Walking signals in Nf=8 QCD on the lattice

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Lattice 2013, 29 July 2013 @ Mainz
LatKMI collaboration

Y. Aoki, T. Aoyama, M. Kurachi, T. Maskawa, K.-i. Nagai, K. Miura, H. Ohki,

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名古屋大学

A. Shibata

KEK
In LatKMI Collaboration;

♦ Lesson from SD about the mass correction in the (finite-size) hyperscaling relation.

*Study of the conformal hyperscaling relation through the Schwinger-Dyson equation*
Yasumichi Aoki, Tatsumi Aoyama, Masafumi Kurachi, Toshihide Maskawa, Kei-ichi Nagai, Hiroshi Ohki, Akihiro Shibata, Koichi Yamawaki, Takeshi Yamazaki, Jan 2012,

♦ Nf=12 is consistent with the conformal behavior, $\gamma_m=0.4$-$0.5$.

*Lattice study of conformality in twelve-flavor QCD*
Yasumichi Aoki, Tatsumi Aoyama, Masafumi Kurachi, Toshihide Maskawa, Kei-ichi Nagai, Hiroshi Ohki, Akihiro Shibata, Koichi Yamawaki, Takeshi Yamazaki, Jul 2012,

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♦ Calculation of the scalar spectrum (flavor-singlet meson and glueball) in Nf=12

*The scalar spectrum of many-flavour QCD*
Yasumichi Aoki, Tatsumi Aoyama, Masafumi Kurachi, Toshihide Maskawa, Kei-ichi Nagai, Hiroshi Ohki, Enrico Rinaldi, Akihiro Shibata, Koichi Yamawaki, Takeshi Yamazaki, Feb 19, 2013,

*Light composite scalar in twelve-flavor QCD on the lattice*
Yasumichi Aoki, Tatsumi Aoyama, Masafumi Kurachi, Toshihide Maskawa, Kei-ichi Nagai, Hiroshi Ohki, Enrico Rinaldi, Akihiro Shibata, Koichi Yamawaki, Takeshi Yamazaki,
e-Print: arXiv:1305.6006 [hep-lat].
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Plan of the Talk:

1. Introduction
2. Simulation setup in Nf=8 QCD
3. Result of Nf=8 case (walking?)
4. Summary

★ Phenomenological (Light scalar (0^{++}), S-param.)
   ⇒ Ohki, Rinaldi, Yamazaki and Aoki
1. Introduction

- LQCD with many fermions
- → Candidate of the (walking) technicolor
Requirements for the successful WTC theory

- **spontaneous chiral symmetry breaking**
- **running coupling “walks”** = slowly changing with $\mu$: $\to$ nearly conformal
- **large mass anomalous dimension**: $\gamma_m \approx 1$
- **light scalar $0^{++}$** ($m_H = 126$ GeV @ LHC!)
  - with input $F_\pi = 246 / \sqrt{N}$ GeV ($N$: # weak doublet in techni-sector)
    - to reproduce $W^\pm$ mass
  - typical QCD like theory: $M_{\text{Had}} \gg F_\pi$ (ex.: QCD: $m_\rho / f_\pi \approx 8$)
    - Naive TC: $M_{\text{Had}} > 1,000$ GeV
    - $0^{++}$ is a special case: pseudo Nambu-Goldstone boson of scale inv.
      - is it really so?
Extended Technicolor (ETC)

- fermion masses → extended technicolor (ETC)
- New strong interaction of \( SU(N_{ETC}) \): \( N_{ETC} > N_{TC} \), \( T_{ETC} = (T, f) \): \( T \in TC, f \in SM \)
- SSB: \( SU(N_{ETC}) \rightarrow SU(N_{TC}) \times SM @ \Lambda_{ETC} (\gg \Lambda_{TC}) \)
  \[ \frac{1}{\Lambda_{ETC}^2} \overline{T}T\bar{f}f \rightarrow m_f = \frac{\langle \overline{T}T \rangle_{ETC}}{\Lambda_{ETC}^2} \]
- \[ \frac{1}{\Lambda_{ETC}^2} \overline{f}f\bar{f}f \text{ FCNC} \]
- FCNC should be small ⇔ top or bottom quark mass should be produced

\[ \rightarrow \text{walking TC} \]
Walking Technicolor

- key: to realize suppressed FCNC and appropriate size of fermion masses
  
  [Holdom, Yamawaki-Bando-Matsumoto]

- renormalized gauge coupling
  - to run very slowly \( \text{walking} \)
  - logarithmically divergent at low energies → to produce techni-pions

- mass anomalous dimension
  - large: \( \gamma_m \approx 1 \)
Walking Technicolor

- key: to realize suppressed FCNC and appropriate size of fermion masses

- renormalized gauge coupling
  - to run very slowly (walking)
  - logarithmically divergent at low energies $\rightarrow$ to produce techni-pions

- mass anomalous dimension
  - large: $\gamma_m \sim 1$

Is it possible to construct such a theory?
conformal window and walking coupling
- non-Abelian gauge theory with $N_f$ massless fermions -

- $N_f$

- $N_f^{AF}$

- $N_f^{crit}$

- Asymptotic non-free

- Conformal window

- Walking Technicolor

- QCD-like

- Walking Technicolor could be realized just below the conformal window

- crucial information: $N_f^{crit}$ & mass anomalous dimension around $N_f^{crit}$
conformal window and walking coupling
- non-Abelian gauge theory with $N_f$ massless fermions -

- Walking Technicolor could be realized just below the conformal window

- crucial information: $N_f^{\text{crit}}$ & mass anomalous dimension around $N_f^{\text{crit}}$
**One-family Extended TechniColor model**

\[
\begin{pmatrix}
    u \\
    d \\
    u \\
    d \\
    e \\
    \nu_e
\end{pmatrix}
\begin{pmatrix}
    c \\
    s \\
    c \\
    s \\
    \mu \\
    \nu_\mu
\end{pmatrix}
\begin{pmatrix}
    t \\
    b \\
    t \\
    b \\
    \tau \\
    \nu_\tau
\end{pmatrix}
\begin{pmatrix}
    U_1 \\
    D_1 \\
    U_1 \\
    D_1 \\
    E_1 \\
    N_1
\end{pmatrix}
\cdots
\begin{pmatrix}
    U_{N_{TC}} \\
    D_{N_{TC}} \\
    U_{N_{TC}} \\
    D_{N_{TC}} \\
    E_{N_{TC}} \\
    N_{N_{TC}}
\end{pmatrix}
\]

- \(SU(3 + N_{TC})\)
- \(SU(2 + N_{TC})\)
- \(SU(1 + N_{TC})\)
- \(SU(N_{TC})\)

8-flavor \(SU(N_{TC})\) technicolor

We consider \(N_{TC} = 3\)

---

**Diagram:**

- Techni fermion
- SM fermion
- ETC gauge boson
Many flavor QCD
⇒ Candidate of walking/conformal

Our investigation in $N_f=12$
⇒ consistent with the conformal with $\gamma=0.4--0.5$.

not favor as WTC (model building)

Thus, we investigate $N_f=8$ QCD.
strong coupling dynamics and non-perturbative

Lattice simulation of $N_f=8$ QCD
Investigation of Nf=8 (direct and indirect):

LHC (J. Kuti et.al.) → $S \chi$ SB

A. Hasenfratz et.al. → If Nf=8 would be walking, ...
   In comparison with Nf=4 and 12.

M. Lombardo et.al. → broken, but near the conformal edge

Y. Iwasaki et.al. → conformal
   (Yukawa-type correlator in Nf=7 and 16)

LatKMI → In this talk (walking?)
2. Simulation setup in Nf=8 case

*Walking signals in Nf=8 QCD on the lattice*

Yasumichi Aoki, Tatsumi Aoyama, Masafumi Kurachi, Toshihide Maskawa, Kei-ichi Nagai, Hiroshi Ohki, Akihiro Shibata, Koichi Yamawaki, Takeshi Yamazaki, Feb 27, 2013,

Published in *Phys.Rev.* D87 (2013) 094511,
e-Print: [arXiv:1302.6859](https://arxiv.org/abs/1302.6859) [hep-lat].
Simulation for $\text{Nf}=8$

lattice action

(8 flavor Hybrid Monte-Carlo simulation)

- Tree-level Symanzik gauge action
- Highly Improved Staggered Quarks = HISQ
  (without tadpole improvement and mass correction in Naik term)

★ parameter set

- $\beta \equiv 6/g^2 = 3.8, (3.7, 3.9, 4.0)$, T/L=4/3 fixed.

<table>
<thead>
<tr>
<th>$V$</th>
<th>$12^3 \times 16$</th>
<th>$18^3 \times 24$</th>
<th>$24^3 \times 32$</th>
<th>$30^3 \times 40$</th>
<th>$36^3 \times 48$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$mf$</td>
<td>0.04~0.16</td>
<td>0.04~0.1</td>
<td>0.02~0.1</td>
<td>0.02~0.07</td>
<td>0.015~0.03</td>
</tr>
</tbody>
</table>

★ Measurements (P+AP method ⇒ double size in T-dir.)

- $M_\pi, F_\pi, M_\rho$, chiral condensate
- $M_\pi L > 4--5$
HISQ with Nf=8:

effective mass for the lowest mf(=0.015)
on the largest size (L=36)
→ good plateau

$M_{\pi}^{PS}=M_{\pi}^{SC}$, $M_{\rho}^{PV}=M_{\rho}^{VT}$
→ good flavor symmetry

**FIG. 2** (color online). Effective masses of PS meson, $M_{\pi}^{\text{eff}}$, at $L=36$. Triangles and other symbols denote results from point sink correlators with random wall source and corner wall source, respectively. Fit results with error band obtained from random wall source correlator are also plotted by solid lines.

**FIG. 4** (color online). Comparisons of $M_{\pi}$ and $M_{\text{SC}}$, and of $M_{\rho}^{(PV)}$ and $M_{\rho}^{(VT)}$ as a function of $m_f$ with largest volume data at each $m_f$. 
3. Result of Nf=8 case
$F_\pi / M_\pi$ for $N_f=8$, 12 and 4 (flat or divergent in $\chi$-limit?)

**FIG. 5.** $F_\pi / M_\pi$ as a function of $M_\pi$ for $N_f = 8$ (left), $N_f = 12$ at $\beta = 3.7$ (center), and $N_f = 4$ at $\beta = 3.7$ (right).

$M_\rho / M_\pi$ for $N_f=8$, 12 and 4 (flat or divergent in $\chi$-limit?)

**FIG. 6.** $M_\rho / M_\pi$ as a function of $M_\pi$ for $N_f = 8$ (left) and $N_f = 12$ at $\beta = 3.7$ (right).
Nf=8 $\Rightarrow$ spontaneous chiral symmetry breaking? (S $\chi$ SB)

Chiral Perturbation Theory test (ChPT)

In S $\chi$ SB; $M_\pi^2 = C_1^m m_f + C_2^m m_f^2 + \cdots$, $F_\pi = F + C_1^F m_f + C_2^F m_f^2 + \cdots$

$\Rightarrow$ Polynomial fit

We regard the data on the largest volume at each mf as the ones on the infinite volume. (Backup figs.)

We don’t discuss the chiral log behavior in this talk.

However, we discussed the chiral log in the published paper.
ChPT analysis (quadratic fit) of $F_\pi$

Chiral limit value, $F$, is zero or non-zero?

Natural chiral expansion parameter

$$\chi = N_f \left( \frac{M_\pi}{4\pi F} \right)^2$$

<table>
<thead>
<tr>
<th>fit range ($m_f$)</th>
<th>$F$</th>
<th>$\chi(m_f = 0.015)$</th>
<th>$\chi(m_f = m_{\text{max}})$</th>
<th>$\chi^2$/$\text{dof}$</th>
<th>dof</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.015–0.04</td>
<td>0.0310(13)</td>
<td>3.74</td>
<td>11.80</td>
<td>0.46</td>
<td>1</td>
</tr>
<tr>
<td>0.015–0.05</td>
<td>0.0278(8)</td>
<td>4.64</td>
<td>19.28</td>
<td>5.56</td>
<td>2</td>
</tr>
<tr>
<td>0.015–0.06</td>
<td>0.0284(6)</td>
<td>4.44</td>
<td>23.2</td>
<td>4.09</td>
<td>3</td>
</tr>
<tr>
<td>0.015–0.07</td>
<td>0.0293(5)</td>
<td>4.18</td>
<td>26.5</td>
<td>4.46</td>
<td>4</td>
</tr>
<tr>
<td>0.015–0.08</td>
<td>0.0296(4)</td>
<td>4.10</td>
<td>30.6</td>
<td>4.06</td>
<td>5</td>
</tr>
<tr>
<td>0.015–0.10</td>
<td>0.0311(3)</td>
<td>3.70</td>
<td>37.0</td>
<td>7.85</td>
<td>6</td>
</tr>
<tr>
<td>0.015–0.16</td>
<td>0.0349(2)</td>
<td>2.94</td>
<td>54.0</td>
<td>34.2</td>
<td>9</td>
</tr>
</tbody>
</table>

The above analysis suggests that our result in $N_f = 8$ is consistent with $S\chi$SB phase with $F = 0.0310(13)$.

The chiral log is not seen in our simulation region of $m_f$. 
Chiral condensate (direct and indirect calc.)

direct: \( \text{Tr}[D_{\text{HI SQ}}^{-1}(x,x)]/4 \)

indirect: \( \Sigma = F_\pi^2 M_\pi^2/(4mf) \)

based on the GMOR relation

In chiral limit \( \text{(quadratic fit in } 0.015 \leq mf \leq 0.04) \)

\[
\langle \bar{\psi}\psi \rangle |_{m_f \to 0} = 0.00052(5), \quad \Sigma |_{m_f \to 0} = 0.00059(13) . \quad F^2 \cdot \left( \frac{M_\pi^2}{4mf} \right) |_{m_f \to 0} = 0.00050(3)
\]
Summary-1. ChPT analysis

- The quadratic fit was done in $0.015 \leq mf \leq 0.04$.
- $F_\pi > 0$, $M_\rho > 0$, Condensate $> 0$, $M_\pi = 0$ in the $\chi$-limit.
- $N_f = 8$ is consistent with $S\chi SB$ in the small $mf$ region.
- The expansion parameter $\chi = O(1)$ in the smallest $mf$ (self-consistent).
- The chiral log is not seen.
- $\Rightarrow$ simple $S\chi SB$ phase?
Scenario of Walking Dynamics (2-loop/ladder);
Case-1: probe $m_f \ll m_D \rightarrow S\chi$ SB-like
Case-2: probe $m_f \gg m_D \rightarrow$ conformal-like

How to translate into
$\Rightarrow \text{Spectrum?}$ $S\chi$ SB and conformal?
$F_\pi$ vs $mf$:

Polynomial-like behavior (ChPT-like)

Power-like behavior? (Remnant (Impact) of conformality?)
$F_\pi$ vs $m_f$:

Polynomial-like behavior (ChPT-like)

Power-like behavior? (Remnant (Impact) of conformality?)
Fπ (left panel) and Mρ (right panel):

quadratic and linear fit in 0.015 ≤ mf ≤ 0.04
power fit in 0.05 ≤ mf ≤ 0.16 (green dotted line)

To confirm the conformal(-like) behavior,
⇒ Finite-size Hyperscaling analysis
Finite size Hyperscaling analysis
(conformal test)
Finite size Hyperscaling analysis

(conformal test)

Finite size Hyperscaling behavior with universal $\gamma$

$$LM_H = F_H(x), LF_H = G_F(x)$$

$$x \equiv L m^{1/(1+\gamma)}$$

(DeGrand, Del Debbio et al.)

\[ r(M_\pi) \neq r(F_\pi), \quad r(F_\pi) \sim 1.0 \]

not conformal;

Nf=8 may be in S\chi SB phase.
Comparison with $N_f=4$
: hyperscaling of $M_\rho L$ (in $S\chi$ SB)

$\beta=3.7$, $\gamma=0.0$

$\beta=3.7$, $\gamma=1.0$

$\beta=3.7$, $\gamma=2.0$

$M_\rho$: no scaling for $0 < \gamma < 2$

$\Rightarrow$ In this sense, it seems that $N_f=8$ is not $S\chi$ SB of ordinary QCD.
(data scaling, but not universal $\gamma$)
Simultaneous fit of hyperscaling with mass corrections

\[ \xi_H = C_0^H + C_1^H X + C_2^H \Lambda m_f^\alpha. \]

(same method with Nf=12)

Hyperscaling? in the middle region of mf (mf\( \geq 0.05 \) and \( \xi_\pi (=M \pi L) \geq 8 \))

The mass corrections might be needed, as done in Nf=12, from the lesson in SD analysis.

**TABLE XI.** Simultaneous FSHS fit with a correction term, \( \xi = C_0^H + C_1^H X + C_2^H \Lambda m_f^\alpha \) using several choices of \( \alpha \). The fitted region is \( m_f \geq 0.05 \) and \( \xi_\pi \geq 8 \).

<table>
<thead>
<tr>
<th>( \alpha = 0.889(55) )</th>
<th>( C_0^H )</th>
<th>( C_1^H )</th>
<th>( C_2^H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_\pi )</td>
<td>-0.005(25)</td>
<td>1.338(96)</td>
<td>1.494(37)</td>
</tr>
<tr>
<td>( \xi_F )</td>
<td>-0.0275(98)</td>
<td>0.4435(36)</td>
<td>—</td>
</tr>
<tr>
<td>( \xi_\rho )</td>
<td>0.53(16)</td>
<td>2.476(39)</td>
<td>—</td>
</tr>
</tbody>
</table>

\( \gamma = 0.9130(76), \chi^2/\text{dof} = 1.73, \text{dof} = 33 \)

---

<table>
<thead>
<tr>
<th>( \alpha = 1 ) fixed</th>
<th>( C_0^H )</th>
<th>( C_1^H )</th>
<th>( C_2^H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_\pi )</td>
<td>-0.014(24)</td>
<td>1.61(10)</td>
<td>1.31(15)</td>
</tr>
<tr>
<td>( \xi_F )</td>
<td>-0.012(10)</td>
<td>0.484(30)</td>
<td>-0.068(44)</td>
</tr>
<tr>
<td>( \xi_\rho )</td>
<td>0.01(19)</td>
<td>2.60(17)</td>
<td>0.25(24)</td>
</tr>
</tbody>
</table>

\( \gamma = 0.874(25), \chi^2/\text{dof} = 0.75, \text{dof} = 32 \)

---

<table>
<thead>
<tr>
<th>( \alpha = \frac{3-2\pi}{1+2\pi} ) fixed</th>
<th>( C_0^H )</th>
<th>( C_1^H )</th>
<th>( C_2^H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_\pi )</td>
<td>0.020(24)</td>
<td>1.52(39)</td>
<td>1.17(35)</td>
</tr>
<tr>
<td>( \xi_F )</td>
<td>-0.011(10)</td>
<td>0.572(34)</td>
<td>-0.158(52)</td>
</tr>
<tr>
<td>( \xi_\rho )</td>
<td>0.03(19)</td>
<td>2.91(30)</td>
<td>-0.15(36)</td>
</tr>
</tbody>
</table>

\( \gamma = 0.775(56), \chi^2/\text{dof} = 0.93, \text{dof} = 32 \)

\( \Rightarrow \) good \( \chi^2/\text{dof} \), but unclear which \( \alpha \) is better.
Summary-2, Finite-size Hyperscaling analysis

- In the region of $mf \geq 0.05$, hyperscaling is seen. (Remnant (“Impact”) of conformality?)
- $\gamma$ for each observable in the finite-size hyperscaling are not universal.
- Mass correction for $mf \geq 0.05$ (heavier region)? ⇔ Lesson from the SD analysis with the (large) mass.
- Simultaneous fit of hyperscaling with mass correction terms gives the common $\gamma = 0.78--0.93 \sim 1$.
- It suggests that in $Nf=8$ there may exists the “remnant” of the conformality in the middle range of $mf$. 
Summary

- SU(3) gauge theories with 4, 12 and 8 HISQ quarks.
- Nf=8; consistent with SχSB in the small mass region of our simulation and the remnant of the conformality in the middle region of mf with γ~1. (In contrast to Nf=4 and 12 cases.)

\[ N_f = 8 \rightarrow \text{Candidate of Walking dynamics} \]
Summary

- SU(3) gauge theories with 4, 12 and 8 HISQ quarks.
- Nf=8; consistent with SχSB in the small mass region of our simulation and the remnant of the conformality in the middle region of mf with γ~1. (In contrast to Nf=4 and 12 cases.)

**Nf=8 → Candidate of Walking dynamics**

Future plan

- Simulation on larger volumes at lighter masses (mf→0)
- Finite Size Scaling (due to the difficulty to take V=∞)
- Lattice spacing dependence (Enhancement) ← many β
- Spectroscopy (Mglueball, M”dilaton”, Mbaryon, Mmeson, Fρ/σ, S-param. etc.)
- Mscalar(singlet), Mglueball, MH ⇒ 126GeV? (O(100GeV))
- → LatKMI (Ohki, Rinaldi, Yamazaki, Aoki), Go!
Backup
KMI computer,

- non GPU nodes
  - 148 nodes
  - 2x Xenon 3.3 GHz
  - 24 TFlops (peak)
- GPU nodes
  - 23 nodes
  - 3x Tesla M2050
  - 39 TFlops (peak)
Spectroscopy (raw data) at $\beta = 3.8$

FIG. 3. Raw data of observables as a function of $m_f$ for $M_\pi$ (top left), $F_\pi$ (top right), $M_\rho$ (bottom left) and $\langle \bar{\psi}\psi \rangle$ (bottom right).
Size dependence of $M_\pi$ and $F_\pi$ at $\beta=3.8$

FIG. 7. $F_\pi$ (left), $M_\pi$ (center) and $M_\rho$ (right) as functions of $L$.

On the larger volume, there is not (or very tiny) size dependence.

We use the data on the largest volume at each mf.
**M_\rho vs mf**

**M_\pi^2 vs mf**

**FIG. 9.** Results of quadratic fit of \(M_\rho\) for various fit ranges.

**FIG. 11.** Quadratic fit of \(M_\pi^2\) for various fit ranges.

**TABLE II.** Chiral fit of \(M_\rho\) with \(M_\rho = C_0^e + C_1^e m_f + C_2^e m_f^2\) for various fit ranges.

<table>
<thead>
<tr>
<th>fit range ((m_f))</th>
<th>(C_0^e)</th>
<th>(\chi^2/\text{dof})</th>
<th>dof</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.015-0.04</td>
<td>0.168(32)</td>
<td>0.0017</td>
<td>1</td>
</tr>
<tr>
<td>0.015-0.05</td>
<td>0.149(33)</td>
<td>0.098</td>
<td>2</td>
</tr>
<tr>
<td>0.015-0.06</td>
<td>0.145(25)</td>
<td>0.084</td>
<td>3</td>
</tr>
<tr>
<td>0.015-0.07</td>
<td>0.144(20)</td>
<td>0.063</td>
<td>4</td>
</tr>
<tr>
<td>0.015-0.08</td>
<td>0.146(16)</td>
<td>0.052</td>
<td>5</td>
</tr>
<tr>
<td>0.015-0.10</td>
<td>0.164(12)</td>
<td>0.57</td>
<td>6</td>
</tr>
<tr>
<td>0.015-0.16</td>
<td>0.189(7)</td>
<td>1.48</td>
<td>9</td>
</tr>
</tbody>
</table>

**TABLE III.** Chiral fit results for \(M_\pi^2\) with \(M_\pi^2 = C_0^\pi + C_1^\pi m_f + C_2^\pi m_f^2\) for various fit ranges.

<table>
<thead>
<tr>
<th>fit range ((m_f))</th>
<th>(C_1^\pi)</th>
<th>(\chi^2/\text{dof})</th>
<th>dof</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.015-0.04</td>
<td>0.0016(13)</td>
<td>1.21</td>
<td>1</td>
</tr>
<tr>
<td>0.015-0.05</td>
<td>-0.0017(9)</td>
<td>5.90</td>
<td>2</td>
</tr>
<tr>
<td>0.015-0.06</td>
<td>-0.0022(6)</td>
<td>4.18</td>
<td>3</td>
</tr>
<tr>
<td>0.015-0.07</td>
<td>-0.0032(5)</td>
<td>5.00</td>
<td>4</td>
</tr>
<tr>
<td>0.015-0.08</td>
<td>-0.0037(5)</td>
<td>5.44</td>
<td>5</td>
</tr>
<tr>
<td>0.015-0.10</td>
<td>-0.0049(4)</td>
<td>7.28</td>
<td>6</td>
</tr>
<tr>
<td>0.015-0.16</td>
<td>-0.0071(3)</td>
<td>14.8</td>
<td>9</td>
</tr>
</tbody>
</table>
Comparison with $N_f = 4$

: hyperscaling of $F_\pi$ (in $S\chi$ SB)

$\beta = 3.7, \gamma = 0.0$

$\beta = 3.7, \gamma = 1.0$

$\beta = 3.7, \gamma = 2.0$

$F_\pi : \text{no scaling for } 0 < \gamma < 2$

⇒ In this sense, it seems that $N_f=8$ is not $S\chi$ SB of ordinary QCD.
Detour: (sideways)

In the previous, the quadratic fit in the wide range of $m_f$ gives worse $\chi^2$/dof.

$\Rightarrow$ Why? What happens?

**Power fit trial:** (conformal-like)

<table>
<thead>
<tr>
<th>fit range ($m_f$)</th>
<th>$C_1$</th>
<th>$\gamma$</th>
<th>$\chi^2$/dof</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.015–0.04</td>
<td>0.415(7)</td>
<td>0.988(19)</td>
<td>14.8</td>
</tr>
<tr>
<td>0.015–0.05</td>
<td>0.414(5)</td>
<td>0.991(15)</td>
<td>9.84</td>
</tr>
<tr>
<td>0.015–0.06</td>
<td>0.418(4)</td>
<td>0.979(12)</td>
<td>7.88</td>
</tr>
<tr>
<td>0.015–0.07</td>
<td>0.424(3)</td>
<td>0.963(9)</td>
<td>7.35</td>
</tr>
<tr>
<td>0.015–0.08</td>
<td>0.425(3)</td>
<td>0.961(8)</td>
<td>6.15</td>
</tr>
<tr>
<td>0.015–0.10</td>
<td>0.426(2)</td>
<td>0.958(7)</td>
<td>5.31</td>
</tr>
<tr>
<td>0.015–0.16</td>
<td>0.428(1)</td>
<td>0.952(4)</td>
<td>3.98</td>
</tr>
</tbody>
</table>

Case including the small $m_f$ region  

**good $\chi^2$/dof**  

Case excluding the small $m_f$ region

Power fit in the middle region of $m_f$ ($0.05 \leq m_f$) shows the good $\chi^2$/dof.
**Linear fit in the hyperscaling relation**

Note: here, Figs. are plotted as a function of \( m_f \), not \( x \).

**FIG. 18.** Linear fits for the FSHS of \( M_\pi \) (left), \( F_\pi \) (center), and \( M_\rho \) (right). The filled symbols are included in the fit, but the open symbols are omitted. The fitted region is \( m_f \geq 0.05 \) and \( \xi_\pi \geq 8 \).

**TABLE VI.** The \( \gamma \) fitted by the linear ansatz. The fitted region is \( m_f \geq 0.05 \) and \( \xi_\pi \geq 8 \).

<table>
<thead>
<tr>
<th></th>
<th>( \gamma )</th>
<th>( C_0^H )</th>
<th>( C_1^H )</th>
<th>( \chi^2/\text{dof} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_\pi )</td>
<td>0.5668(26)</td>
<td>0.049(22)</td>
<td>2.57766(99)</td>
<td>2.52</td>
</tr>
<tr>
<td>( \xi_F )</td>
<td>0.9279(79)</td>
<td>-0.17(10)</td>
<td>0.4372(38)</td>
<td>0.73</td>
</tr>
<tr>
<td>( \xi_\rho )</td>
<td>0.798(20)</td>
<td>0.04(19)</td>
<td>2.779(69)</td>
<td>0.66</td>
</tr>
</tbody>
</table>

\( \gamma \) is not universal. \( \gamma (M_\pi) \neq \gamma (F_\pi) \), \( \gamma (F_\pi) \sim 1.0 \)