Recent progress in the effective string theory description of LGTs. [1] [2]

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[1] M. Billó, M. Caselle, F. Gliozzi, M. Meineri, R. Pellegrini JHEP05(2012)130

[2] M. Caselle, D. Fioravanti, F. Gliozzi, R. Tateo JHEP07(2013)071

Summary:

1. Introduction and motivation
2. Lorentz invariance
3. The effective string theory as the $T\bar{T}$ perturbation of the 2d free bosonic model
4. Application: the boundary term of the effective action

A new approach to effective string theories.

In the past few years two major breackthrough changed our understanding of effective string theories

- The Effective String action is strongly constrained by Lorentz invariance. The first few orders of the action are universal and coincide with those of the Nambu-Goto action. This explains why N.-G. describes so well the infrared regime of Wilson loops or Polyakov Loop correlators.[1] [2] [3]
- The Nambu-Goto effective theory can be described as a free 2d bosonic theory perturbed by the irrelevant operator $T\bar{T}$ (where T and $\bar{T}$ are the two chiral components of the energy momentum tensor). This perturbation turns out to be quantum integrable and yields, using the Thermodynamic Bethe Ansatz (TBA), a spectrum which, in a suitable limit, coincides with the Nambu-Goto one. [4]

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[1] M. Luscher and P. Weisz JHEP07(2004)014

[2] H. B. Meyer JHEP05(2006)066

[3] O. Aharony and M. Field JHEP01(2011)065

[4] M. Caselle, D. Fioravanti, F. Gliozzi, R. Tateo JHEP07(2013)071

Effective string action (naive...)

The action of the effective string theory can be written as low energy expansion in the number of derivatives of the transverse fields ("physical gauge").

$$S = S_{cl} + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + c_2(\partial_\alpha X \cdot \partial^\alpha X)^2 + c_3(\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right] + S_b \,,$$

where:

- S_{cl} describes the usual ("classical") perimeter-area term.
- S_b is the boundary contribution characterizing the open string
- $X_i(\xi_0, \xi_1)$ $(i = 1, \dots, d-2)$ parametrize the displacements orthogonal to the surface of minimal area representing the configuration around which we expand
- ξ_0, ξ_1 are the world-sheet coordinates.

Effective string and spacetime symmetries.

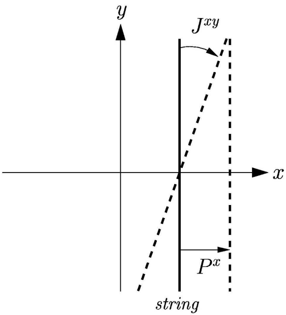
- Symmetries of the action must hold in the low energy regime.
- String vacuum is not Poincaré invariant.

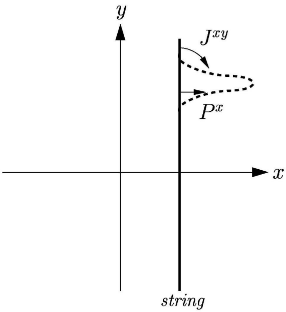
$\Longrightarrow$ Poincaré symmetry is broken spontaneously.

$$ISO(D-1,1) \to SO(D-2) \otimes ISO(1,1). \implies 3(D-2) \text{ Goldstone bosons?}$$

Just $D-2$ tranverse fluctuations of the string.

The remaining $2(D-2)$ Lorentz transformations are realized non-linearly





Non-linear realization and long-string expansion.

An internal transformation of the fields realizes the Poincaré group:

- Broken translations:
  $X^i \to X^i + a^i$. $\Longrightarrow$ Only field derivatives in the effective action.
- Broken rotation in the plane $(1,2)$:

$$\delta_\epsilon^{bj} X_i = \epsilon \left(-\delta_{ij} \xi_b - X_j \partial_b X_i \right)$$

Number of derivatives minus number of fields (scaling) preserved.

Fields and coordinates rescaling $\Longrightarrow$ Derivative expansion:

$$\partial_a X^i \longrightarrow \frac{1}{\sqrt{\sigma} R} \partial_a X^i .$$

Variations by broken rotation mix orders $\Longrightarrow$ Recurrence relations.

$ISO(1,1)$ and $SO(D-2)$ invariance $\Longrightarrow$ Contraction of indices.

Effective string action is strongly constrained! [1] [2] [3]

- the terms with only first derivatives coincide with the Nambu-Goto action to all orders in the derivative expansion.
- The first allowed correction to the Nambu-Goto action turns out to be the the six derivative term

$$c_4 \left(\partial_\alpha \partial_\beta X \cdot \partial^\alpha \partial^\beta X\right) \left(\partial_\gamma X \cdot \partial^\gamma X\right)$$

  with arbitrary coefficient c_4
- however this term is non-trivial only when $d > 3$. For $d = 3$ the first non-trivial deviation of the Nambu-Goto action is an eight-derivative term
- The fact that the first deviations from the Nambu-Goto string are of high order, especially in $d = 3$, explains why in early Monte Carlo calculations a good agreement with the Nambu-Goto string was observed.

[1] M. Luscher and P. Weisz JHEP07(2004)014
[2] H. B. Meyer JHEP05(2006)066
[3] O. Aharony and M. Field JHEP01(2011)065

The effective string theory as the $T\bar{T}$ perturbation of the 2d free bosonic model [1]

The first term of the effective string expansion is simply a 2d free field theory

$$S_0[X] = \frac{\sigma}{2} \int d^2\xi \, (\partial_\alpha X \cdot \partial^\alpha X) \ .$$

the corresponding energy-momentum tensor can be written as

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\delta_{\alpha\beta} \, (\partial^\gamma X \cdot \partial_\gamma X) \ .$$

Setting the Nambu-Goto (universal!) values for c_2 and c_3 in

$$S = S_{cl} + S_0[X] + \frac{\sigma}{2} \int d^2\xi \left[c_2(\partial_\alpha X \cdot \partial^\alpha X)^2 + c_3(\partial_\alpha X \cdot \partial^\beta X)(\partial_\beta X \cdot \partial^\alpha X) \right] + S_b + \dots ,$$

we find:

$$S = S_{cl} + S_0[X] - \frac{\sigma}{4} \int d^2\xi \, T_{\alpha\beta} T^{\alpha\beta} + S_b + \dots$$

[1] M. Caselle, D. Fioravanti, F. Gliozzi, R. Tateo JHEP07(2013)071

The effective string theory as the $T\bar{T}$ perturbation of the 2d free bosonic model

Setting (as usual in 2d CFTs) $T_{zz} = \frac{1}{2}(T_{11} - iT_{12})$ and $T_{\bar{z}\bar{z}} = \frac{1}{2}(T_{11} + iT_{12})$ and choosing the normalization as $T = -2\pi\sigma T_{zz}$, $\bar{T} = -2\pi\sigma T_{\bar{z}\bar{z}}$ so as to have the OPE

$$T(z)T(w) = \frac{D-2}{2}\frac{1}{(z-w)^4} + \dots$$

we find:

$$S = S_{cl} + S_0[X] - \frac{1}{2\pi^2\sigma}\int d^2\xi \, T\bar{T} + S_b + \dots$$

The effective string theory at the NLO is nothing else that the $T\bar{T}$ perturbation of the free field theory!

This massless perturbation can be studied with TBA techinques

Zamolodchikov solution.[1]

- Massless excitations confined on a infinite line or a ring naturally separate into right and left movers (in this example only one species of particles is present).
- The right-right and left-left mover scattering is trivial, while the left-right scattering is described by the amplitude

$$S(p,q) = \frac{2\sigma + ipq}{2\sigma - ipq} \ ,$$

- σ sets the scale of the perturbation p is the momentum of the right mover and $-q$ the momentum of the left mover.
- In the limit $\sigma \to \infty$, $S(p,q) \to 1$, right and left mover excitations decouple and the scale invariance of the model is fully restored at $\sigma = \infty$.

Starting from this S matrix, Zamolodchikov [2]was able to derive the Thermodynamic Bethe Ansatz (TBA) equations for the vacuum energy of the theory defined on a infinite cylinder with circumference R.

[2] A. B. Zamolodchikov [hep-th/0401146]

[1] A. B. Zamolodchikov Nucl. Phys. **B358**, 524 (1991)

The TBA equation.

The relevant equations are

$$\epsilon(p) = Rp - \int_0^\infty \frac{dq}{2\pi}\phi(p,q)\,\bar{L}(q), \quad \bar{\epsilon}(p) = Rp\ - \int_0^\infty \frac{dq}{2\pi}\phi(p,q)\,L(q)\ ,$$

where $\epsilon(p)$ and $\bar{\epsilon}(p)$ are the pseudoenergies for the right and the left movers, respectively,

$$\phi(p,q) = -i\partial_q \ln S(p,q)\ ,$$

and

$$L(p) = \ln(1 + e^{-\epsilon(p)})\ , \quad \bar{L}(p) = \ln(1 + e^{-\bar{\epsilon}(p)})\ .$$

Relation with the Nambu-Goto action.

The $T\bar{T}$ perturbation can be solved only perturbatively, but if we select the leading part $S_1(p,q)$ of the Zamolodchikov's S matrix at large σ

$$S(p,q) = e^{ipq/\sigma - i(pq/\sigma)^3/12 + \dots} = S_1(p,q)e^{-i(pq/\sigma)^3/12+\dots}$$

then the kernel in the TBA equations becomes much simpler

$$\phi(p,q) = -i\partial_q \ln S_1(p,q) = p/\sigma \ ,$$

the TBA equations decouple and can be solved to all orders leading exactly to the Nambu-Goto energy spectrum! [1]

[1] S. Dubovsky, R. Flauger and V. Gorbenko JHEP09(2012)044

Solving the TBA equations.

The TBA equations can be rewritten as follows:

$$E_{(n,\bar{n})}(R) = \mathcal{E} + \bar{\mathcal{E}} + \sigma R$$

with

$$\mathcal{E} = -\frac{\pi(\tilde{c}_{IR} - 24n)}{12(R + \bar{\mathcal{E}}/\sigma)} , \quad \bar{\mathcal{E}} = -\frac{\pi(\tilde{c}_{IR} - 24\bar{n})}{12(R + \mathcal{E}/\sigma)} . \qquad (1)$$

these equations can be easily solved, giving

$$E_{(n,\bar{n})}(R) = \pm\sqrt{\sigma^2 R^2 + 4\pi\sigma\left(n + \bar{n} - \frac{\tilde{c}_{IR}}{12}\right) + \left(\frac{2\pi(n-\bar{n})}{R}\right)^2} .$$

Setting $\tilde{c}_{IR} = D - 2$ this becomes exactly the N-G spectrum
But we have much more. The derivation holds for any infrared CFT ![1]
the only change is that $\tilde{c}_{IR} = c_{IR} - 24h$ where h may be anyone of the conformal weights of the theory (and can be tuned using suitable boundary conditions).

[1] M. Caselle, D. Fioravanti, F. Gliozzi, R. Tateo JHEP07(2013)071

Application: the boundary term of the effective action: Constraints imposed by the Lorentz invariance

If the boundary is a Polyakov line in the ξ_0 direction placed at $\xi_1 = 0$, on which we assume Dirichlet boundary conditions $X_i(\xi_0, 0) = 0$, the most general boundary action should be of this type

$$S_b = \int d\xi_0 \left[b_1 \partial_1 X \cdot \partial_1 X + b_2 \partial_1 \partial_0 X \cdot \partial_1 \partial_0 X + b_3 (\partial_1 X \cdot \partial_1 X)^2 + \dots \right] .$$

Imposing Lorentz invariance one finds that $b_1 = 0$ and that the b_2 term is only the first term of a Lorentz invariant expression[1] :

$$b_2 \int d\xi_0 \left[\frac{\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} - \frac{(\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^2} \right] .$$

which is the analogous in the case of the boundary action of the Nambu-Goto action for the "bulk" effective action.

[1] M. Billo, M. Caselle, F. Gliozzi, M. Meineri, R. Pellegrini JHEP05(2012)130

Simulation I: Polyakov loops

- In order to eliminate the non-universal perimeter and constant terms from the expectation value of Polyakov loop correlators $P(R, L)$ (where L is the length of the two loops and R their distance) we measured the following ratio:

$$R_P(R, L) = \frac{P(R + 1, L)}{P(R, L)} \ .$$

- Due to the peculiar nature of our algorithm, based on the dual transformation to the 3d spin Ising model, this ratio can be evaluated for large values of R and L with very high precision.

Simulation I: Polyakov loops

The effective string prediction for this observable reads, up to the second loop order,

$$R_P(R,L) = \mathrm{e}^{-\sigma L}\,\frac{\eta(\mathrm{i}\frac{L}{2R})}{\eta(\mathrm{i}\frac{L}{2R+2})}\,(1 + F_2(R+1,L) + F_P(R+1,L) - F_2(R,L) - F_P(R,L))$$

where the first term is the leading effective string correction (the "Lüscher term") and is given by the Dedekind function η,

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\ ,$$

where $q = \exp(2\pi\mathrm{i}\tau)$.

Simulation I: Polyakov loops

$F_2(R, L)$ is the next to leading "bulk" correction

$$F_2(R, L) = -\frac{\pi^2 L}{1156 R^3 \sigma}\left(2E_4(\mathrm{i}\frac{L}{2R}) - E_2^2(\mathrm{i}\frac{L}{2R})\right)$$

where the E_{2k}, with $k = 1, 2, \ldots$, are the Eisenstein series:

$$E_{2k}(\tau) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \ ,$$

where $\sigma_p(n)$ is the sum of the p-th powers of the divisors of n:

$$\sigma_p(n) = \sum_{m|n} m^p \ .$$

Simulation I: Polyakov loops

Finally $F_P = \langle S^{(1)}_{b,2} \rangle_P$ is the leading correction coming from the boundary[1]

$$F_P(R,L) = -b_2 \frac{\pi^3 L}{60 R^4} E_4(\mathrm{i}\frac{L}{2R}) \ .$$

[1] O. Aharony and M. Field JHEP01(2011)065

Simulation settings

- We performed our simulations in the 3d gauge Ising model, using a dual algorithm

data set	β	L	σ	$1/T_c$
1	0.743543	68	0.0228068(15)	5
2	0.751805	100	0.0105255(11)	8
3	0.754700	125	0.0067269(17)	10

Table: Some information on the data sample

Results

- The values of b_2 extracted from the data show the expected scaling behaviour $b_2 \sim \frac{1}{\sqrt{\sigma}^3}$

data set	b_2	$b_2\sqrt{\sigma}^3$	χ^2
1	7.25(15)	0.0250(5)	1.2
2	26.8(8)	0.0289(9)	1.8
3	57.9(12)	0.0319(7)	1.3

Table: Values of b_2 as a function of β

Simulation II: Wilson loops

As a check of our analysis we performed the same simulation for the Wilson loops fixing the value of b_2 obtained above. In this case there is no more parameter to fit and we can directly compare our predictions with the results of the simulations. To eliminate all the non-universal parameters we constructed the following combination:

$$R_W^{'}(L, Lu) = \frac{W(L,R)}{W(L+1,R-1)} - \exp\{-\sigma(1+L(1-u))\} \;, \qquad u = R/L$$

Simulation II: Wilson loops

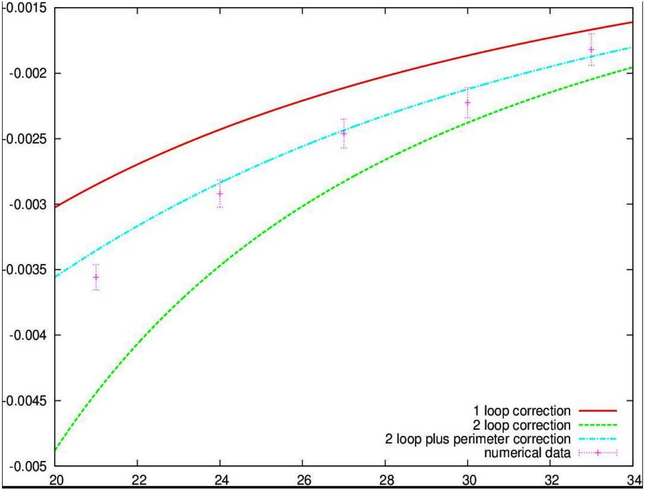


Figure: $R'_W(L, L\frac{4}{3})$ at $\beta = 0.754700$.

Conclusions

- The Effective String action is strongly constrained by Lorentz invariance. The first few orders of the bulk and of the boundary action are universal This explains why the Nambu-Goto effective theory describes so well the infrared regime of the interquark potential
- In the 3d Ising model also the first universal boundary correction can be reliably estimated and agrees with predictions
- The NLO correction to the Nambu-Goto action can be described as a free 2d bosonic theory perturbed by the irrelevant operator $T\bar{T}$
- This perturbation is quantum integrable and yields, via TBA, a spectrum which, in a suitable limit, coincides with the Nambu-Goto one
- A whole class of "Nambu-Goto like theories" can be constructed, as $T\bar{T}$ perturbations of generic 2d CFTs

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