Transport Coefficients of the QGP

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Outline

1. Transport Coefficients
   - Heavy Ion Collisions
   - Strongly Coupled QGP
   - Conductivity from Lattice QCD
2. On the Lattice
   - Details on the Action
   - Conserved Current
   - MEM
3. Results
   - Spectral functions
   - Stability Tests
   - Conductivity

Introduction

- Quark gluon plasma: a phase of matter when T is raised up to $150,000$ times the one at the core of the sun.
- When the temperature reaches $T > T_c$ quarks and gluons becomes the degrees of freedom $\rightarrow$ QGP

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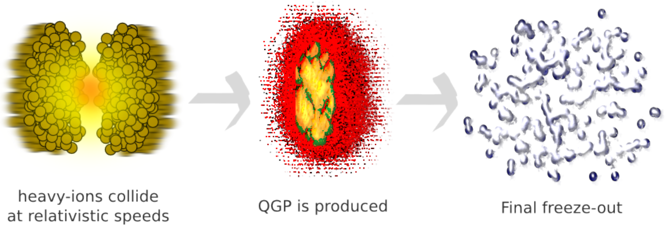
Why do we study the QGP?

- Dynamic properties of QGP are relevant to constrain early universe cosmology models
- Understand the output of heavy ion collisions experiments at RHIC and CERN

Heavy Ion Collisions



- Effective theories to study the evolution of the QGP:
  $\rightarrow$ Input parameters: transport coefficients.
- Experimental evidence for a strongly coupled QGP:
  $\rightarrow$ perturbation theory fails (see results for η).
- First principles calculation is needed:
  $\rightarrow$ Lattice QCD.

Electrical Conductivity

- Electromagnetic current (only up/down contribution)

$$j_{em}^{\mu} = \frac{2}{3}\bar{u}\gamma^{\mu}u - \frac{1}{3}\bar{d}\gamma^{\mu}d$$

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- Euclidean Correlator

$$G_{\mu\nu}(\tau) = \int d^3x \, \langle \, j^{\mu}_{em}(\tau, \mathbf{x}) j^{\nu}_{em}(0, \mathbf{0})^{\dagger} \, \rangle$$

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- Euclidean Correlator $\Rightarrow$ spectral function

$$\begin{aligned} G_{\mu\nu}(\tau) &= \int d^3x \, \langle \, j^{\mu}_{em}(\tau, \mathbf{x}) j^{\nu}_{em}(0, \mathbf{0})^{\dagger} \, \rangle \\ &= \int_0^{\infty} \frac{d\omega}{2\pi} \, K(\omega, \tau) \, \rho^{\mu\nu}(\omega, \mathbf{p}), \end{aligned}$$

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$$= \int_0^{\infty} \frac{d\omega}{2\pi} \, K(\omega, \tau) \, \rho^{\mu\nu}(\omega, \mathbf{p}),$$

- Kubo's Formula for Conductivity σ

$$\sigma = \lim_{\omega \to 0} \frac{1}{6} \frac{\rho^{ii}(\omega)}{\omega}$$

⇒ Important for evolution of EM fields in the QGP

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- Kubo's Formula for Conductivity σ

$$\sigma = \lim_{\omega \to 0} \frac{1}{6} \frac{\rho^{ii}(\omega)}{\omega} \quad \Rightarrow$$

Important for evolution of EM fields in the QGP

- Non-zero σ forces magnetic fields to freeze in the plasma.

[K. Tuchin, 2013]

Outline

Clover Action - $N_f = 2 + 1$

$$\begin{aligned}\hat{M}[U] &= \hat{m}_0 + \gamma_t \hat{W}_t + \frac{1}{\gamma_f} \sum_s \gamma_s \hat{W}_s \\ &- \frac{c_t}{2} \sum_s \sigma_{ts} \hat{F}_{ts} - \frac{c_s}{2\gamma_g} \sum_{s<s'} \sigma_{ss'} \hat{F}_{ss'}\end{aligned}$$

[2009, Lin, Edwards, Joo]

- Bare gauge/fermion anisotropy γ_g, γ_f
  $\rightarrow$ Tuned to give a fixed value for $\xi = a_s/a_t = 3.5$

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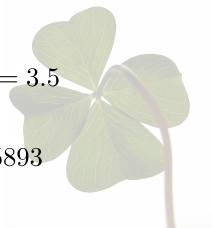
  $\rightarrow$ Tuned to give a fixed value for $\xi = a_s/a_t = 3.5$
- $O(a)$ Improved with clover term
  $\rightarrow$Tree-level conditions: $c_t = 0.9027$, $c_s = 1.5893$



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- $O(a)$ Improved with clover term
  $\rightarrow$Tree-level conditions: $c_t = 0.9027$, $c_s = 1.5893$
- Stout-smeared gauge links:
  $\rightarrow \rho = 0.15$, $n_\rho = 2$

Configurations

N_s	N_τ	T [MeV]	T/T_c	$N_{\texttt{CFG}}$	$N_{\texttt{SRC}}$
32	16	350	1.89	1059	4
24	20	280	1.52	1001	4
32	24	235	1.26	500	4
32	28	201	1.08	502	4
32	32	176	0.95	501	4
24	36	156	0.84	501	4
24	40	140	0.76	523	4
32	48	117	0.63	601	1

- Two spatial lattice extension available $N_s = 24, 32$
- $a_s = 0.1227(8)$ fm and $a_t = 0.03506(23)$ fm
- m_s physical and $m_{u,d}$ with $M_\pi/M_\rho = 0.446(3)$
- $T = 120 \sim 350$ MeV, with $T_c = 186(2)$ MeV

Conserved Current on the Lattice

- Conserved vector current – $\kappa_4 = \frac{1}{2}, \kappa_i = \frac{1}{2\gamma_f}$

$$V_\mu^{\mathrm{C}}(n) = \kappa_\mu \Big[\bar\psi(n+\hat\mu)(1+\gamma_\mu)\,U_\mu^\dagger(n)\,\psi(n) - \bar\psi(n)(1-\gamma_\mu)\,U_\mu(n)\,\psi(n+\hat\mu)\Big]$$

- Ward identity protects the current from renormalization

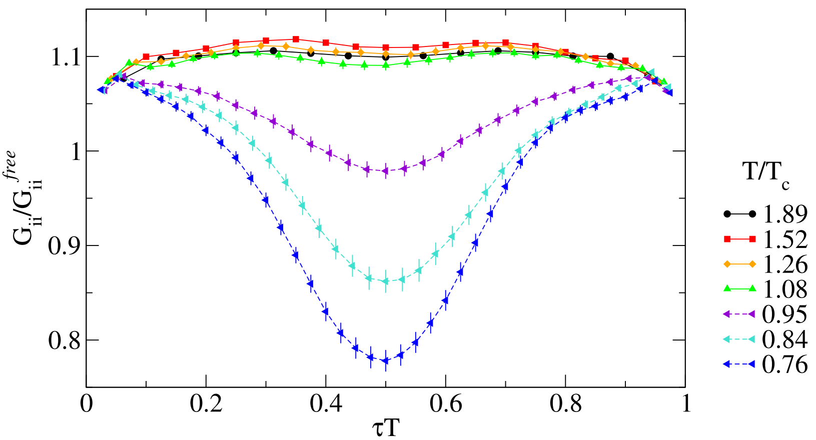
$$Z_{V^C} \equiv 1$$

- Improvement pattern given by

$$V_\mu^{\mathrm{CI}} - V_\mu^{\mathrm{C}} \equiv \frac{1}{4}\sum_\rho(\delta_{\rho,0} + \nu\delta_{\rho,i})a_\rho\partial_\rho^-\bar\psi(x)\,\sigma_{\mu\rho}\,\psi(x) \xrightarrow{p\to 0} 0$$

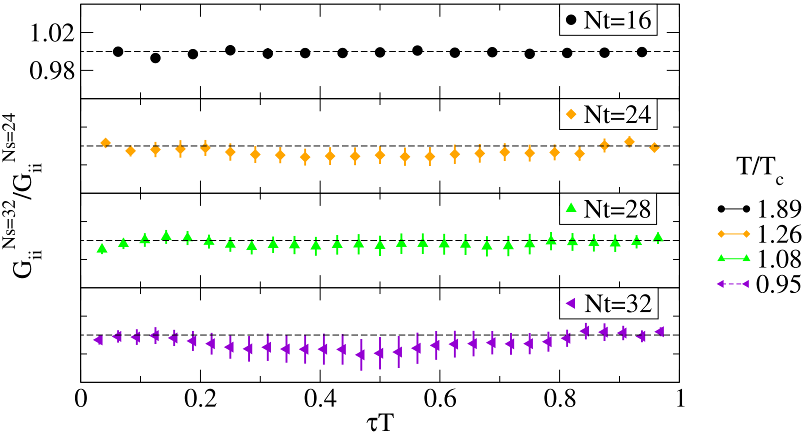
Conserved Current - Spatial Component

$$G_{ii}(\tau) = \sum_{\vec{y}} \langle\, V_i^{\mathrm{C}}(\vec{x}, x_0)\, V_i^{\mathrm{C}}(\vec{y}, \tau + x_0)^\dagger \,\rangle$$



Conserved Current - Volume Effects

$$G_{ii}(\tau) = \sum_{\vec{y}} \langle\, V_i^{\mathrm{C}}(\vec{x}, x_0)\, V_i^{\mathrm{C}}(\vec{y}, \tau + x_0)^\dagger \,\rangle$$



An ill posed problem

$$G_{ii}(\tau) = \int_0^\infty \mathrm{d}\omega\, \rho(\omega)\, \frac{\cosh \omega(\frac{\beta}{2} - \tau)}{\sinh \beta\omega/2}$$

$$\sim O(10) \qquad \sim O(1000)$$

An ill posed problem

$$G_{ii}(\tau_j) = \Delta\omega \sum_{i=0}^{N_\omega} \rho_i \, K_{ij}$$

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An ill posed problem

$$G_{ii}(\tau_j) = \Delta\omega \sum_{i=0}^{N_\omega} \rho_i K_{ij}$$

$$\sim O(10) \qquad \sim O(1000)$$

- Standard χ^2-fit fails: non unique solution.
- Need to use Bayesian probability theory.

  [Karsch et al. 2002] [Gupta, 2004]

Bayesian Probability Theory

Conditional Probability

$$P[\rho|DH] = \frac{P[D|\rho H]P[\rho|H]}{P[D|H]}$$

Bayesian Probability Theory

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$$P[\rho|DH] = \frac{P[D|\rho H]P[\rho|H]}{P[D|H]}$$

- $P[D|\rho H]$ - likelihood function $\exp(-L)$
- $P[D|H]$ - normalization

$$L = \frac{1}{2}\sum_{i,j}\left(G(\tau_i) - F_i\right) C_{ij}^{-1}\left(G(\tau_j) - F_j\right)$$

$$F_j = \Delta\omega \sum_i^{N_\omega} \rho_i K_{ij}$$

Bayesian Probability Theory

Conditional Probability

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- $P[D|\rho H]$ - likelihood function $\exp(-L)$
- $P[D|H]$ - normalization
- $P[\rho|H]$ - prior probability: Entropy $\exp(-\alpha S)$

$$S = \int_0^\infty \frac{d\omega}{2\pi}\left[\rho(\omega) - m(\omega) - \rho(\omega)\ln\frac{\rho(\omega)}{m(\omega)}\right]$$

Default Model: $m(\omega) = m_0\omega(b+\omega)$ finite intercept of $\rho(\omega)/\omega$

Bayesian Probability Theory

Conditional Probability

$$P[\rho|DH] \propto \exp\left(-L + \alpha S\right)$$

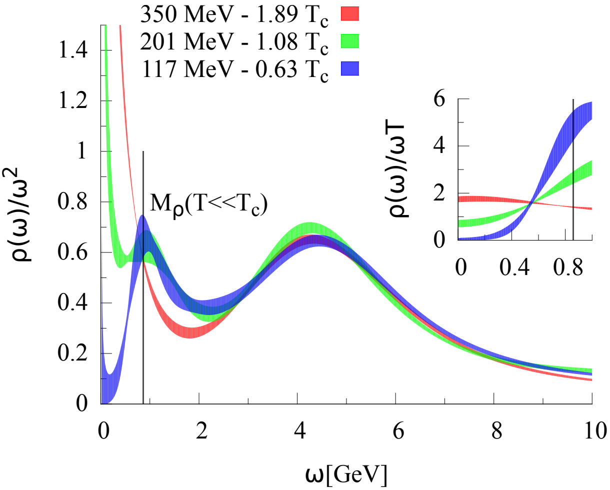
- $P[D|\rho H]$ - likelihood function $\exp(-L)$
- $P[D|H]$ - normalization
- $P[\rho|H]$ - prior probability: Entropy $\exp(-\alpha S)$

<u>Solution</u> given by $\delta P[\rho|DH] = 0$:

- Modification of Bryan's algorithm [Aarts et al. - 2007]
  $\Rightarrow$ Fixes kernel instabilities at low ω

Outline

Spectral Functions from MEM



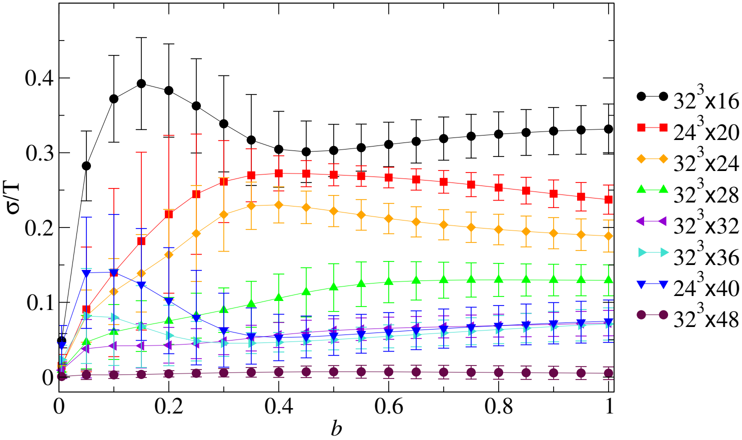
- First Peak: ρ-particle
- Structures at $3-6$ GeV: lattice artefacts
- Inset: intercept shows a T-dependent conductivity

$$\sigma = \lim_{\omega\to 0}\frac{1}{6}\frac{\rho^{ii}(\omega)}{\omega}$$

Default Model Dependence

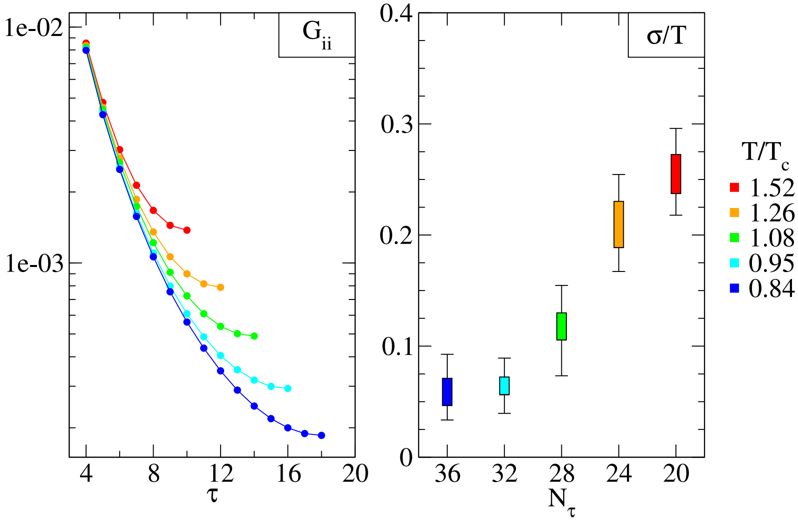
- We check the dependence of the result on b:

$$m(\omega) = m_0\omega(b + a_t\omega)$$



Stability Tests - $\Delta\tau$

- Comparing results with same τ-slices but different T



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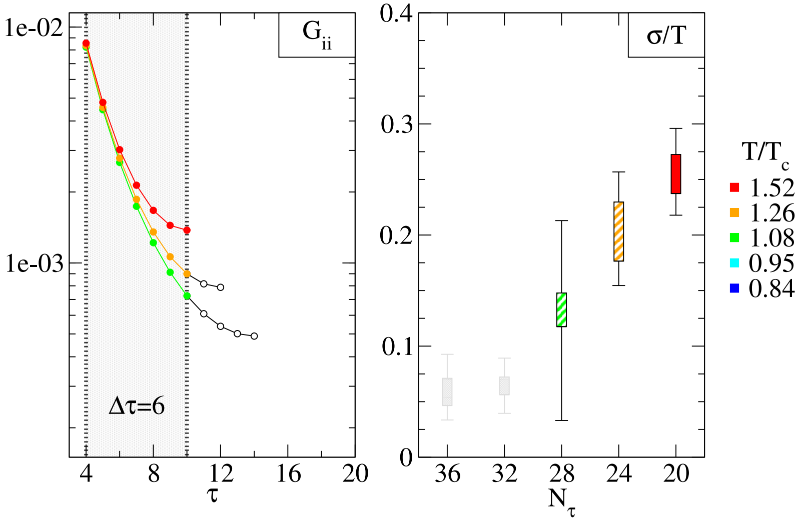

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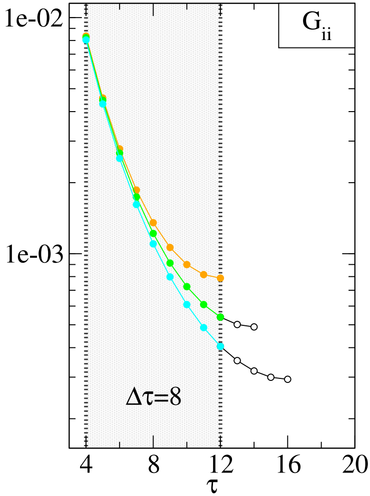

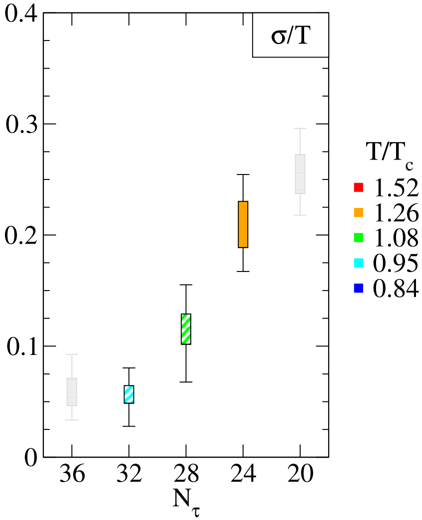

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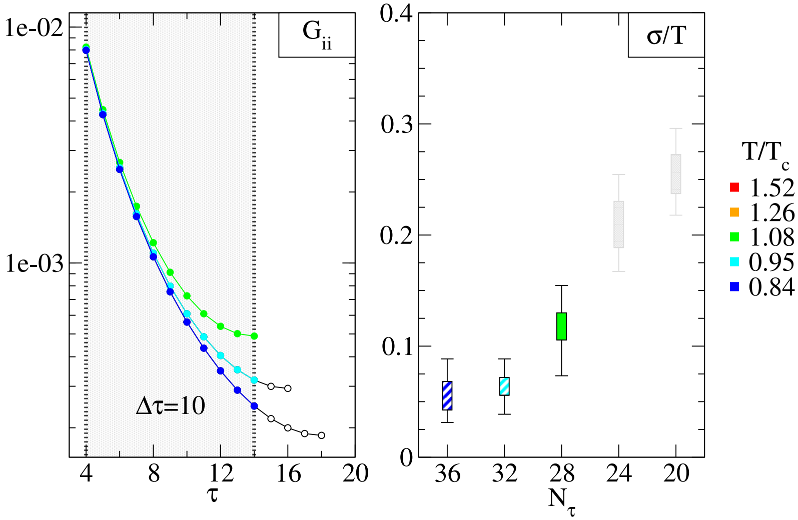
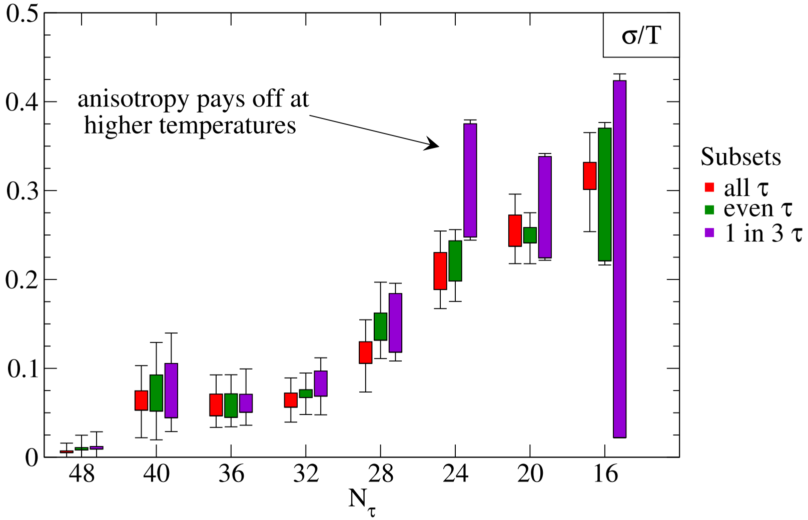
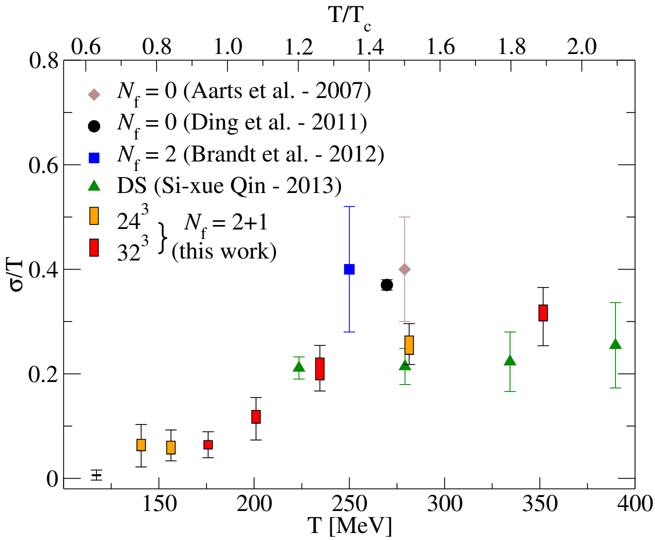


Stability Tests - Anisotropy

- Comparing results when using only a subset of τ-slices



Conductivity - Final Result



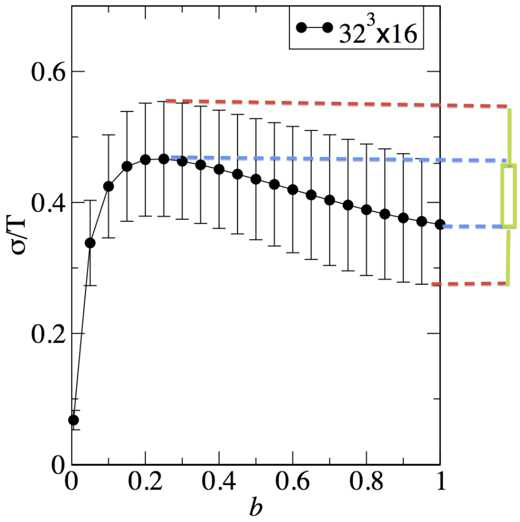
Conclusions

- Electrical Conductivity plays an important role in the evolution of EM fields in Heavy Ion Collisions;
- Inside QGP phase results comparable with previous ones;
- First analysis with conserved current of conductivity with different temperatures;
- New observation: increase of σ/T already in the confined phase.

Next:

- Strange quark contribution;
- Magnetic Field influence.

Thanks



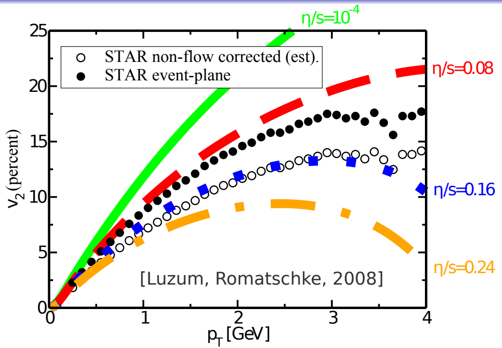
32³x16
σ/T
b
0
0.2
0.4
0.6
0.8
1

Strongly Coupled QGP

- Elliptic flow

$$v_2 = \langle \frac{p_X^2 - p_Y^2}{p_X^2 + p_Y^2} \rangle$$

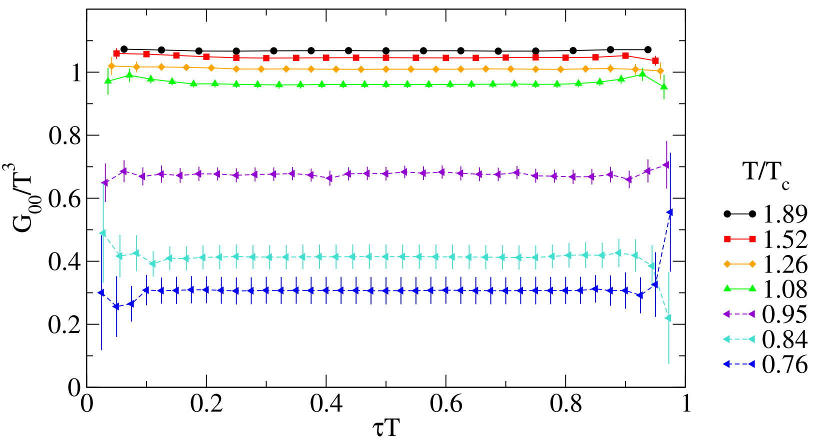
- v_2 found very large: a direct measure of collectivity.



- Dissipative hydrodynamic: $v_2(P_T) \leftrightarrow \eta$ shear viscosity
- η/s found smaller than other system:
  $\rightarrow$ strongly interacting.
  $\rightarrow$ perturbation theory fails.
- First principle calculation of transport coefficients is needed:
  $\rightarrow$ Lattice QCD.

Conserved Current - Temporal Component

$$G_{00}(\tau) = \sum_{\vec{y}} \langle\, V_0^{\mathrm{C}}(\vec{x}, x_0)\, V_0^{\mathrm{C}}(\vec{y}, \tau + x_0)^{\dagger}\, \rangle$$



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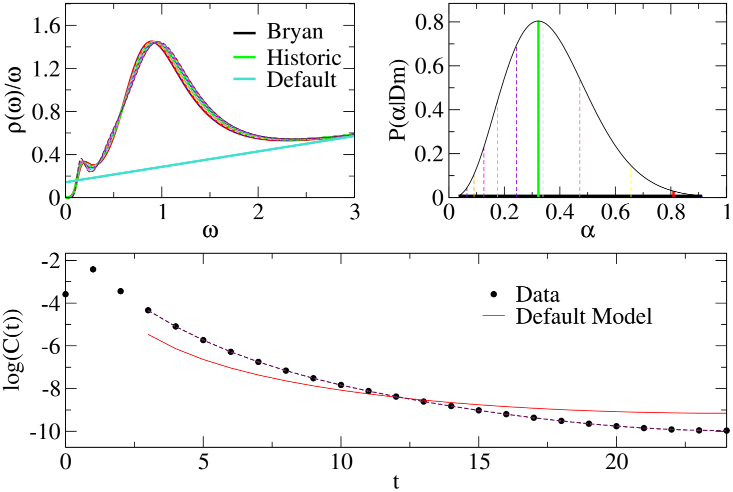

Outline

- Retarded Correlator
- Hydrodynamics Evolution

Linear Response

- Classical external source coupled to O

$$H(t) = H_0 - H_{ext}(t) \qquad \text{with } H_{ext} = \int \mathrm{d}\boldsymbol{x}\, f(\boldsymbol{x},t)\, O(\boldsymbol{x},t)$$

- The evolution for O is

$$i\frac{\partial}{\partial t}O(t) = -[H(t), O(t)]$$

- To linear order in f

$$\delta\left\langle O(\boldsymbol{x},t)\right\rangle = \int_{-\infty}^{t} \mathrm{d}t'\mathrm{d}\boldsymbol{x}' G(\boldsymbol{x}-\boldsymbol{x}', t-t') f(\boldsymbol{x}', t') + \mathrm{O}(f^2)$$

where $G(\boldsymbol{x},t) = i\left\langle [O(\boldsymbol{x},t), O(0,0)]\right\rangle$

Linear Response

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$$i\frac{\partial}{\partial t} O(t) = -[H(t), O(t)]$$

- To linear order in f $\quad + \int \mathrm{d}t\mathrm{d}\boldsymbol{x}\, e^{i\omega t - \boldsymbol{x}\cdot\boldsymbol{k}}$

$$\delta \left\langle \tilde{O}(\boldsymbol{k},\omega)\right\rangle = \tilde{G}_R(\omega,\boldsymbol{k})\tilde{f}(\omega,\boldsymbol{k})$$

where $\tilde{G}_R(\omega,\boldsymbol{k}) = i\int_0^\infty \mathrm{d}t\, e^{i\omega t - \boldsymbol{x}\cdot\boldsymbol{k}} \left\langle [O(t,\boldsymbol{x}), O(0,0)]\right\rangle$

Particle Number Diffusion

- Perturbation of Particle number

$$H_\mu = H_0 - \int \mathrm{d}\boldsymbol{x}\, \mu(\boldsymbol{x},t)\, n(\boldsymbol{x},t) \quad \text{with} \quad \mu(\boldsymbol{x},t) = \mu(\boldsymbol{x}) e^{\epsilon t} \theta(-t)$$

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- Hydrodynamics: conservation law + *constitutive* equation

$$\partial_t n + \nabla \cdot \boldsymbol{j} = 0 \qquad \boldsymbol{j} = -D\nabla n \text{ (Fick's Law)}$$

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- Diffusion equation

$$\partial_t n(\boldsymbol{x}) = D\nabla^2 n(\boldsymbol{x}) \quad \to \quad \tilde{n}(\omega,\boldsymbol{k}) = \frac{n(0,\boldsymbol{k}) D\boldsymbol{k}^2}{-i\omega + D\boldsymbol{k}^2}$$

- Static susceptibility

$$\chi_s^N = n(0,\boldsymbol{x}) = \int_0^\infty \mathrm{d}t\, e^{-\epsilon t} \int \mathrm{d}\boldsymbol{x}' G^{nn}(\boldsymbol{x}-\boldsymbol{x}',t)\mu(\boldsymbol{x}')$$

Kubo's Formula

- Substituting $f \leftarrow \mu$ and $O \leftarrow n$ in the linear response for $\delta \langle \tilde{O}(\boldsymbol{k}, \omega) \rangle$

$$\tilde{G}_R^{nn}(\omega, \boldsymbol{k}) = \frac{(D\boldsymbol{k}^2)^2 + i\omega D\boldsymbol{k}^2}{\omega^2 + (D\boldsymbol{k}^2)^2}$$

- Spectral function

$$\rho^{nn}(\omega, \boldsymbol{x}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \langle [n(\boldsymbol{x}, t), n(0,0)] \rangle = \frac{1}{\pi} \mathrm{Im} \, G_R^{nn}(\omega, \boldsymbol{x})$$

- Diffusion coefficient extracted by

$$D\chi_s^N = \pi \lim_{\omega \to 0} \lim_{\boldsymbol{k} \to 0} \frac{\rho_L(\omega, \boldsymbol{k})}{\omega}$$

Technical Issues

- SVD Decomposition of the Kernel [Bryan, 1989]

$$K^T = UWV^T \text{ with } W = \text{diag}(w_1, \dots, w_{N_\omega})$$

$$\text{but} \quad w_{N_s+i} \ll 1 \quad \Rightarrow \quad \vec{\rho^*} = \sum_{i=0}^{N_s} b_i \vec{u}_i \quad \text{with} \quad N_s < N_\tau \ll N_\omega$$

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- Bryan approach: integrate over all α

$$\rho_{out} = \int d\alpha \, \rho_\alpha(\omega) \, P[\alpha | DHm]$$

$\Rightarrow$ Fix Kernel instabilities at low ω [Aarts, Allton, Hands, Foley, 2007]

$$K \underset{\omega \to 0}{\sim} \frac{1}{\omega} \quad \Rightarrow \quad \overline{K}(\omega, \tau) = \frac{\omega}{2T} K(\omega, \tau), \quad \overline{\rho}(\omega) = \frac{2T}{\omega} \rho(\omega)$$

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Estimates of η

- Experiments [Teaney, 2009]

$$\left(\frac{\eta}{s}\right)_{pheno} \lesssim 0.40$$

- Perturbative [Arnold, Moore, Yaffe, 2000]

$$\left(\frac{\eta}{s}\right)_{\text{leading log}} = \frac{c}{g^4 \log(1/g)} \underset{\alpha_s=0.15}{\approx} 2.0$$

- SYM at infinite coupling [Policastro, Son, Starinets, 2001]

$$\left(\frac{\eta}{s}\right)_{\mathcal{N}=4,\lambda=\infty} = \frac{1}{4\pi}$$