The sign problem and Abelian lattice duality

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Lattice 2013
Mainz, Germany
July 29-August 3 2013

Abelian lattice models

Meisinger & mco arxiv:1306.1495

- Lattice duality solves the sign problem for a broad class of Abelian models, mapping complex actions into dual models with real actions.

- For extended regions of parameter space, calculable for each model, duality resolves the sign problem for both analytic methods and computer simulations.

- Explicit duality relations are given for models for spin and gauge models based on Z(N) and U(1) symmetry groups.

- The dual forms are generalizations of the chiral Z(N) model and the lattice Frenkel-Kontorova model, respectively.

- From this equivalence, a rich set of spatially-modulated phases is found in the strong-coupling region of the original models.

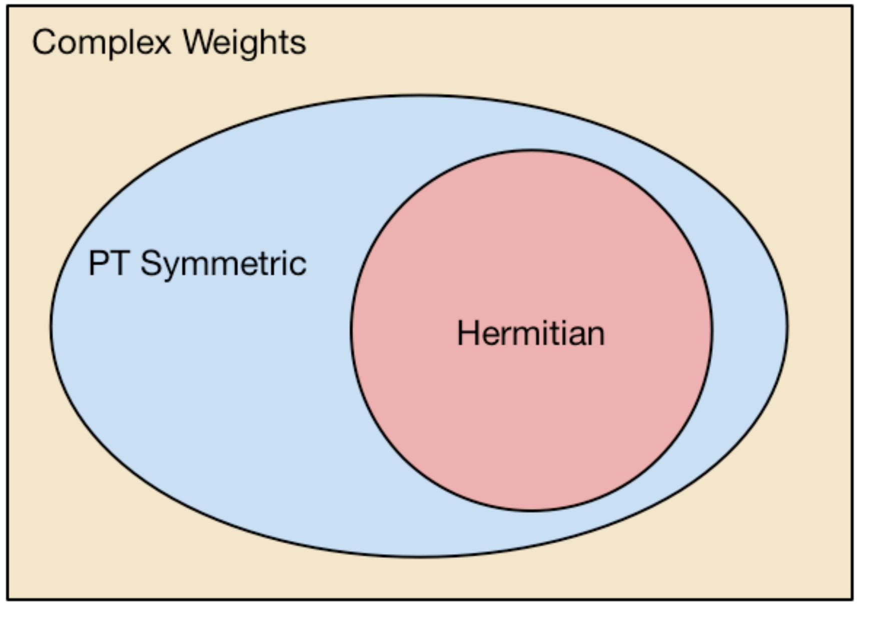
Key results from Generalized PT symmetry

Finite density models have CT symmetry:

$$\mathcal{C} : A_\mu \to -A_\mu \quad \mathcal{T} : i \to -i \qquad\qquad \mathcal{CT} : P \to P$$

- Eigenvalues real or in conjugate pairs Bender and Boettcher (1998)
- Equivalence to hermitian model when all eigenvalues are real Mostafazadeh (2003)
- Constructibility of a real representation Meisinger and mco (2013)
- PT models have real partition functions Meisinger and mco (2013)

$$Z = \sum_r e^{-\beta E_r} + \sum_p \left(e^{-\beta E_p} + e^{-\beta E_p^*} \right)$$



Lattice-oriented review:
Meisinger and mco arXiv:1208.5077
Phil.Trans.Roy.Soc.Lond. A371 (2013) 20120058

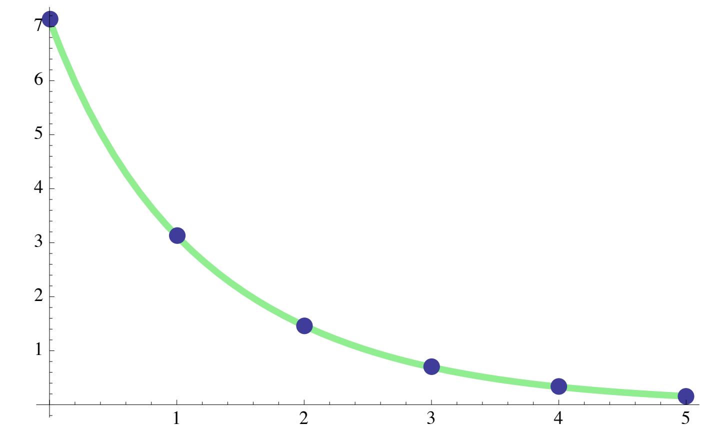
Classification of Phases via $\mathcal{PT}$ Symmetry

Example: d=1 Z(3) spin chain with complex action (Meisinger, mco & Wiser, 2010)

- Region I: PT symmetry is unbroken, and all eigenvalues are real. Behavior of correlation functions similar to a Hermitian system.

$$E_j \in R \;\; \forall j$$

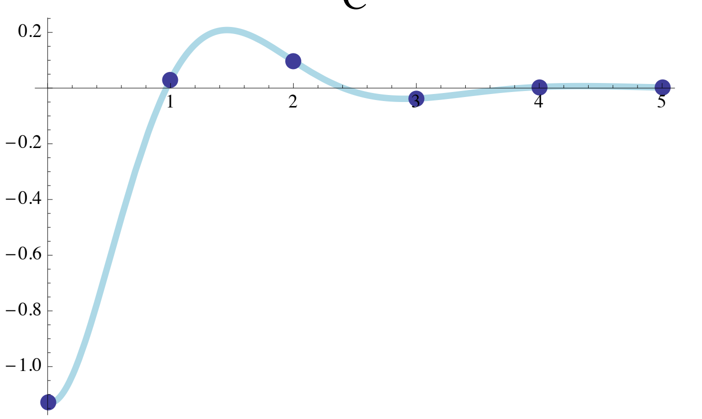
A



- Region II: PT symmetry is broken by a one or more pairs of excited states becoming complex. Thermodynamic properties are unaffected, but oscillatory behaviors appears in correlation functions.

$$E_j \neq E_j^* \;\; j > 0$$

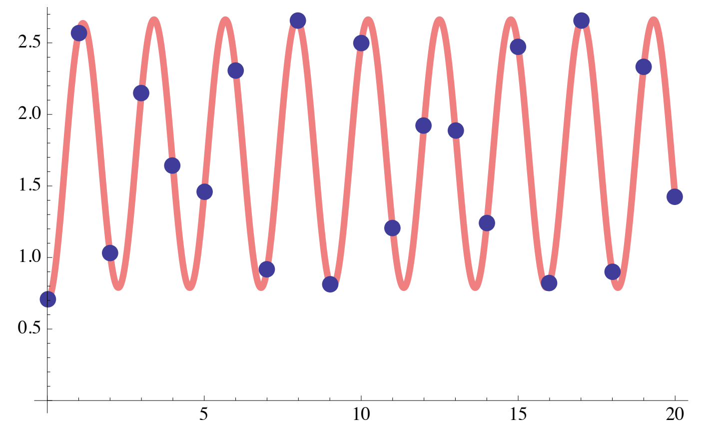
C



- Region III: PT symmetry is broken by the ground state becoming complex. The system is in a spatially modulated phase.

$$E_0 \neq E_0^*$$

D



d=2 U(1): derivation

Villain form of XY model with a chemical potential following JKKN, 1977

Heat kernel action

$$Z = \int_{S^1} [d\theta] \sum_{n_\nu} \exp\left[-\frac{K}{2} \sum_{x,\nu} \left(\partial_\nu \theta(x) - i\mu\delta_{\nu 2} - 2\pi n_\nu(x) \right)^2 \right]$$

Duality transform of action

$$Z = \int_{S^1} [d\theta] \prod_{x,\nu} \sum_{p_\nu(x)\in Z} \frac{1}{\sqrt{2\pi K}} e^{-p_\nu^2(x)/2K} e^{ip_\nu(x)(\nabla_\nu \theta(x) - i\delta_{\nu 2}\mu)}$$

Introduction of dual variables

$p_\rho(x) = \epsilon_{\rho\nu} \nabla_\nu m(X)$

$$Z = \sum_{\{m(X)\}\in Z} \frac{1}{\sqrt{2\pi K}} e^{K - \sum_{X,\nu}\left[(\nabla_\nu m(X))^2/2K + \mu\delta_{\nu 2}\epsilon_{\nu\rho}\nabla_\rho m(X)\right]}$$

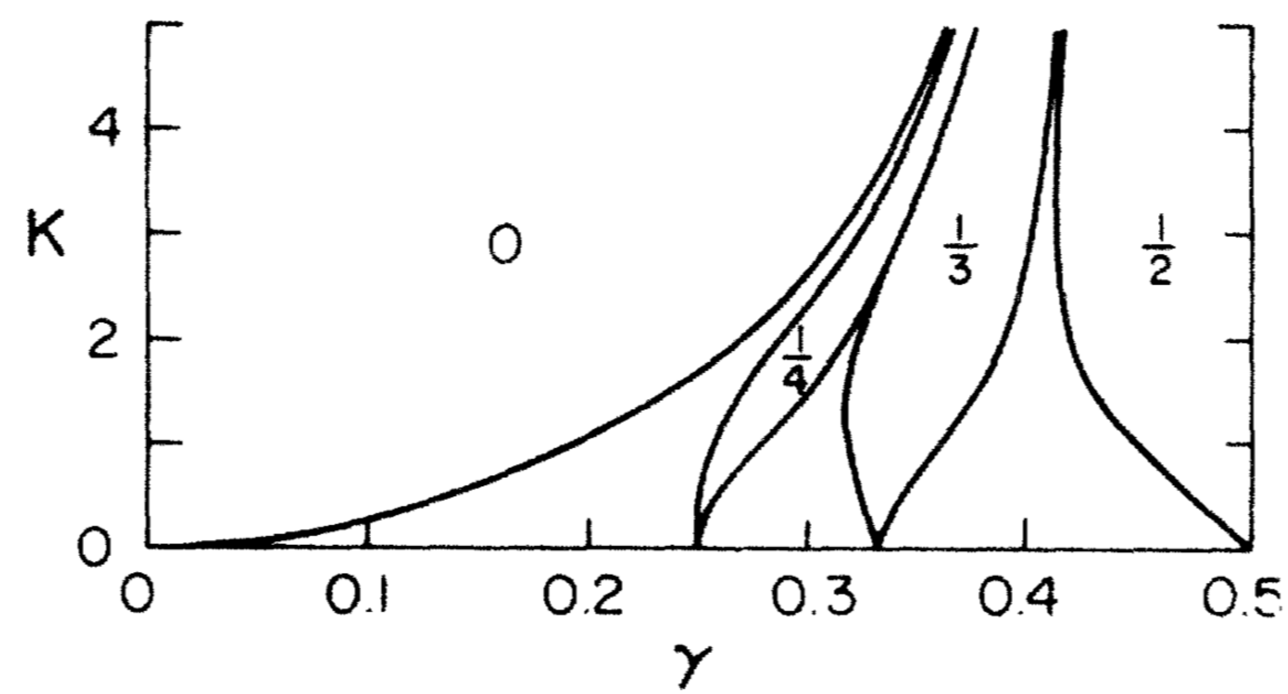
Introduction of dual scalar field

$$Z = \int_R [d\phi(X)]\, e^{-\sum_{X,\nu}\left[(\nabla_\nu \phi(X))^2/2K - \mu\nabla_1\phi(X)\right]} \sum_{\{m(X)\}\in Z} e^{2\pi i m(X)\phi(X)}$$

d=2 U(1): interpretation

m=1 only $$Z=\int_R [d\phi(X)]\exp\left[-\sum_{X,\mu}\frac{1}{2K}\left(\nabla_\mu\phi(X)\right)^2-\sum_X \mu\nabla_1\phi(X)+\sum_X 2y\cos\left(2\pi\phi(X)\right)\right]$$

- m=1 contributions only gives a lattice sine-Gordon model with an extra term: lattice form of Frenkel-Kontorova model.
- For fixed X_2, derivative term counts the number of kinks on that slice.
- Continuum form equivalent to a massive Thirring model with a chemical potential.
- Frenkel-Kontorova model has rich modulated phase structure.



Chou and Griffiths, 1986

This d=1 phase diagram is valid for d=2 at high T by dimensional reduction.

$\gamma \propto \mu$

Duality: why it works

- PT symmetry is analogous to a reality condition on the Fourier transform

$$H(p,x) = p^2 + ix^3 \qquad H^*(p,x) = H(p,-x)$$

In Abelian lattice models: $\mathcal{F}^2 = \mathcal{C}$

- Lattice duality for Abelian systems uses the Fourier transform on the group

Z(3) model: $e^{J_1 z + J_2 z^2} = a_0 + a_1 z + a_2 z^2$

coefficients all real!

Plenary talk by Gattringer; see also many papers in the last few years.

- In the dual representation, particles are lattice topological excitations.

4d: particle worldlines become magnetic monopole worldlines under duality; μ couples to the monopole current density

d=2 Z(N)

- Villain form again using methods of Elitzur *et al.* (1979)

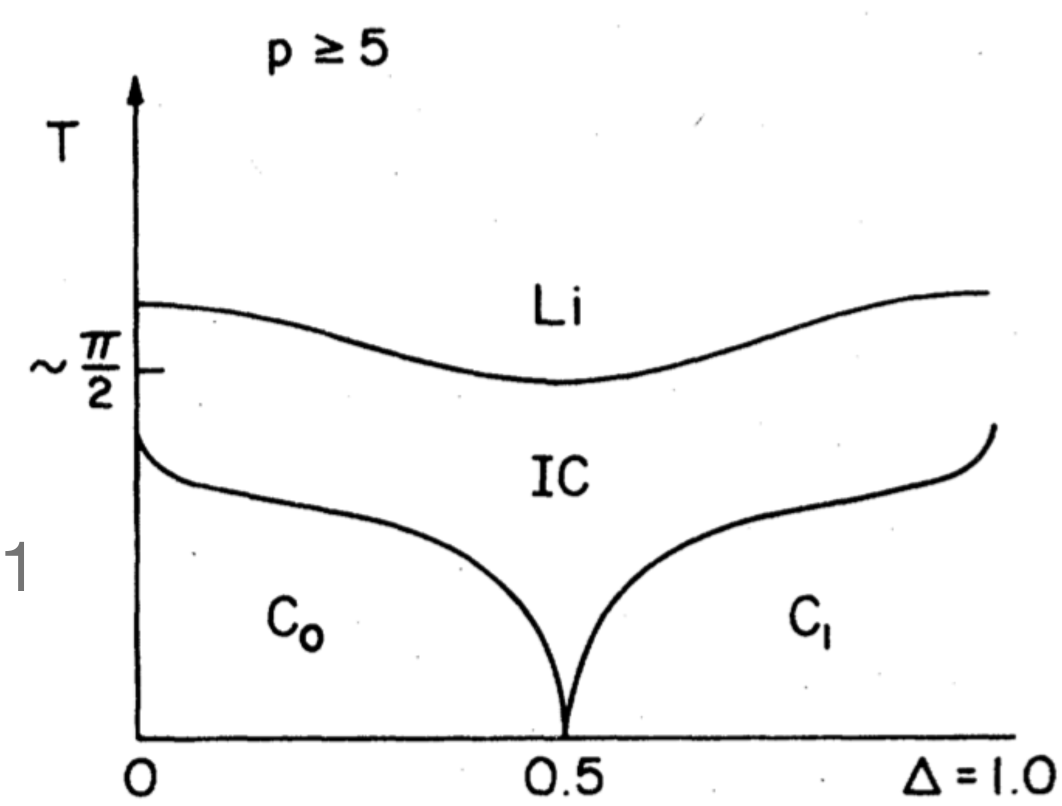
$$Z = \sum_m \sum_{n_\nu} \exp\left[-\frac{J}{2}\sum_{x,\nu}\left(\frac{2\pi}{N}\partial_\nu m(x) - i\mu\delta_{\nu 2} - 2\pi n_\nu(x)\right)^2\right]$$

- Exact duality statements

$$J \leftrightarrow \tilde{J} = \frac{N^2}{4\pi^2 J}$$

$$\mu\delta_{\nu,2} \leftrightarrow \tilde{\mu} = -i\frac{2\pi J\mu}{N}\delta_{\nu,1}$$

- Incommensurate phase (IC) for J small (Ostlund, 1981) that extends the Coulomb phase for N>4. Li is the ordered phase. C0 is the dual ordered phase. C1 is a high-density phase. There are N different C phases.



Ostlund, 1981

$$\Delta = J\mu$$

d=3 Z(N): duality

Villain action for gauge and spins.

$$Z = \sum_{m,n_\nu,p_\nu,q_{\nu\rho}} \exp\left[-\frac{J}{2}\sum_{x,\nu}\left(\frac{2\pi}{N}\partial_\nu m\left(x\right) - \frac{2\pi}{N}p_\nu - i\mu_\nu - 2\pi n_\nu\left(x\right)\right)^2\right]$$

$$\times \exp\left[-\frac{K}{2}\sum_{x,\nu>\rho}\left(\frac{2\pi}{N}\left(\partial_\nu p_\rho - \partial_\rho p_\nu\right) - iG_{\nu\rho} - 2\pi q_{\nu\rho}\right)^2\right]$$

G is a background field, but corresponds to an electric field in Minkowski space. This is again a sign problem (Shintani *et al.* 2006; Alexandru 2008).

$$J \to \tilde{J} = \frac{N^2}{4\pi^2 K}$$

$$K \to \tilde{K} = \frac{N^2}{4\pi^2 J}$$

Duality swaps between gauge and spin degrees of freedom.

$$\mu_\nu \to \tilde{\mu}_\nu = -i\frac{2\pi K}{N}\epsilon_{\nu\rho\sigma}G_{\rho\sigma}$$

$$G_{\nu\rho} \to \tilde{G}_{\nu\rho} = -i\frac{2\pi J}{N}\epsilon_{\nu\rho\sigma}\mu_\sigma$$

d=3 Z(N): gauge theory

Duality in d=3 maps Z(N) gauge theory to the chiral Z(N) spin model

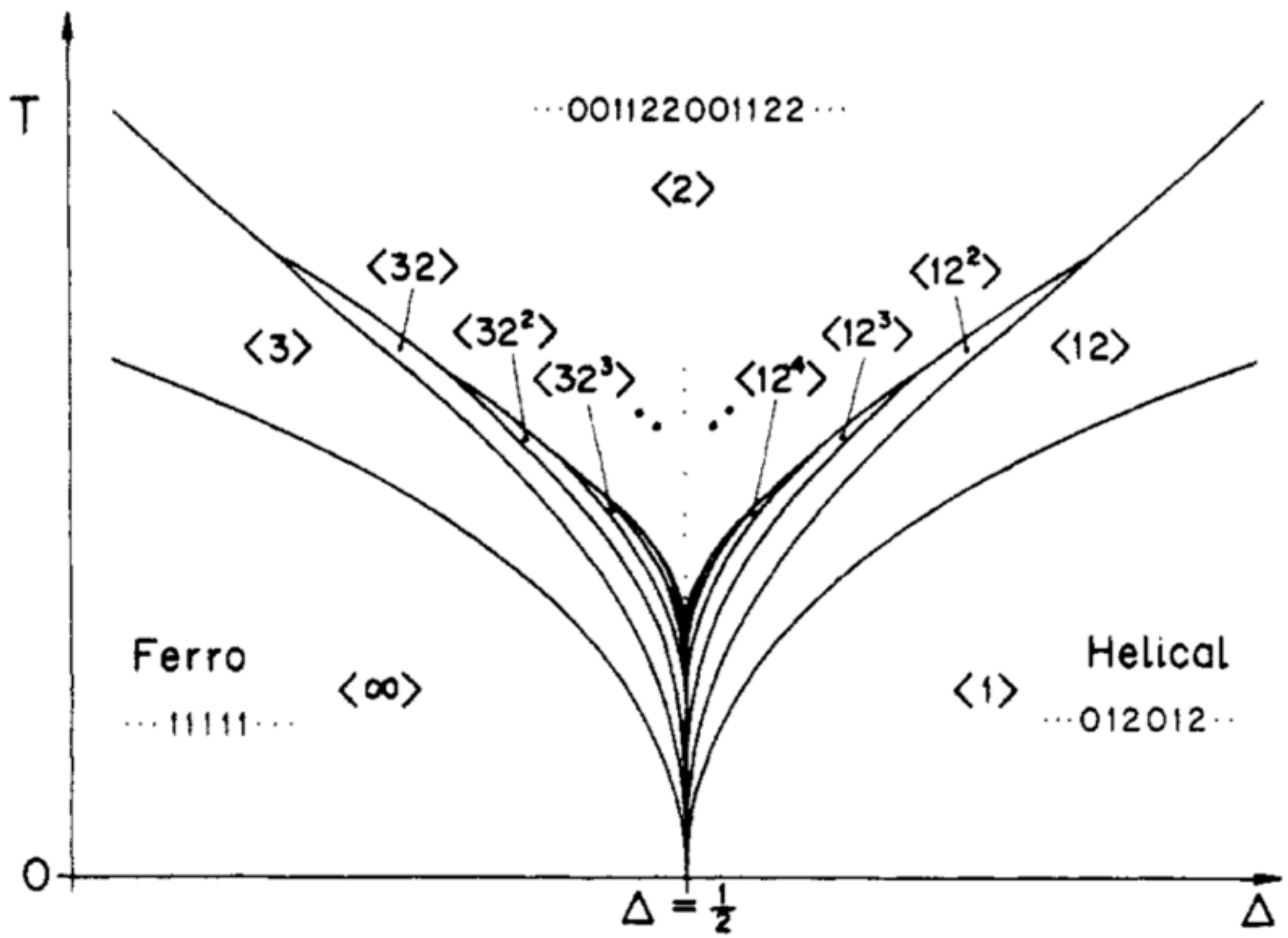


Figure 1. Schematic representation (not to scale) of the (T, Δ) phase diagram of the chiral Potts or asymmetric clock model illustrating the unbounded sequences of commensurate phases of character $\langle 32^k \rangle$ and $\langle 12^k \rangle$ for $k = 0, 1, 2, \ldots$.

Fisher and Yeomans, 1981

The dual model has a rich set of commensurate modulated phases.

$$\Delta = KG_{12}$$

General result for Z(N)

Arbitrary PT-symmetric Z(N) potential has N real parameters v_j

$$V(z) = \sum_{j=0}^{N-1} v_j z^j$$

$$V_j = V(\omega^j)$$

$$\omega = \exp(2\pi i/N)$$

CT invariance $V_{N-j} = V_j^*$

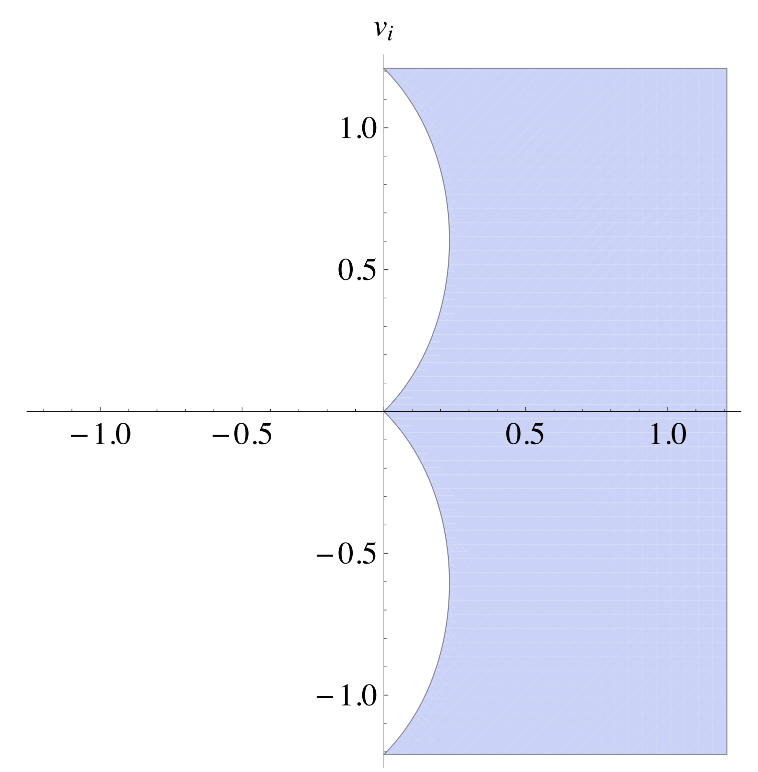
Duality is a Z(N) Fourier transform:

$$\exp\left(-\tilde{V}_j\right) = \sum_{k=0}^{N-1} \omega^{jk} \exp(-V_k)$$

Dual positive weight region for Z(3)

$$v_r = v_1 + v_2$$

$$v_i = v_1 - v_2$$



In the dual positive weight region

$$\tilde{V}(w) = \sum_{j=0}^{N-1} \tilde{v}_j w^j$$

dual potential is real with chiral phases;
N real parameters including phases

$$\tilde{v}_j = \tilde{v}_{N-j}^*$$

Conclusion and Prospects

- Solution of the sign problem for Abelian lattice models: both analytical and simulation methods can be applied to a large class of Abelian models
- Rich phase structure for Abelian systems seems typical
- Prospects for non-Abelian systems
  - Real representation exists!
  - Lack of full non-Abelian duality a problem
  - SU(N) deformed to $U(1)^{N-1}$ can be treated