The colour adjoint static potential from Wilson loops with generator insertions and its physical interpretation

31st International Symposium on Lattice Field Theory – Mainz, Germany

Marc Wagner, Owe Philipsen
Goethe-Universität Frankfurt am Main, Institut für Theoretische Physik
mwagner@th.physik.uni-frankfurt.de
http://th.physik.uni-frankfurt.de/~mwagner/

July 29, 2013

The (singlet) static potential $V^1$ is a very common and important observable in lattice gauge theory.

It is the energy of a static antiquark $\bar{Q}(x)$ and a static quark $Q(y)$ in a colour singlet (i.e. a gauge invariant) orientation as a function of the separation $r \equiv |x - y|$.

The spin of a static quark is irrelevant, i.e. in the following

- no spin indices or $\gamma$ matrices,
- only spinless colour charges,

\[ \bar{Q}^a_A(x) = (Q^{a,\dagger}(x)\gamma_0)A \rightarrow Q^{a,\dagger}(x), \]
\[ Q^a_A(y) \rightarrow Q^a(y), \]

where $a$ denotes a colour index and $A$ a spin index.
The singlet static potential for gauge group $SU(N)$ can be obtained as follows:

1. Define a trial state
   $$|\Phi^1\rangle \equiv \bar{Q}(x)U(x,y)Q(y)|0\rangle.$$

2. The temporal correlation function of this trial state simplifies to the well known Wilson loop,
   $$\langle \Phi^1(t_2)|\Phi^1(t_1)\rangle = e^{-2M\Delta t}N\left\langle W_1(r, \Delta t) \right\rangle, \quad \Delta t \equiv t_2 - t_1 > 0.$$

3. The singlet static potential $V^1 \equiv V_0^1$ can be obtained from the asymptotic exponential behaviour,
   $$\left\langle W_1(r, \Delta t) \right\rangle = \sum_{n=0}^{\infty} c_n \exp \left( - V^1_n(r)\Delta t \right) \xrightarrow{\Delta t \to \infty} \exp \left( - V^1(r)\Delta t \right).$$
   $$V^1(r) = - \lim_{\Delta t \to \infty} \frac{\langle \dot{W}_1(r, \Delta t) \rangle}{\langle W_1(r, \Delta t) \rangle}.$$
Goal of this work: compute and interpret the potential of a static antiquark $\bar{Q}(x)$ and a static quark $Q(y)$ in a colour adjoint (i.e. a gauge variant) orientation in various gauges as a function of the separation $r \equiv |x - y|$.

A colour adjoint orientation of a static antiquark and a static quark can be obtained by inserting the generators of the colour group $T^a$ (e.g. for $SU(3)$, $T^a = \lambda^a/2$), i.e. $\bar{Q}T^aQ|0\rangle$.

If the static antiquark and the static quark are separated in space, a straightforward generalisation is

$|\Phi^{T^a}\rangle \equiv \bar{Q}(x)U(x, x_0)T^aU(x_0, y)Q(y)|0\rangle$.

A corresponding definition of the colour adjoint static potential has been proposed and used in pNRQCD (a framework based on perturbation theory).

• We discuss non-perturbative calculations analogous as for the singlet static potential in various gauges,

\[
\langle \Phi^T_a(t_2)|\Phi^T_a(t_1)\rangle = e^{-2M\Delta t}N \langle W^T_a(r, \Delta t) \rangle ,
\]

\[
W^T_a(r, \Delta t) \equiv \frac{1}{N} \text{Tr} \left( T^a U_R T^a, U_L \right)
\]

\[
\langle W^T_a(r, \Delta t) \rangle = \sum_{n=0}^{\infty} c_n \exp \left( - V^T_a(r) \Delta t \right) \Delta t \rightarrow \infty \propto \exp \left( - V^T_a(r) \Delta t \right).
\]

• In particular we are interested,

  – whether the colour adjoint static potential \( V^T_a \equiv V^T_0^a \) is gauge invariant (i.e. whether the obvious gauge dependence of the correlation function \( \langle W^T_a(r, \Delta t) \rangle \) only appears in the matrix elements \( c_n \)),

  – whether \( V^T_a \) indeed corresponds to the potential of a static antiquark and a static quark in a colour adjoint orientation, or whether it has to be interpreted differently.
Without gauge fixing

\[ \langle W_{T^a}(r, \Delta t) \rangle = 0, \]

because this correlation function is gauge variant (and does not contain any gauge invariant contribution).

→ Without gauge fixing the calculation of a colour adjoint static potential fails.
\[ V^{T^a} \text{ in Coulomb gauge} \]

- Coulomb gauge: \( \nabla A^g(x) = 0 \), which amounts to an independent condition on every time slice \( t \).

- The remaining residual gauge symmetry corresponds to global independent colour rotations \( h^{\text{res}}(t) \in SU(N) \) on every time slice \( t \); with respect to this residual gauge symmetry the colour adjoint Wilson loop transforms as

\[
\langle W_{T^a}(r, \Delta t) \rangle = \frac{1}{\sqrt{N}} \text{Tr} \left( T^a U_R T^{a,\dagger} U_L \right) \rightarrow h^{\text{res}} \\
\rightarrow h^{\text{res}} \frac{1}{\sqrt{N}} \text{Tr} \left( h^{\text{res},\dagger}(t_1) T^a h^{\text{res}}(t_1) U_R h^{\text{res}}(t_2) T^{a,\dagger} h^{\text{res},\dagger}(t_2) U_L \right).
\]

- Since \( h^{\text{res}}(t_1) \) and \( h^{\text{res}}(t_2) \) are independent, the situation is analogous to that without gauge fixing, i.e.

\[
\langle W_{T^a}(r, \Delta t) \rangle_{\text{Coulomb gauge}} = 0,
\]

\[ \rightarrow \text{In Coulomb gauge the calculation of a colour adjoint static potential fails.} \]

Marc Wagner, Owe Philipsen, “The colour adjoint static potential from Wilson loops with generator insertions and its physical interpretation”, Jul 29, 2013
• Lorenz gauge: $\partial_\mu A^g_\mu(x) = 0$.

• In Lorenz gauge a Hamiltonian or a transfer matrix does not exist.

• Only gauge invariant correlation functions like the ordinary Wilson loop $\langle W_1(r, \Delta t) \rangle$ exhibit an asymptotic exponential behaviour and, therefore, allow the determination of energy eigenvalues.

• The colour adjoint Wilson loop $\langle W^{T^a}(r, \Delta t) \rangle_{\text{Lorenz gauge}}$ does not decay exponentially in the limit of large $\Delta t$.

$\rightarrow$ The physical meaning of a colour adjoint static potential determined from $\langle W^{T^a}(r, \Delta t) \rangle_{\text{Lorenz gauge}}$ (as frequently done in perturbation theory) is unclear.
$V_T^a$ in temporal gauge (1)

- Temporal gauge: $\partial_\mu A_0^g(x) = 0$ or equivalently $U_0^g(x) = 1$.

- Temporal links gauge transform as

$$U_0^g(t, x) = g(t, x)U_0(t, x)g^\dagger(t + a, x), \quad g(t, x) \in SU(N).$$

- A possible choice to implement temporal gauge is

$$
\begin{align*}
g(t = 2a, x) &= U_0(t = a, x), \\
g(t = 3a, x) &= g(t = 2a, x)U_0(t = 2a, x) = U_0(t = a, x)U_0(t = 2a, x), \\
g(t = 4a, x) &= g(t = 3a, x)U_0(t = 3a, x) = \ldots, \\
\ldots &= \ldots
\end{align*}$$
By inserting the transformation to temporal gauge \( g(t, x) \), the gauge variant colour adjoint Wilson loop turns into a gauge invariant observable:

\[
\left\langle W_{T^a}(r, \Delta t) \right\rangle_{\text{temporal gauge}} = \frac{1}{N} \left\langle \text{Tr} \left( U^{T^a, g(t_1; x, y) U^{T^a, t, g(t_2; y, x)} \right) \right\rangle_{\text{temporal gauge}} = \ldots =
\]

\[
= \frac{2}{N(N^2 - 1)} \sum_a \sum_b \left\langle \text{Tr} \left( T^a U_R T^b U_L \right) \text{Tr} \left( T^a U(t_1, t_2; x_0) T^b U(t_2, t_1; x_0) \right) \right\rangle
\]

\( (U^{T^a}(x, y) = U(x, x_0) T^a U(x_0, y)) \).

- \( \text{Tr}(T^a U_R T^b U_L) \): Wilson loop with generator insertions.
- \( \text{Tr}(T^a U(t_1, t_2; x_0) T^b U(t_2, t_1; x_0)) \): propagator of a static adjoint quark.

The colour adjoint Wilson loop in temporal gauge is a correlation function of a gauge invariant three-quark state, one fundamental static quark, one fundamental static anti-quark, one adjoint static quark.
$V^{T^a}$ in temporal gauge (3)

- Equivalently, after defining

\[ |\Phi^{Q\bar{Q}Q^{ad}}\rangle \equiv Q^{ad,a}(x_0)(\bar{Q}(x)U^{T^a}(x,y)Q(y))|0\rangle, \]

one can verify

\[ \langle \Phi^{Q\bar{Q}Q^{ad}}(t_2)|\Phi^{Q\bar{Q}Q^{ad}}(t_1)\rangle \propto \langle W_{T^a}(r, \Delta t) \rangle_{\text{temporal gauge}}. \]

$V^{T^a}$ in temporal gauge should not be interpreted as the potential of a static quark and a static anti-quark, which form a colour-adjoint state.

$V^{T^a}$ in temporal gauge is the potential of a colour-singlet three-quark state.

$V^{T^a}$ in temporal gauge does not only depend on the $Q\bar{Q}$ separation $r = |x - y|$, but also on the position $s = |x - x_0|/2 - |y - x_0|/2$ of the static adjoint quark $Q^{ad}$, i.e. $V^{T^a}(r, s)$ (in the following we work with the symmetric alignment $x_0 = (x + y)/2$).
A different approach, leading to the same result, is the transfer matrix formalism.


One can perform a spectral analysis of the colour adjoint Wilson loop:

\[
\langle W_{Ta}(r, \Delta t) \rangle_{\text{temporal gauge}} = \frac{1}{N} \sum_{k} e^{-(V_{k}^{Ta}(r)-E_{0})\Delta t} \sum_{\alpha,\beta} \langle k_{a_{\alpha\beta}}^{a_{\alpha\beta}} | U_{a_{\alpha\beta}}^{Ta}(x, y) | 0 \rangle^2,
\]

where \( |k_{a_{\alpha\beta}}^{a_{\alpha\beta}} \rangle \) denotes states containing three static quarks (one fundamental static quark, one fundamental static anti-quark, one adjoint static quark).

→ Again the conclusion is that \( V_{Ta} \) in temporal gauge is the potential of a colour-singlet three-quark state.
In the literature one can also find a proposal of a gauge invariant quantity, from which the colour adjoint static potential can possibly be determined,

\[ W_B(r, \Delta t) \equiv \frac{1}{N} \text{Tr} \left( T^a U_R T^{b,\dagger} U_L \right) B^a(x_0, t_1) B^b(x_0, t_2), \]

i.e. open colour indices are saturated by colour magnetic fields.


However, using the transfer matrix formalism one can again perform a spectral analysis and show that only states with a fundamental quark and a fundamental antiquark \( |k_{\alpha\beta}\rangle \) (i.e. singlet static potentials) contribute to the correlation function:

\[ \langle W_B(r, \Delta t) \rangle = \sum_k e^{-(V_{k,-}^1(r)-\epsilon_0)\Delta t} \sum_{\alpha,\beta} \left| \langle k_{\alpha\beta} | U_{\alpha\beta}^{T^a B^a} (x, y) | 0 \rangle \right|^2. \]

\[ \rightarrow \langle W_B(r, \Delta t) \rangle \text{ is not suited to extract a colour adjoint static potential.} \]
Numerical lattice results for $SU(2)$

- $SU(2)$ colour group, four different lattice spacings $a = 0.038 \text{ fm} \ldots 0.102 \text{ fm}$.

- In temporal gauge the colour adjoint (or rather $Q\bar{Q}Q^{\text{ad}}$) static potential $V^{T_a}$ is attractive,
  - for small separations stronger than the singlet static potential $V^1$,
  - for large separations the slope is the same as for the singlet static potential $V^1$ (indicates flux tube formation between $QQ^{\text{ad}}$ and $\bar{Q}Q^{\text{ad}}$).
Perturbation theory for static potentials is a good approximation for small quark separations and should agree in that region with corresponding non-perturbative results.

Singlet static potential (gauge invariant, i.e. the gauge is not important):

\[ V^1(r) = -\frac{(N^2 - 1)g^2}{8N\pi r} + \text{const} + \mathcal{O}(g^4). \]

Colour adjoint static potential (in Lorenz gauge):

\[ V^{T^a}(r) = +\frac{g^2}{8N\pi r} + \text{const} + \mathcal{O}(g^4). \]

- In Lorenz gauge a Hamiltonian or a transfer matrix does not exist, i.e. the physical meaning is unclear; appears frequently in the literature.
- The repulsive behaviour is not reproduced by any of the presented non-perturbative considerations or computations.
LO perturbative calculations (2)

• Colour adjoint static potential ("in temporal gauge"; more precisely: perturbative calculation in Lorenz gauge of the gauge invariant observable, which is equivalent to the colour adjoint Wilson loop in temporal gauge):

\[ V_{Q\bar{Q}Q^{\text{ad}}} (r, s = 0) = -\frac{(4N^2 - 1)g^2}{8N\pi r} + \text{const} + \mathcal{O}(g^4). \]

- Attractive and stronger by a factor 4…5 than the singlet static potential (depending on \( N \)).
- Qualitative agreement with numerical lattice results for \( SU(2) \).
Conclusions

- We have discussed the non-perturbative definition of a static potential $V^{Ta}$ for a quark antiquark pair in a colour adjoint orientation, based on Wilson loops with generator insertions $\langle W_{Ta}(r, \Delta t) \rangle$ in various gauges:
  - **Without gauge fixing/Coulomb gauge**: $\langle W_{Ta}(r, \Delta t) \rangle = 0$, i.e. the calculation of a potential $V^{Ta}$ fails.
  - **Lorenz gauge**: a Hamiltonian or a transfer matrix does not exist, the physical meaning of a corresponding potential $V^{Ta}$ is unclear.
  - **Temporal gauge**: a strongly attractive potential $V^{Ta}$, which should be interpreted as the potential of three quarks, i.e. $V^{Ta} = V\bar{Q}QQ^{ad}$.

Clearly the resulting potential $V^{Ta}$ is gauge dependent.

- Saturating open colour indices with $B^a$, yields a singlet static potential.

- LO perturbation theory in Lorenz gauge has long predicted $V^{Ta}$ to be repulsive; it appears impossible, to reproduce this repulsive behaviour by a non-perturbative computation based on Wilson loops.