Relativistic, model-independent, three-particle quantization condition: (I) Derivation

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based on unpublished work with
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Introduction

Maiani Testa no-go theorem says that one cannot get S-matrix (above threshold) from infinite-volume Euclidean-time correlators.[1]

In finite volume the no-go theorem does not apply.

Indeed, Lüscher derived a relation between

finite-volume spectrum of QCD Hamiltonian (below four pion masses)
and
phase shift for elastic two-pion scattering.[2][3][4]

This method has been used extensively, providing scattering predictions from first principles QCD.

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[1] Maiani, L. & Testa, M. *Phys.Lett.* **B245,** 585–590 (1990).
[2] Luescher, M. *Commun. Math. Phys.* **104,** 177 (1986).
[3] Luescher, M. *Commun. Math. Phys.* **105,** 153–188 (1986).
[4] Luescher, M. *Nucl. Phys.* **B354,** 531–578 (1991).

Introduction

There has, however, so far been no lattice calculation of S-matrix elements above inelastic threshold.

Here one should distinguish between
a) systems with multiple, strongly-coupled, **two-particle** channels
b) systems with one or more, strongly-coupled, $(N > 2)$-**particle** channels

In the first case, the formalism for determining S-matrix from finite-volume spectrum is well understood.[5][6][7]

[5] Bernard, V. *et al.* *JHEP* **1101,** 019 (2011).
[6] Briceno, R. A. & Davoudi, Z. arXiv:1204.1110 [hep-lat] (2012).
[7] Hansen, M. T. & Sharpe, S. R. *Phys.Rev.* **D86,** 016007 (2012).

Introduction

Important progress has also been made for the simplest ($N > 2$)-particle cases:

two-to-three and **three-to-three scattering**.[8][9]

However, a **relativistic**, **model-independent** relation between finite-volume spectrum and S-matrix for three-particle states is still unavailable.

This is the subject of this talk.

[8] Polejaeva, K. & Rusetsky, A. *Eur.Phys.J.* **A48,** 67 (2012).
[9] Briceno, R. A. & Davoudi, Z. arXiv:1212.3398 [hep-lat] (2012).

Finite-volume set-up

Here finite volume means...

- finite, cubic spatial volume (extent L)
- periodic boundary conditions [$\vec{p} \in (2\pi/L)\mathbb{Z}^3$]
- time direction infinite.

Assume L large enough to ignore exponentially suppressed (e^{-mL}) corrections. **Neglect** e^{-mL} **throughout.**

Assume continuum field theory throughout.

Allow non-zero total momentum in finite-volume frame...

- total energy E
- total momentum $\vec{P}$ $\quad \left(\vec{P} = (2\pi/L)\vec{n}_P \qquad \vec{n}_P \in \mathbb{Z}^3 \right)$
- CM frame energy E^* $\quad \left(E^{*2} = E^2 - \vec{P}^2 \right)$

Particle content set-up

Restrict particle content to

- single scalar with mass m. So all results for **identical particles**.
- interactions governed by local relativistic field theory, with $\mathbb{Z}_2$ **symmetry**. (G-parity for pions)

Restrict CM energy, $m < E^* < 5m$.

Theory is otherwise arbitrary...

- include all operators with an even number of scalar fields
- make no assumptions about relative coupling strength

Derivation

We relate spectrum to scattering via finite-volume correlator[10]

$$C_L(E,\vec{P}) \equiv \int_L d^4x e^{i(-\vec{P}\cdot\vec{x}+Ex^0)} \langle 0| T\sigma(x)\sigma^\dagger(0)|0\rangle \,,$$

where σ is odd-particle interpolating field.

All E for which $C_L(E,\vec{P})$ diverges are in the finite-volume spectrum. So, we **determine a condition of divergence, to all orders in perturbation theory.**

Result depends on
two-to-two scattering amplitude: $i\mathcal{M}$
three-to-three scattering amplitude: $i\mathcal{M}_{3\to 3}$

[10] Kim, Sachrajda and Sharpe. *Nucl.Phys.* **B727**, 218–243 (2005).

Finite-volume correlator

Due to finite-volume condition, loop-momenta in diagrams are summed

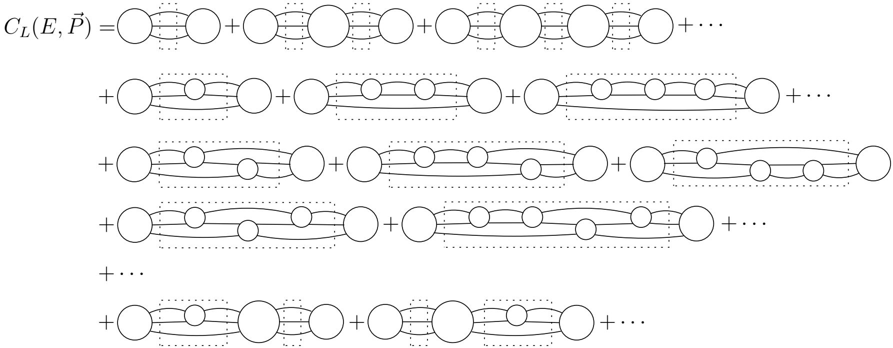
$$\frac{1}{L^3}\sum_{\vec{p}} \quad \text{for} \quad \vec{p} \in (2\pi/L)\mathbb{Z}^3 \,.$$

For smooth function, difference between sum and integral is exponentially suppressed.

Only keep sums when summand diverges. This happens when intermediate states go on-shell, which is **only possible for three-particle states**.

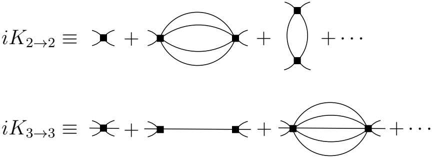
Skeleton expansion

Deduce skeleton expansion, which keeps all $1/L^n$ corrections to $C_L(E, \vec{P})$



Here boxes indicate remaining summed loops.

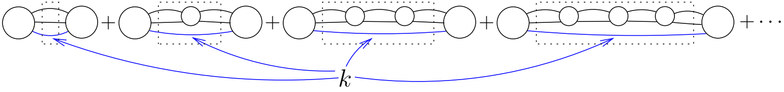
All other loops inside kernels, with $\sum \to \int$



Detailed analysis: No switches

Begin by considering diagrams with $iK_{2\to 2}$ insertions all on same pair

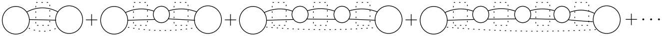
$C_L^{(1)} \equiv$



Important finite-volume effects from $k^0 = \omega_k = \sqrt{\vec{k}^2 + m^2}$, which gives singularity from on-shell state.

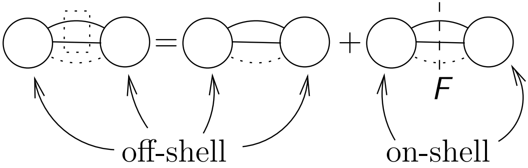
Setting $k^0 = \omega_k$ generates **two-particle diagrams** with energy-momentum $(E - \omega_k, \vec{P} - \vec{k})$

$C_L^{(1)} = \frac{1}{L^3} \sum_{\vec{k}} \frac{1}{2\omega_k}$



Detailed analysis: No switches

Next use the identity

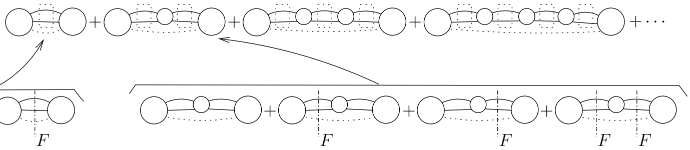


Third term represents

$$\sigma i F \sigma^\dagger = \text{(row vector)}\times\text{(matrix)}\times\text{(column vector)}$$

where entries of σ, $\sigma^\dagger$ are coefficients of $Y_{\ell,m}$ decomposition.

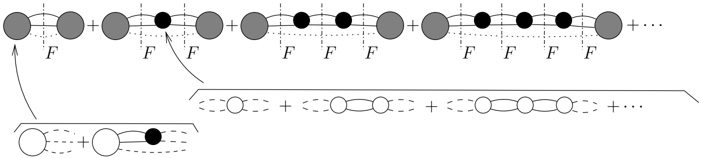
$$C_L^{(1)} = \frac{1}{L^3}\sum_{\vec{k}}\frac{1}{2\omega_k}$$



Detailed analysis: No switches

Regroup terms by number of iF insertions

$$C_L^{(1)} = C_\infty^{(1)} + \frac{1}{L^3}\sum_{\vec{k}} \frac{1}{2\omega_k}$$



We deduce

$$C_L^{(1)} - C_\infty^{(1)} = (\sigma + A'^{(1,u)}) \frac{iF}{2\omega L^3} \frac{1}{1 - i\mathcal{M} iF} (\sigma^\dagger + A^{(1,u)}) - \sigma \frac{iF}{3\omega L^3} \sigma^\dagger \,,$$

where second term is from extra symmetry factor.

Detailed analysis: No switches

$$C_L^{(1)} - C_\infty^{(1)} = (\sigma + A'^{(1,u)}) \frac{iF}{2\omega L^3} \frac{1}{1 - i\mathcal{M} iF} (\sigma^\dagger + A^{(1,u)}) - \sigma \frac{iF}{3\omega L^3} \sigma^\dagger \,.$$

Lots of notation here!

Structure is (row vector)×(matrix)×(column vector) on product space

[finite-volume momentum]×[angular momentum]

Detailed analysis: No switches

$$C_L^{(1)} - C_\infty^{(1)} = (\sigma + A'^{(1,u)}) \frac{iF}{2\omega L^3} \frac{1}{1 - i\mathcal{M} iF} (\sigma^\dagger + A^{(1,u)}) - \sigma \frac{iF}{3\omega L^3} \sigma^\dagger \,.$$

Lots of notation here!

Structure is (row vector)×(matrix)×(column vector) on product space

[finite-volume momentum]×[angular momentum]

For example $i\mathcal{M}$ is short for the diagonal matrix $i\mathcal{M}_{k',\ell',m';k,\ell,m}$ (with $k, k' \in (2\pi/L)\mathbb{Z}^3$)

$$4\pi Y^*_{\ell',m'}(\hat{k}'^*) i\mathcal{M}_{k',\ell',m';k,\ell,m} Y_{\ell,m}(\hat{k}^*) \equiv \delta_{kk'} i\mathcal{M}(E - \omega_k, \vec{P} - \vec{k}, \hat{k}'^*, \hat{k}^*)$$

Understanding matrix structure is crucial to understanding result.

Detailed analysis: No switches

$$C_L^{(1)} - C_\infty^{(1)} = (\sigma + A'^{(1,u)}) \frac{iF}{2\omega L^3} \frac{1}{1 - i\mathcal{M} iF} (\sigma^\dagger + A^{(1,u)}) - \sigma \frac{iF}{3\omega L^3} \sigma^\dagger .$$

Other matrix entering the result is

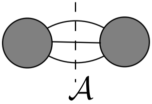
$$iF_{k',k} \equiv \delta_{k',k} \frac{1}{2} \left[\frac{1}{L^3} \sum_{\vec a} - \int_{\vec a} \right] \frac{i4\pi Y(\hat a^*) Y^*(\hat a^*)}{2\omega_a 2\omega_{P-k-a} (E - \omega_k - \omega_a - \omega_{P-k-a} + i\epsilon)} .$$

Now introduce shorthand

$$[\mathcal{A}] \equiv \frac{iF}{2\omega L^3} \frac{1}{1 - i\mathcal{M} iF} .$$

Think of this as a new kind of cut

$$(\sigma + A'^{(1,u)}) [\mathcal{A}] (\sigma^\dagger + A^{(1,u)}) \equiv$$



Bottom particle free

Compare this to theory with bottom particle non-interacting

Then $C_L^{(1)} - C_\infty^{(1)}$ is the full correlator, but with second term omitted.

Correlator diverges whenever $\det[1 - i\mathcal{M}iF] = 0$. Or when

$$\det_{ang\ \ mom}[1 - i\mathcal{M}iF]_{k=(0,0,0)} \times \det_{ang\ \ mom}[1 - i\mathcal{M}iF]_{(2\pi/L,0,0)} \times \cdots = 0\,,$$

for all $\vec{k} \in (2\pi/L)\mathbb{Z}^3$. **Just as expected!**

Detailed analysis: No switches

Returning to our identical-particle theory

$$C_L^{(1)} - C_\infty^{(1)} = (\sigma + A'^{(1,u)})\big[\mathcal{A}\big](\sigma^\dagger + A^{(1,u)}) - \sigma \frac{iF}{3\omega L^3}\sigma^\dagger \,,$$

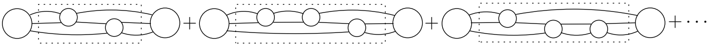
we stress that the sum over $\vec{k}$ includes terms for which

$$E_2^{*2} \equiv (E - \omega_k)^2 - (\vec{P} - \vec{k})^2 < 4m^2 \,.$$

Values below but close to $E_2^* = 2m$ must be included.[11]
However, when $E_2^* \lesssim m$ then iF is suppressed so that contributions can be neglected.

[11] Polejaeva, K. & Rusetsky, A. *Eur.Phys.J.* **A48,** 67 (2012).

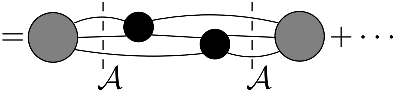
Detailed analysis: One switch

$C_L^{(2)} \equiv$ 

In this case we have two "spectator momenta"
[two momenta that do not appear in two-particle loops].

Evaluating contour integrals and separating out infinite-volume gives

$$C_L^{(2)} - C_\infty^{(2)} = (\sigma + A'^{(1,u)}) \big[\mathcal{A}\big] \, i\mathcal{M}_{3\to 3}^{(2,\text{unsym.})} \big[\mathcal{A}\big] (\sigma^\dagger + A^{(1,u)}) + \cdots \,,$$



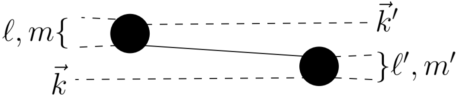
where the ellipsis represents terms that modify the endcaps of $C_L^{(1)} - C_\infty^{(1)}$.

Detailed analysis: One switch

$$C_L^{(2)} - C_\infty^{(2)} = (\sigma + A'^{(1,u)})\left[\mathcal{A}\right] i\mathcal{M}_{3\to3}^{(2,\text{unsym.})}\left[\mathcal{A}\right](\sigma^\dagger + A^{(1,u)}) + \cdots$$

Here

$$i\mathcal{M}^{(2,\text{unsym.})}_{3\to3;k',\ell',m';k,\ell,m} \equiv$$



Observe that certain $\vec{k}$ and $\vec{k}'$ put intermediate propagator on-shell.

Implies that $i\mathcal{M}_{3\to3}$ has physical pole above threshold

Detailed analysis: One switch

$$C_L^{(2)} - C_\infty^{(2)} = (\sigma + A'^{(1,u)}) \big[\mathcal{A}\big] \, i\mathcal{M}_{3\to3}^{(2,\text{unsym.})} \big[\mathcal{A}\big] (\sigma^\dagger + A^{(1,u)}) + \cdots$$

Here

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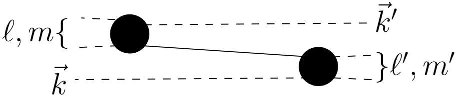

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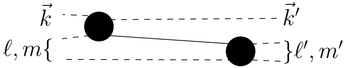
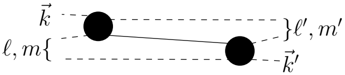
To reach a physical result this diagram must combine with





But $Y_{\ell,m}$ decomposition fails here. Amounts to

$$\frac{1}{\cos\theta} \approx A + B\cos\theta \qquad \text{garbage!}$$

Divergence free three-to-three amplitude

Resolution is to introduce

$$i\mathcal{M}^{(2,\mathrm{unsym.})}_{df,3\to3} \equiv i\mathcal{M}^{(2,\mathrm{unsym.})}_{3\to3} - i\mathcal{M}\frac{i}{2\omega(E-3\omega)}i\mathcal{M}\,.$$

$i\mathcal{M}^{(2,\mathrm{unsym.})}_{df,3\to3}$ is finite

- Decompose in $Y_{\ell,m}$ (even after symmetrization)
- For low energies, truncation of decomposition is good approximation

The approach of separating out singularities like this was first suggested over 40 years ago.[12]

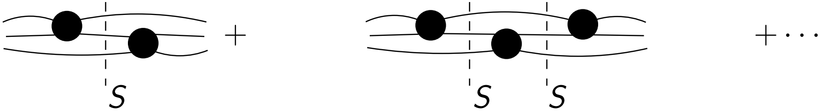
Makes sense to recover singularity-free quantity from finite-volume spectrum! Then add singular terms back.

[12] Rubin *et al.* *PR 146-4* (1966).

Divergence free three-to-three amplitude

Define

$$i\mathcal{M}_{df,3\to 3} \equiv i\mathcal{M}_{3\to 3}$$
$$-\left[i\mathcal{M}\frac{i}{2\omega(E-3\omega)}i\mathcal{M} + \int i\mathcal{M}\frac{i}{2\omega(E-3\omega)}\frac{1}{2\omega}i\mathcal{M}\frac{i}{2\omega(E-3\omega)}i\mathcal{M} + \cdots \right],$$



where $\cdots$ indicates infinite series of with additional $i\mathcal{M}$.

This definition of $i\mathcal{M}_{df,3\to 3}$ arises naturally in our investigation of the finite-volume theory.

It is the observable to be extracted from the spectrum.

We stress that, once extracted, it can be combined with the $i\mathcal{M}$ dependent terms to recover the usual three-to-three scattering amplitude $i\mathcal{M}_{3\to 3}$.

The **relativistic**, **model-independent** relation between finite-volume spectrum and scattering amplitudes

$$\det[F_{\rm three}^{-1} + i\mathcal{M}_{df,3\to3}] = 0\,,$$

where

$$F_{\rm three} \equiv \frac{1}{2\omega L^3}\left[(2/3)iF - \frac{1}{[iF]^{-1} - [1 - i\mathcal{M}iG]^{-1}\, i\mathcal{M}}\right]$$

$$iG_{k,p} = \frac{1}{2\omega_p L^3}\frac{i4\pi Y(\hat{p}^*)Y^*(\hat{k}^*)}{2\omega_{P-p-k}(E-\omega_p-\omega_k-\omega_{P-p-k})}\,,$$

$$iF_{k,k'} = \delta_{k,k'}\frac{1}{2}\left[\frac{1}{L^3}\sum_{\vec{a}} - \int_{\vec{a}}\right]\frac{i4\pi Y(\hat{a}^*)Y^*(\hat{a}^*)}{2\omega_a 2\omega_{P-k-a}(E-\omega_k-\omega_a-\omega_{P-k-a}+i\epsilon)}\,.$$

Here harmonic indices have been left implicit.

This is the main result of the talk.

Conclusion

We have given a relativistic, model-independent relation between three-particle S-matrix elements and the finite-volume spectrum.

The next step is to map out the spectrum in the full range $3m < E^* < 5m$ for realistic scattering amplitude inputs.

Also interesting would be an attempt to weakly perturb our relation, in order to get a generalization of the Lellouch-Lüscher relation between finite- and infinite-volume weak decay matrix elements.

Supported by the Fermilab Fellowship in Theoretical Physics

Backup Slides

Suggestive comment concerning $i\mathcal{M}_{df,3\to3}$

From *Three particle scattering rates and singularities of the T-matrix* by Potapov and Taylor, PRA 16-6, 1977

It is well known, of course, that the three-particle T matrix has singularities in its physical region. [For example there is] a doublescattering singularity; these come about because the three particles can undergo two. separate collisions in pairs.

Suggestive comment concerning $i\mathcal{M}_{df,3\to3}$

From *Dispersion Relations for Three-Particle Scattering Amplitudes* by Rubin, et.al, PR 146-4, 1966

[Physical singularities] will not, of course, prevent us from projecting states of definite total angular momentum... On the other hand, because of the rescattering singularities, the partial-wave expansion is not expected to converge uniformly. It is interesting to realize that in this respect the case of two-body scattering via short-range forces is unique...If we denote by T_R the contribution of the rescattering singularities to the amplitude T, so that $T - T_R$ is free of singularities in the physical region, we can expand $T - T_R$ in partial waves and write

$$T = T_R + \sum_J (T - T_R)_J \,. \tag{1}$$

We can then approximate T by truncating the series for $T - T_R$. **This approximation scheme is feasible because, as we have seen, T_R is given in terms of the two-body amplitudes.**

Summing maximally singular terms

Let us focus on one of the pieces that appears inside

$$F_{\rm three} \equiv \frac{1}{2\omega L^3}\left[(2/3)iF - \frac{1}{[iF]^{-1} - [1 - i\mathcal{M}iG]^{-1}\, i\mathcal{M}}\right]$$

namely

$$X \equiv \frac{1}{1 - i\mathcal{M}iG} i\mathcal{M}\,,$$

where

$$iG_{k,p} = \frac{1}{2\omega_p L^3}\frac{i4\pi Y(\hat{p}^*)Y(\hat{k}^*)}{2\omega_{P-p-k}(E - \omega_p - \omega_k - \omega_{P-p-k})}\,.$$

Summing maximally singular terms

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$$F_{\rm three} \equiv \frac{1}{2\omega L^3}\left[(2/3)iF - \frac{1}{[iF]^{-1} - [1 - i\mathcal{M}iG]^{-1}\, i\mathcal{M}}\right]$$

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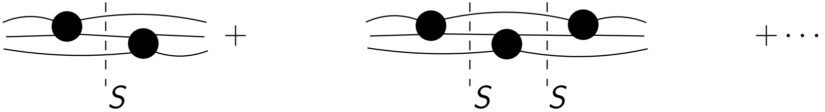
X is sum of all maximally singular diagrams.
Cannot decompose singularities in harmonics, so use a matrix in momentum space.
Legitimate to truncate the matrix with function which smoothly goes to zero below threshold. Must put the same truncation in $i\mathcal{M}_{df,3\to3}$.

Divergence free three-to-three amplitude

Define

$$i\mathcal{M}_{df,3\to3} \equiv i\mathcal{M}_{3\to3}$$
$$-\left[i\mathcal{M}\frac{i}{2\omega(E-3\omega)}i\mathcal{M} + \int i\mathcal{M}\frac{i}{2\omega(E-3\omega)}\frac{1}{2\omega}i\mathcal{M}\frac{i}{2\omega(E-3\omega)}i\mathcal{M} + \cdots\right].$$



For degenerate particles, subtracting the first two terms is sufficient to render $i\mathcal{M}_{df,3\to3}$ finite.

For arbitrary particle masses, the infinite set must be subtracted to get a finite quantity.