The static self-energy (and the plaquette) at large orders in perturbation theory

Based on
Bauer, Bali, Pineda: Phys. Rev. Lett. 108 (2012) 242002
Bali, Bauer, Pineda, Torrero: Phys. Rev. D87 (2013) 094517
Bali, Bauer, Pineda: in preparation

See also hep-ph/0208031, hep-ph/0310130, hep-lat/0509022

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Lattice2013, Mainz, July 2013

Renormalon: Asymptotic behavior of the perturbative series associated to the OPE.

Current-current correlator ($c_{n*}\alpha_s^{n*+1} \sim \Lambda_{\rm QCD}^4/Q^4 \rightarrow c_n \sim n!$):

$$\int d^4x e^{iqx} \langle vac|J(x)J(0)|vac\rangle = \sum_{n=0}^{N} c_n \alpha_s^{n+1}(Q) + \frac{\Lambda_{\rm QCD}^4}{Q^4} + \cdots$$

Heavy quark physics:

$$\sum_{n=0}^{N} c_n \alpha_s^{n+1}(m_Q) + \frac{\Lambda_{\rm QCD}}{m_Q} + \cdots$$

The natural place to look for these effects.

$$m_{\rm OS} = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}\,,$$

$$M_B = m_{\rm OS} + \bar{\Lambda}_B + \mathcal{O}(1/m_{\rm OS})\,, \quad m_{\tilde{G}} = m_{\tilde{g},{\rm OS}} + \Lambda_H + \mathcal{O}\left(1/m_{\tilde{g},{\rm OS}}\right)$$

($b = \beta_1/(2\beta_0^2)$ Beneke)

$$r_n \overset{n\rightarrow\infty}{=} N_m\, \nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

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Over the years a lot of evidence in favour of the existence of the renormalon. Particularly important for heavy quark physics.

- $$2m_{\rm OS} + E_s(r) = 2m_{\rm OS} + V_s(r) + \mathcal{O}(r^2)$$

  $2m_{\rm OS} + V_s = 2m_{\rm RS} + V_{s,\rm RS}$ is renormalon free. Good description of the singlet static potential at short distances.

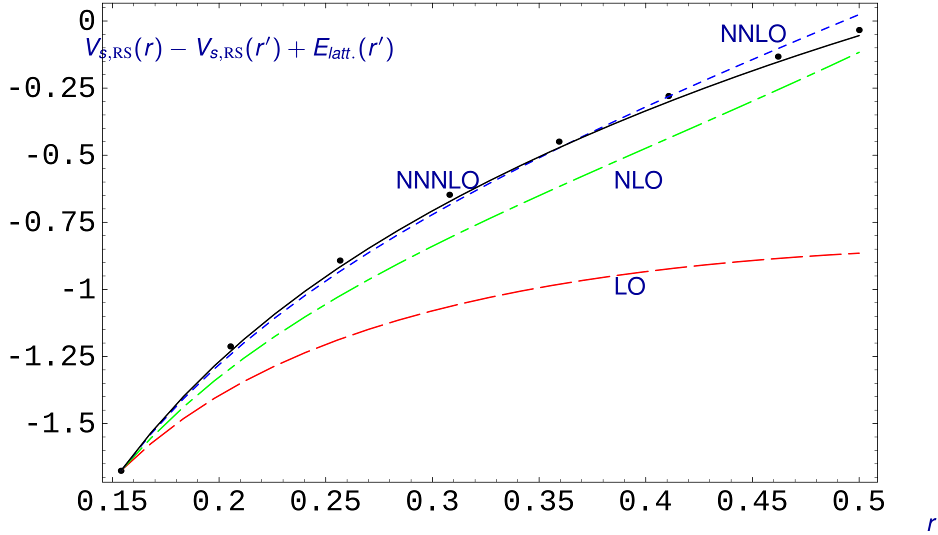

Figure: *From Pineda, hep-ph/0208031.*

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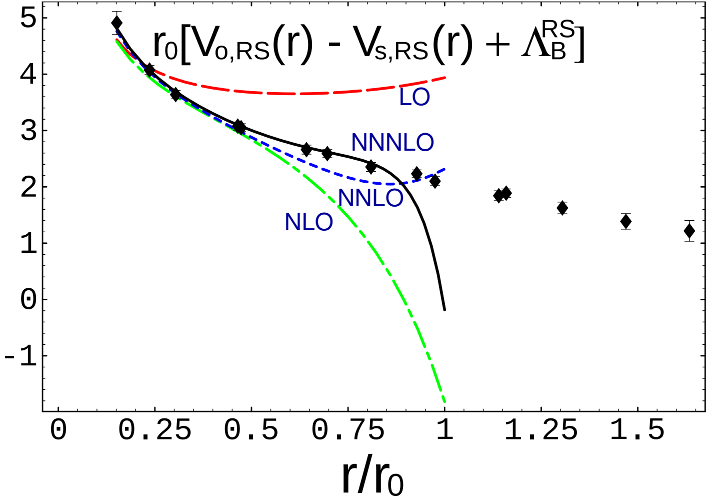

Figure: Splitting between the Π_u and the Σ_g^+ potentials and the comparison with the theoretical prediction. From Bali-Pineda, hep-ph/0310130.

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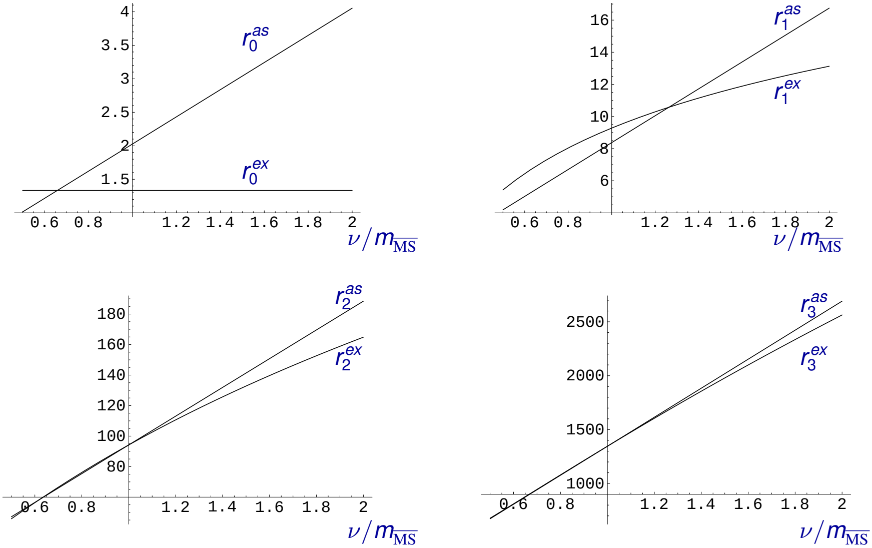

Figure: Plots of the exact (r_n^{ex}) and asymptotic (r_n^{as}) value of $r_n(\nu)$ at different orders in perturbation theory as a function of $\nu/m_{\overline{\rm MS}}$. From Pineda, hep-lat/0509022.

Yet...

- Not possible to compute using known semiclassical analysis.
- Based on few orders in perturbation theory ($\sim 3, 4$)
- Against renormalon existence (Suslov), or against renormalon dominance (Zakharov and followers).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD (and in heavy quarkonium physics).

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POLYAKOV LOOP versus δm (and m)

Possible to compute the energy of an static source in the lattice: δm of HQET.
We use Numerical Stochastic Perturbation Theory

$$\delta m = \frac{1}{a}\sum_{n=0}^{\infty} c_n^{(3,\rho)} \alpha^{n+1}(1/a)\,(\text{fundamental}), \quad \delta m_{\tilde{g}} = \frac{1}{a}\sum_{n=0}^{\infty} c_n^{(8,\rho)} \alpha^{n+1}(1/a)\,(\text{adjoint})$$

$$\lim_{n\to\infty} c_n^{(R,\rho)} = r_n(\nu)/\nu$$

$$L^{(R)}(N_S,N_T) = \frac{1}{N_S^3}\sum_{\mathbf{n}} \frac{1}{d_R}\mathrm{tr}\left[\prod_{n_4=0}^{N_T-1} U_4^R(n)\right] \qquad U_\mu^R(n) \approx e^{iA_\mu^R[(n+1/2)a]}$$

We implement triplet and octet representations R ($d_R = 3, 8$).

$$P^{(R,\rho)}(N_S,N_T) = -\frac{\ln\langle L^{(R,\rho)}(N_S,N_T)\rangle}{aN_T} = \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S,N_T)\alpha^{n+1},$$

$$\delta m = \lim_{N_S,N_T\to\infty} P^{(3,\rho)}(N_S,N_T), \qquad \delta m_{\tilde{g}} = \lim_{N_S,N_T\to\infty} P^{(8,\rho)}(N_S,N_T),$$

$$c_n^{(R,\rho)} = \lim_{N_S,N_T\to\infty} c_n^{(R,\rho)}(N_S,N_T).$$

	$\mathcal{O}(\alpha^4)$	$\mathcal{O}(\alpha^{20})$	$\mathcal{O}(\alpha^{32})$
$N_S(N_T)$	4(4)	8(8, 10, 12, 14)	4(8)

Table: *The first arrow states to which order in α the coefficients of $c_n^{(R)}(N_T, N_S)$ have been computed for each specific lattice volume for PBC.*

$\mathcal{O}(\alpha^3)$	$N_S(N_T)$	5(5, 6, 7, 8, 10)			
$\mathcal{O}(\alpha^4)$	$N_S(N_T)$	4(5, 6, 7, 8, 10, 12, 16, 20, 24)	12(16, 20)		
$\mathcal{O}(\alpha^{12})$	$N_S(N_T)$	6(6, 8, 10, 12, 16)	8(12, 16)		
$\mathcal{O}(\alpha^{12})$	$N_S(N_T)$	10(8, 12, 16, 20)	16(12, 16, 20)		
$\mathcal{O}(\alpha^{20})$	$N_S(N_T)$	7(7, 8)	8(8, 10)	9(12)	10(10)
$\mathcal{O}(\alpha^{20})$	$N_S(N_T)$	11(16)	12(12)	14(14)	

Table: *The first column states to which order in α the coefficients of $c_n^{(R)}(N_T, N_S)$ and the associated ratios have been computed for each specific lattice volume for TBC.*

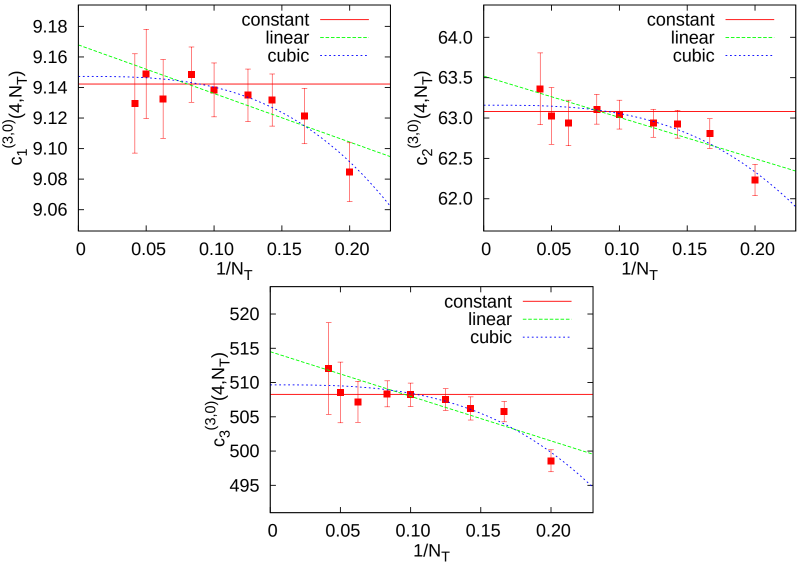

Figure: $c^{(3,0)}_{1,2,3}(4, N_T)$ *as a function of* $1/N_T$*, in comparison to a constant plus linear fit, a constant plus cubic fit, and a constant fitted only to the* $N_T > 10$ *points.*

Perturbative OPE at finite volume

$$\delta m(N_S) = \lim_{N_T\to\infty} P(N_S, N_T) \qquad \text{and} \qquad c_n(N_S) = \lim_{N_T\to\infty} c_n(N_S, N_T)\,.$$

For large N_S, we write (OPE: $\frac{1}{a} \gg \frac{1}{N_S a} \gg \Lambda_{\rm QCD}$)

$$\delta m(N_S) = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_{n=0}^{\infty} f_n \alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right).$$

Taylor expansion of $\alpha\Big((aN_S)^{-1}\Big)$ in powers of $\alpha(a^{-1})$:

$$c_n(N_S) = c_n - \frac{f_n(N_S)}{N_S} + \mathcal{O}\left(\frac{1}{N_S^2}\right); \qquad f_n(N_S) = \sum_{i=0}^{n} f_n^{(i)} \ln^i(N_S)\,,$$

$f_n^{(0)} = f_n$ and the coefficients $f_n^{(i)}$ for $i > 0$ are determined by f_m with $m < n$ and β_j with $j \leq n-1$.

$$f_1(N_S) = f_1 + f_0\frac{\beta_0}{2\pi}\ln(N_S)\,,$$

$$f_2(N_S) = f_2 + \left[2f_1\frac{\beta_0}{2\pi} + f_0\frac{\beta_1}{8\pi^2}\right]\ln(N_S) + f_0\left(\frac{\beta_0}{2\pi}\right)^2\ln^2(N_S)\,,$$

and so on.

"Physical interpretation"

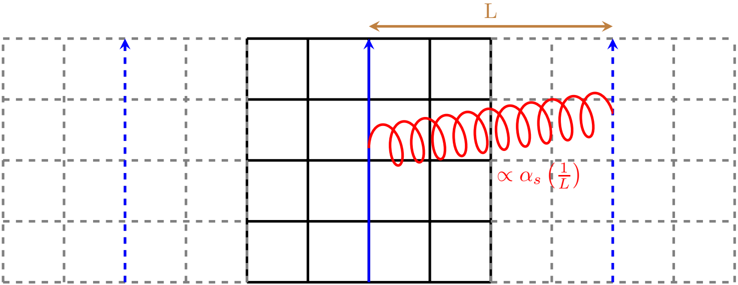

Figure: *Self-interactions with replicas producing* $1/L = 1/(aN_S)$ *Coulomb terms.*

$$P \propto \int_{1/(aN_S)}^{1/a} dk\, \alpha(k) \sim \frac{1}{a}\sum_n c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_n c_n \alpha^{n+1}\left((aN_S)^{-1}\right) ,$$

$$c_n \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n n! , \qquad f_n^{(i)}(N_S) \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{n!}{i!} .$$

	$c_n^{(3,0)}$	$c_n^{(3,1/6)}$	$c_n^{(8,0)} C_F/C_A$	$c_n^{(8,1/6)} C_F/C_A$
c_0	2.117274357	0.72181(99)	2.117274357	0.72181(99)
c_1	11.136(11)	6.385(10)	11.140(12)	6.387(10)
$c_2/10$	8.610(13)	8.124(12)	8.587(14)	8.129(12)
$c_3/10^2$	7.945(16)	7.670(13)	7.917(20)	7.682(15)
$c_4/10^3$	8.215(34)	8.017(33)	8.197(42)	8.017(36)
$c_5/10^4$	9.322(59)	9.160(59)	9.295(76)	9.139(64)
$c_6/10^6$	1.153(11)	1.138(11)	1.144(13)	1.134(12)
$c_7/10^7$	1.558(21)	1.541(22)	1.533(25)	1.535(22)
$c_8/10^8$	2.304(43)	2.284(45)	2.254(51)	2.275(45)
$c_9/10^9$	3.747(95)	3.717(97)	3.64(11)	3.703(98)
$c_{10}/10^{10}$	6.70(22)	6.65(22)	6.49(25)	6.63(22)
$c_{11}/10^{12}$	1.316(52)	1.306(53)	1.269(59)	1.303(53)
$c_{12}/10^{13}$	2.81(13)	2.79(13)	2.71(14)	2.78(13)
$c_{13}/10^{14}$	6.51(35)	6.46(35)	6.29(37)	6.45(35)
$c_{14}/10^{16}$	1.628(96)	1.613(97)	1.57(10)	1.614(97)
$c_{15}/10^{17}$	4.36(28)	4.32(28)	4.22(29)	4.33(28)
$c_{16}/10^{19}$	1.247(86)	1.235(86)	1.206(89)	1.236(86)
$c_{17}/10^{20}$	3.78(28)	3.75(28)	3.66(28)	3.75(28)
$c_{18}/10^{22}$	1.215(93)	1.204(94)	1.176(95)	1.205(94)
$c_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

	$f_n^{(3,0)}$	$f_n^{(3,1/6)}$	$f_n^{(8,0)} C_F/C_A$	$f_n^{(8,1/6)} C_F/C_A$
f_0	0.7696256328	0.7810(59)	0.7696256328	0.7810(69)
f_1	6.075(78)	6.046(58)	6.124(87)	6.063(68)
$f_2/10$	5.628(91)	5.644(62)	5.60(11)	5.691(78)
$f_3/10^2$	5.87(11)	5.858(76)	6.00(18)	5.946(91)
$f_4/10^3$	6.33(22)	6.29(17)	6.57(40)	6.26(23)
$f_5/10^4$	7.73(35)	7.71(26)	7.67(66)	7.78(42)
$f_6/10^5$	9.86(53)	9.80(42)	9.68(99)	9.79(69)
$f_7/10^7$	1.388(81)	1.378(71)	1.35(15)	1.38(11)
$f_8/10^8$	2.12(12)	2.11(12)	2.06(22)	2.10(17)
$f_9/10^9$	3.54(20)	3.52(20)	3.40(37)	3.51(27)
$f_{10}/10^{10}$	6.49(33)	6.44(34)	6.23(67)	6.44(43)
$f_{11}/10^{12}$	1.296(64)	1.286(66)	1.24(13)	1.286(74)
$f_{12}/10^{13}$	2.68(19)	2.64(18)	2.65(33)	2.65(21)
$f_{13}/10^{14}$	6.70(54)	6.68(52)	6.36(90)	6.66(57)
$f_{14}/10^{16}$	1.58(14)	1.56(14)	1.55(22)	1.57(15)
$f_{15}/10^{17}$	4.41(34)	4.37(33)	4.24(47)	4.37(35)
$f_{16}/10^{19}$	1.241(92)	1.230(91)	1.20(11)	1.231(94)
$f_{17}/10^{20}$	3.79(28)	3.75(28)	3.67(30)	3.76(28)
$f_{18}/10^{22}$	1.215(94)	1.204(94)	1.176(97)	1.205(94)
$f_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

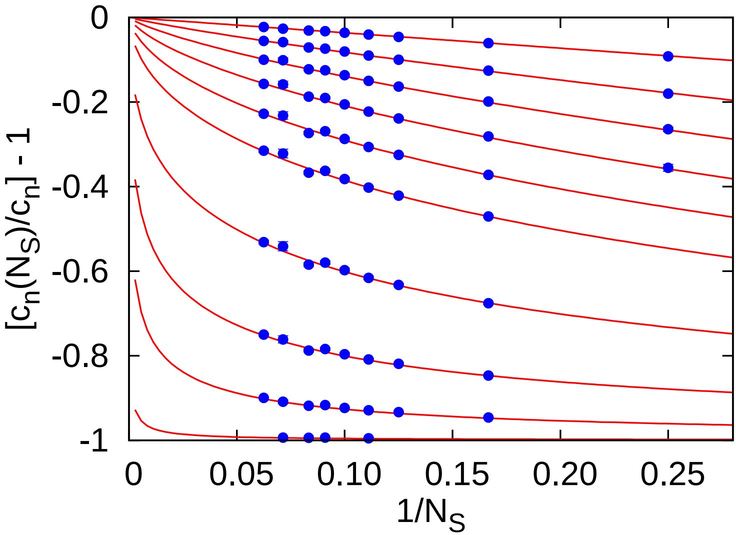

Figure: $c_n^{(3,0)}(N_S)/c_n^{(3,0)} - 1$ *for* $n \in \{0, 1, 2, 3, 4, 5, 7, 9, 11, 15\}$ *(top to bottom). For each value of* N_S *we have plotted the data point with the maximum value of* N_T*. The curves represent the global fit.* $-(1/N_S) f^{(3,0)}_{0,\mathrm{DLPT}}/c^{(3,0)}_{0,\mathrm{DLPT}}$ *is shown for* $n = 0$.

Ratios

$$\frac{c_n^{(3,\rho)}}{c_{n-1}^{(3,\rho)}}\frac{1}{n}=\frac{c_n^{(8,\rho)}}{c_{n-1}^{(8,\rho)}}\frac{1}{n}$$
$$=\frac{\beta_0}{2\pi}\left\{1+\frac{b}{n}-\frac{bs_1}{n^2}+\frac{1}{n^3}\left[b^2s_1^2+b(b-1)(s_1-2s_2)\right]+\mathcal{O}\left(\frac{1}{n^4}\right)\right\} \;.$$

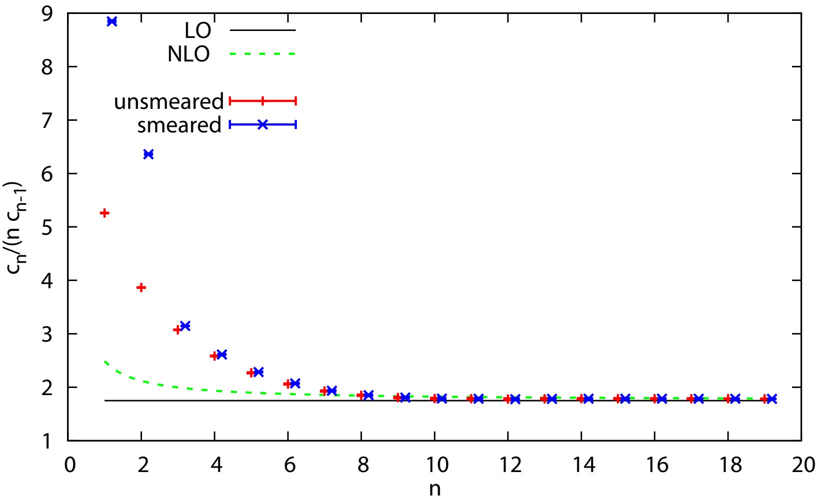

Figure: Ratios $c_n/(nc_{n-1})$ of the smeared (blue) and unsmeared (red) triplet static self-energy coefficients c_n in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

N_m

$$c_n^{fitted} = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)}c_1 + \frac{b(b-1)}{(n+b)(n+b-1)}c_2 + \cdots\right).$$

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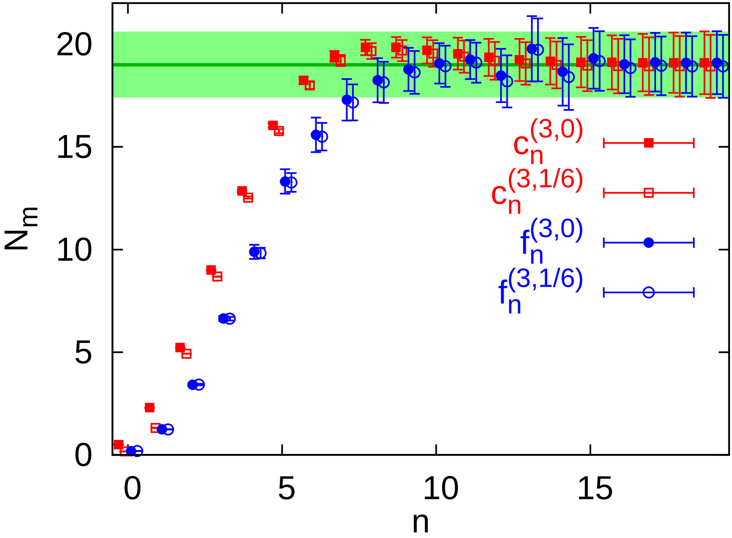

Figure: *N_m, determined from the coefficients $c_n^{(3,0)}$, $c_n^{(3,1/6)}$, $f_n^{(3,0)}$ and $f_n^{(3,1/6)}$ at NNLO. The horizontal band is our final result: $N_m^{\rm latt} = 19.0 \pm 1.6$.*

From lattice to $\overline{\mathrm{MS}}$ scheme

$$\alpha_{\overline{\mathrm{MS}}}(\mu) = \alpha_{\mathrm{latt}}(\mu)\left(1 + d_1\alpha_{\mathrm{latt}}(\mu) + d_2\alpha^2_{\mathrm{latt}}(\mu) + d_3\alpha^3_{\mathrm{latt}}(\mu) + \mathcal{O}(\alpha^4_{\mathrm{latt}})\right)\,,$$

$$N^{\overline{\mathrm{MS}}}_{m,m_{\tilde g}} = N^{\mathrm{latt}}_{m,m_{\tilde g}}\Lambda_{\mathrm{latt}}/\Lambda_{\overline{\mathrm{MS}}}\,,\quad \text{where}\quad \Lambda_{\overline{\mathrm{MS}}} = e^{\frac{2\pi d_1}{\beta_0}}\Lambda_{\mathrm{latt}} \approx 28.809338139488\,\Lambda_{\mathrm{latt}}\,.$$

This yields the numerical values

$$N^{\overline{\mathrm{MS}}}_m = 0.660(56)\,,\quad C_F/C_A\,N^{\overline{\mathrm{MS}}}_{m_{\tilde g}} = -C_F/C_A\,N^{\overline{\mathrm{MS}}}_{\Lambda} = 0.649(62)\,.$$

Other combinations of interest are

$$N^{\overline{\mathrm{MS}}}_{V_s} = -1.32(11)\,,\quad N^{\overline{\mathrm{MS}}}_{V_o} = 0.14(18)\,.$$

Assuming that

$$c_{3,\overline{\mathrm{MS}}} \simeq N^{\overline{\mathrm{MS}}}_m\left(\frac{\beta_0}{2\pi}\right)^3\frac{\Gamma(4+b)}{\Gamma(1+b)}\left(1+\frac{b}{(3+b)}s_1+\frac{b(b-1)}{(3+b)(2+b)}s_2+\cdots\right),$$

and using our central value $c^{(3,0)}_{3,\mathrm{latt}} = 794.5$, we obtain

$$d_3 \simeq 365\,,\qquad \beta_3^{\mathrm{latt}} \simeq -1.7\times 10^6\,.$$

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CONCLUSIONS I

Renormalons go beyond large-β_0 analysis: $\rightarrow$ OPE

Clearly seen in heavy quark physics from $\overline{\mathrm{MS}}$-like computations: Pole mass, static potential, hybrid potential, binding energy, $\cdots$

It is compulsory to take into account renormalon effects in order to do precision computations in heavy quark physics.

For the first time, it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}.

$$N_m^{\mathrm{latt}} = 19.0 \pm 1.6\,, \quad C_F/C_A\, N_\Lambda^{\mathrm{latt}} = -18.7 \pm 1.8\,,$$
$$N_m^{\overline{\mathrm{MS}}} = 0.660 \pm 0.056\,, \quad C_F/C_A\, N_\Lambda^{\overline{\mathrm{MS}}} = -0.649 \pm 0.062\,.$$

Completely consistent with continuum-like determinations ($N_m^{\overline{\mathrm{MS}}} = 0.62$).

We have (numerically) proven, beyond any reasonable doubt (10 standard deviations), the existence of the renormalon in QCD.

Nonperturbative quantities ($\bar{\Lambda}$, Λ_H, $\langle G^2\rangle$, $\cdots$) can only be defined after subtracting the divergent perturbative series.

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Plaquette (Bali, Bauer, Pineda) PRELIMINARY

$$\langle P \rangle = \sum_{n=0}^{N} p_n \alpha^{n+1}(a^{-1}) + a^4 \frac{\pi^2}{36} \langle G^2 \rangle + \cdots$$

$$d = 1 (n_0 \sim 7) \longrightarrow d = 4 (n_0 \sim 28)$$

$$N + 1 = 35$$

$$p_n = N_P \left(\frac{\beta_0}{d\, 2\pi} \right)^n \frac{\Gamma(n+1+d\, b)}{\Gamma(1+d\, b)} (1 + \cdots) \,.$$

PRELIMINARY

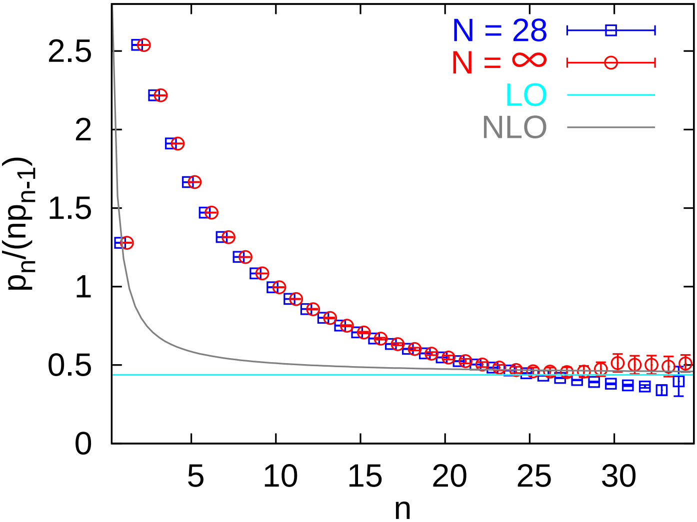

Figure: Ratios $p_n/(np_{n-1})$ of the plaquette coefficients p_n ($N = \infty$, $N = 28$) in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

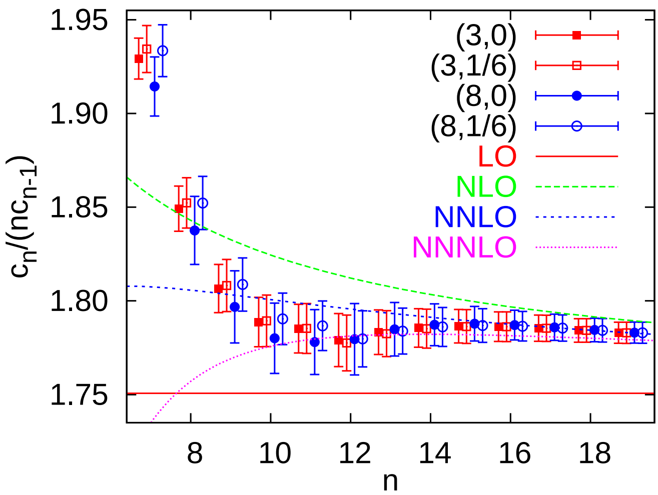

Figure: *The ratios $c_n/(nc_{n-1})$ for the smeared and unsmeared, triplet and octet fundamental static self-energies, compared to the prediction for the LO, next-to-leading order (NLO), NNLO and NNNLO of the $1/n$ expansion.*

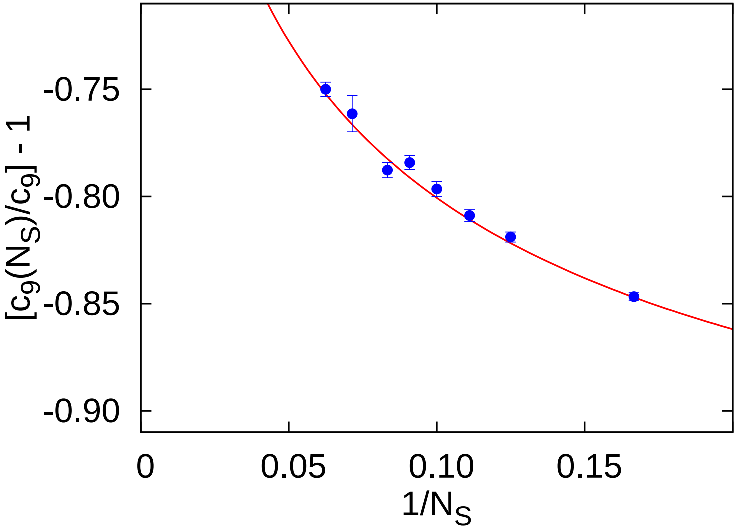

Figure: $c_9^{(3,0)}(N_S)/c_9^{(3,0)} - 1$. *For each value of N_S we have plotted the data point with the maximum value of N_T. The curve represents the global fit.*