T}_{0k}$ belongs to a six-dimensional representation of H(4) together with off-diagonal elements and since no other gauge invariant operators with equal or less dimension exist, it renormalizes multiplicatively!

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GOAL:

Non-perturbative calculation of Z_T

$$\boxed{(\hat{T}^{\mathrm{clov}}_{\mu\nu})_R = Z_T (\hat{T}^{\mathrm{clov}}_{\mu\nu})_0 \quad (\mu \neq \nu)}$$

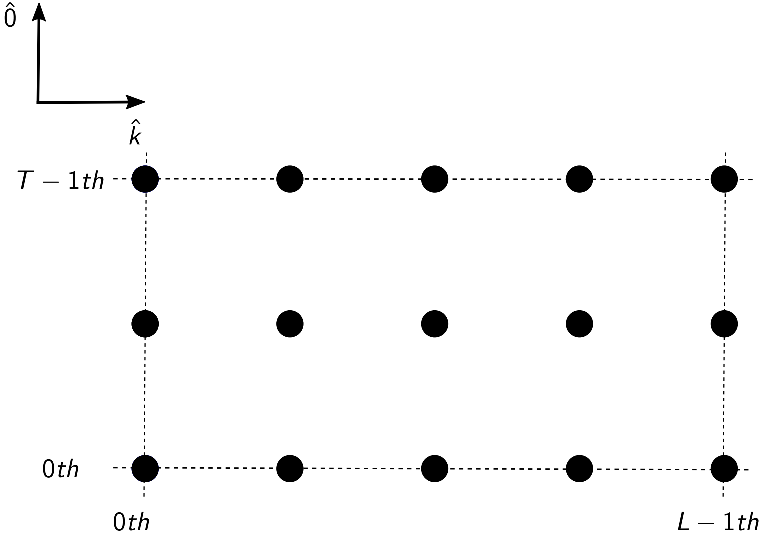
STRATEGY:

Shifted boundary conditions...

"The matching will occur by calculating the entropy of the system in a renormalized and unrenormalized way ... "

Shifted boundary conditions

H. B. Meyer and L. Giusti, *JHEP* **1301** (2013) 140 [arXiv:1211.6669]



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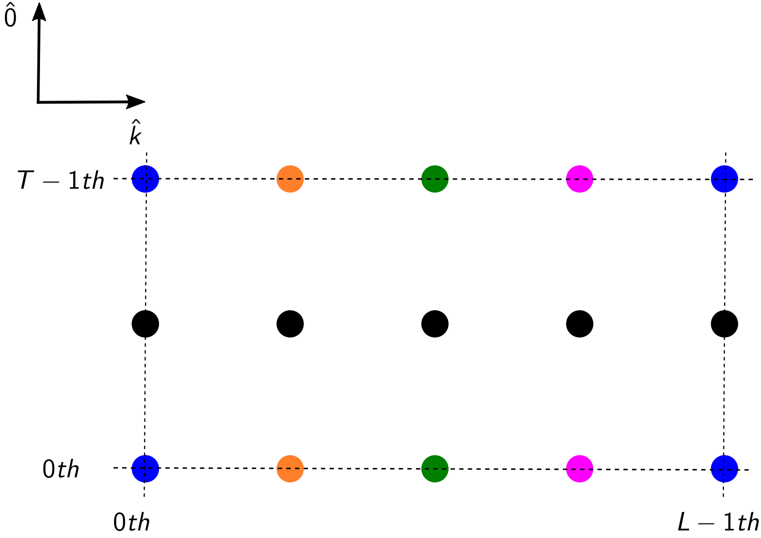

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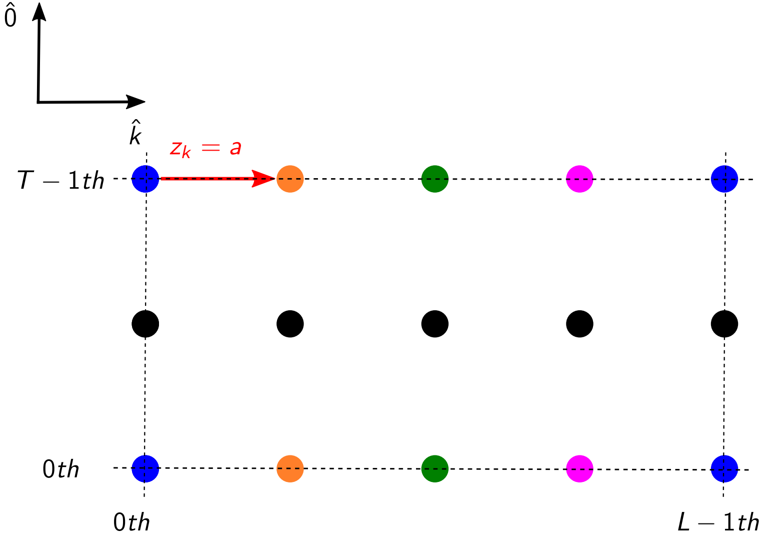

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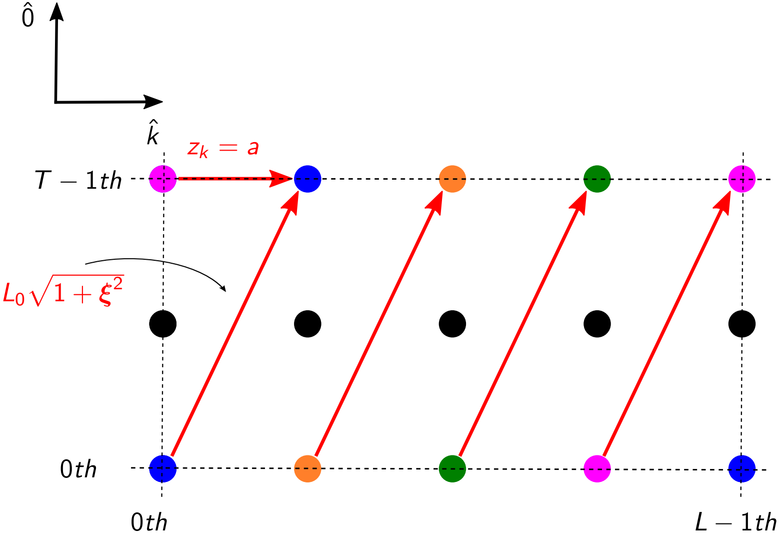

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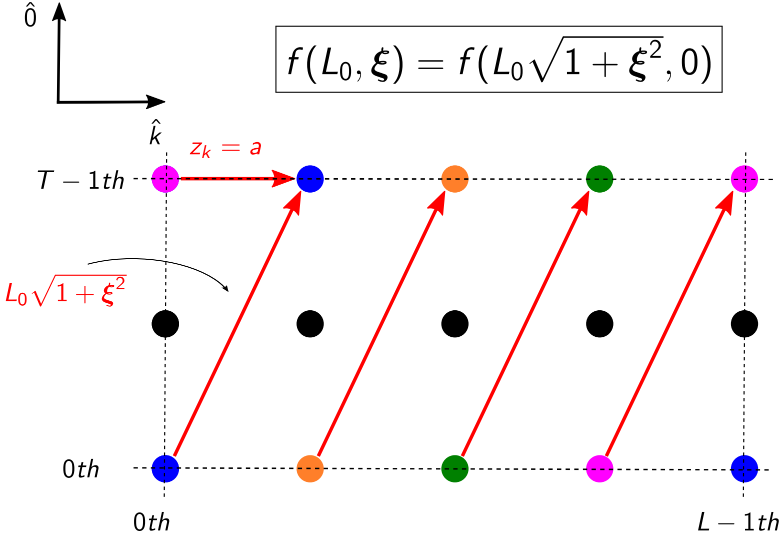

Consequences of $f(L_0,\xi) = f(L_0\sqrt{1+\xi^2},0)$

L. Giusti, *Wed 31th, 11:40, Seminar Room B*

- Apart from the usual WI's that relate correlation functions of $\hat{T}_{0k}$ and $\hat{T}_{00}$ ...

  $\Longrightarrow$ there are non-trivial WI's connecting both systems (shifted and not shifted).

- There is a way of extracting termodynamic potentials by measuring the response of the system to the shift:

$$\boxed{s = -\frac{L_0(1+\xi^2)^{3/2}}{\xi_k} Z_T \left\langle \hat{T}_{0k} \right\rangle_{\boldsymbol{\xi}\neq\mathbf{0}}} \tag{1}$$

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- The shift parameter ξ can also be used to vary the temperature in small steps without altering the UV behavior of the theory ($T = 1/N_T a$).

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Consistency checks (free theory) ...

analytic calculations ...

$\left\langle \hat{T}_{0k}^{\text{clov}} \right\rangle_{\xi}^{\text{free,lat}}$ calculated in PT:

$$\Longrightarrow (s/T^3)^{\text{free,lat}} \text{ via Eq. (1).}$$

$$\Longrightarrow \frac{(s/T^3)^{\text{free,lat}}}{s_{SB}/T^3} = 1 + c_1 (a/L_0)^2$$

where,

$$s_{SB}/T^3 = 4\pi^2 (N_c^2 - 1)/45$$

H. B. Meyer and L. Giusti, *JHEP* **1301** '13 140

simulation ...

Difficult to simulate the free theory:

1. critical slowdown
2. finite size effects $\sim e^{-g_0^2 L/\beta}$

We chose $N_T = 2$:

g_0^2	L
0.429	32
0.3	48
0.24	60
0.2	72

$\xi = (1,0,0)$

$\xi = (1/2,0,0)$

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Extrapolating $g_0^2 \to 0$ ...

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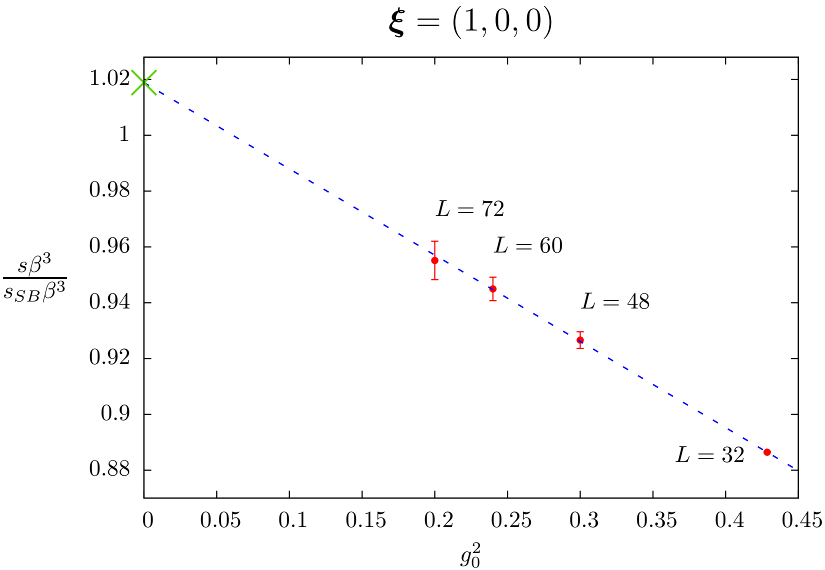
Simulating the free lattice theory ...



Extracting Z_T (I) ...

- First, we need a renormalized quantity related to $\left\langle \hat{T}_{0k} \right\rangle_\xi$ to compare our data with ...

Generating function associated to the momentum distribution

Definition:

$$
\begin{aligned}
K(L_0,\xi) &= -\log \frac{Z(L_0,\xi)}{Z(L_0)} = -\log \frac{\mathrm{Tr}\{e^{-L_0(\hat{H}-i\xi\hat{P})}\}}{\mathrm{Tr}\{e^{-L_0\hat{H}}\}} \\
\left.\frac{\partial K(L_0,\xi)}{\partial \xi}\right|_{\xi=\xi'} &= -L_0 Z_T \left\langle \int d^3x\, \hat{T}_{0k}(x) \right\rangle_{\xi'}
\end{aligned}
$$

"Since it is a ratio of partition functions, it has a finite and universal continuum limit. All derived quantities are renormalized as well!"

Extracting Z_T (II) ...

- How to systematically compute ratios of partition functions was first investigated by M. Della Morte and L. Giusti, *Comput. Phys. Commun.* **180** (2009) 819

- L. Giusti and H. B. Meyer, *PRL* **106** (2011) 131601 computed $K(L_0, \xi)$ for several values of the inverse coupling between $[5.9, 7.584]$.
- By a limiting procedure they extracted the entropy as the second cumulant of $K(L_0, \xi)$:

$$\left(\frac{s}{T^3}\right)_R = \frac{\partial^2}{\partial \xi_1^2} \left. \frac{K(L_0, \xi)}{T^3 L^3} \right|_{\xi=0} = \lim_{a \to 0} \frac{2K(L_0, \xi)}{|\xi|^2 T^3 L^3}$$

Surprisingly mild cutoff effects!

Extracting Z_T (III) ...

Measuring $\left\langle \hat{T}^{\rm clov}_{0k} \right\rangle_{\boldsymbol{\xi}}$ and taking the first derivative with respect to ξ_1 one gets an *unrenormalized* prediction for $(s/T^3)_0$

$$\left(\frac{s}{T^3}\right)_0 = \frac{\partial}{\partial \xi_1} \left. \frac{\left\langle \hat{T}^{\rm clov}_{0k} \right\rangle_{\boldsymbol{\xi}}}{T^4} \right|_{\boldsymbol{\xi}=0} = \lim_{\xi_1 \to 0} \left[\frac{\left\langle \hat{T}^{\rm clov}_{0k} \right\rangle_{\boldsymbol{\xi}}}{T^4 \xi_1} - \frac{\cancelto{0}{\left\langle \hat{T}^{\rm clov}_{0k} \right\rangle_{\boldsymbol{\xi}=0}}}{T^4 \xi_1} \right]$$

$$Z_T^{\rm clov} = \frac{\left(\frac{s}{T^3}\right)_R}{\left(\frac{s}{T^3}\right)_0}$$

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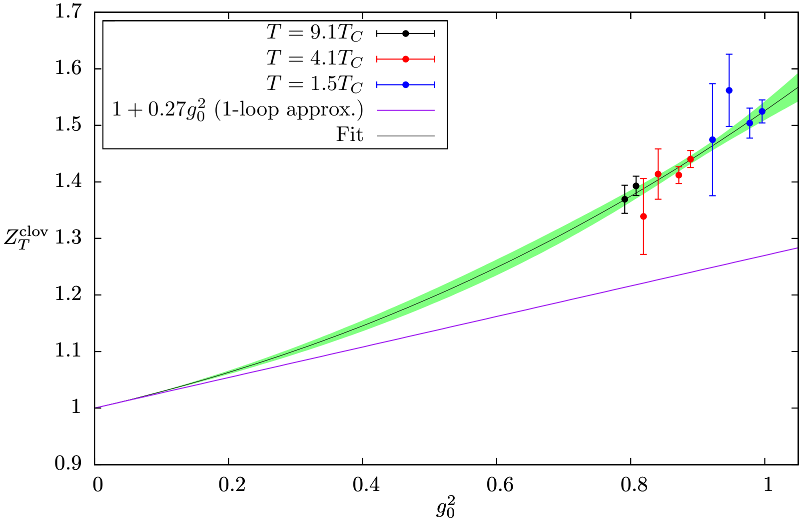
$$\boxed{Z^{\mathrm{clov}}_T = \frac{\left(\frac{s}{T^3}\right)_R}{\left(\frac{s}{T^3}\right)_0}}$$

Results (I) ...

Lat	$6/g_0^2$	L_0/a	L/a	$\left(\frac{s}{T^3}\right)_{Giusti,Meyer}$	$\left(\frac{s}{T^3}\right)_0$	Z_T^{clov}
A_2	6.024	5	16	4.98(4)	3.27(3)	1.52(2)
A_3	6.137	6	18	4.88(6)	3.24(4)	1.50(3)
A_4	6.337	8	24	5.12(19)	3.28(6)	1.56(6)
A_5	6.507	10	30	4.9(3)	3.32(9)	1.47(10)
B_2	6.747	5	16	6.53(6)	4.53(2)	1.44(2)
B_3	6.883	6	18	6.40(6)	4.53(2)	1.41(2)
B_4	7.135	8	24	6.42(20)	4.54(2)	1.41(4)
B_5	7.325	10	30	6.1(3)	4.55(4)	1.33(7)
C_2	7.426	5	20	7.13(8)	5.11(3)	1.39(2)
C_3	7.584	6	24	6.94(12)	5.07(3)	1.36(3)

Results (II) ...



One loop approximation taken from S. Caracciolo, *et al.*, *Physics Letter B* **260** (1991) 3

Conclusions and Outlook ...

- The fit function is given by

$$Z_T^{\mathrm{clov}} = \frac{1 + 0.1368 g_0^2 + 0.1858 g_0^4}{1 - 0.1332 g_0^2}$$

- the error is ~ 0.008 when $g_0^2 \in [0.8, 1.0]$

Possible improvements ...

1. Try to match better the one loop approximation expression from S. Caracciolo, *et al.* $\Longrightarrow$ smaller values of g_0^2.
2. Include fermions (full QCD, other gauge groups SU(N), ...)
3. Calculate the diagonal components renormalization constant z_T via

$$Z_T \left\langle \hat{T}_{0k}^{\mathrm{clov}} \right\rangle_\xi = \frac{\xi_k}{1 - \xi_k^2} z_T \left\langle \hat{T}_{00}^{\mathrm{clov}} - \hat{T}_{kk}^{\mathrm{clov}} \right\rangle_\xi$$

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