Relativistic, model-independent 3-particle quantization condition: (2) Threshold expansion

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Based on unpublished work with Max Hansen

Relativistic, model-independent 3-particle quantization condition: (2) Explication & utility

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Outline

* New features compared to 2-particle case
* How to truncate and make practical
* Important check: threshold expansion compared to results from NR EFT
* Closing comments

New features compared to 2-particle case

3-particle quant. condition

- For given **P**, adjust total energy E until:

$$\det[F_{\rm three}^{-1} + i\mathcal{M}_{df,3\to3}] = 0$$

$$F_{\rm three} \equiv \frac{1}{2\omega L^3}\left[(2/3)iF - \frac{1}{[iF]^{-1} - [1 - i\mathcal{M}iG]^{-1}\, i\mathcal{M}}\right]$$

- Entries are infinite dim. matrices with "indices"

["spectator" momentum: $\mathbf{k}=2\pi\mathbf{n}/L$] x [2-particle CM angular momentum: l,m]

- $\mathcal{M}=\mathcal{M}_{2\to2}$ and $\mathcal{M}_{df,3\to3}$ are on-shell amplitudes (analytically continued if below threshold)

- F and G are kinematical, finite-volume factors

2-particle quant. condition

- For given **P**, adjust total energy E until:

$$\det\left(F^{-1} + i\mathcal{M}\right) = 0$$

Form of result given by [Kim, Sachrajda & SRS] ; equivalent to earlier results of [Luscher; Rummukainen & Gottlieb]

- Entries are infinite dim. matrices with "indices"

[2-particle CM angular momentum: *l,m*]

- $\mathcal{M}=\mathcal{M}_{2\to2}$ is on-shell amplitude (analytically continued if below threshold)
- *F* is kinematical, finite-volume factor

Comparison

- Overall forms are (superficially) similar
- *F* is essentially the same finite-volume kinematical factor in both cases (with trivial spectator momentum dependence in the 3-particle case)
- Differences for 3 particles:
  - Enlarged matrix index space
  - Need to introduce divergence-free 3→3 amplitude [see Max's talk]
  - Presence of "switch factor" G
  - Necessarily includes subthreshold 2→2 scattering [see Max's talk]

Enlarged index space

[2-particle CM angular momentum: *l*,*m*]

↓

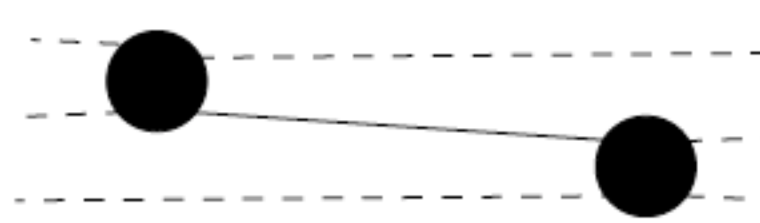
[“spectator” momentum: $\mathbf{k}=2\pi\mathbf{n}/L$] x [2-particle CM angular momentum: *l*,*m*]

- Reflects larger on-shell phase space
- Finite volume restricts index space for 3 particles
- This restriction to quantized **k** essential to obtain the correct result if third particle is non-interacting [see Max’s talk]

Presence of switch-factor G

- Enters because of subtraction of divergent part of 3→3 amplitude

$$i\mathcal{M}^{(2,\,\mathrm{unsym})}_{3\to 3;k',\ell',m';k,\ell,m} \equiv$$



Singular!

$$i\mathcal{M}^{(2,\,\mathrm{unsym})}_{df,3\to 3} \equiv i\mathcal{M}^{(2,\,\mathrm{unsym})}_{3\to 3} - i\mathcal{M}\underline{\frac{i}{2\omega(E-3\omega)}}\,i\mathcal{M}$$

Non-singular!

Proportional to G

- Obtain G when add back in subtracted part

$$+i\mathcal{M}\; iG\; i\mathcal{M}$$

- Arises when switch from 2→2 scatterings of one pair to a different pair
    - Switches which particle is spectator in coordinate system

Relation to dimer approach

- Roles of F and G are almost symmetrical

Previous form: $$F_{\rm three} \equiv \frac{iF}{2\omega L^3}\left[(2/3) - \frac{1}{1-\underline{[1-i\mathcal{M}iG]^{-1}}i\mathcal{M}iF}\right]$$

Sum of subtractions in finite volume
$1 + i\mathcal{M}i\mathcal{G} + (i\mathcal{M}i\mathcal{G})^2 + \ldots$

"Dimer form": $$-F_{\rm three} \equiv \frac{iF}{6\omega L^3} + \frac{1}{2\omega L^3} iF \frac{1}{1 - i\mathcal{D}iG} i\mathcal{D}iF$$

$$i\mathcal{D} \equiv \frac{1}{1-i\mathcal{M}iF} i\mathcal{M}$$

Finite-volume scattering amplitude a.k.a. dimer propagator

$$= i\mathcal{M} + i\mathcal{M}i\mathcal{F}i\mathcal{M} + i\mathcal{M}i\mathcal{F}i\mathcal{M}i\mathcal{F}i\mathcal{M} + \ldots$$

- May allow relation to dimer approach of [Briceno & Davoudi, arXiv:1212.3398] to be worked out

How to truncate & make practical

Truncation in 2 particle case

$$\det\left(F^{-1} + i\mathcal{M}\right) = 0$$

- Entries are infinite dim. matrices with "indices" [CM angular momentum: l,m]
- If $\mathcal{M}$ (which is diagonal in l,m) vanishes for $l > l_{max}$ then can show that need only keep $l \leq l_{max}$ in F (which is not diagonal) and so have finite matrix condition which can be inverted to find $\mathcal{M}(E)$ from energy levels

Truncation in 3 particle case

$$\det[F_{\rm three}^{-1} + i\mathcal{M}_{df,3\to3}] = 0$$

$$F_{\rm three} \equiv \frac{1}{2\omega L^3}\left[(2/3)iF - \frac{1}{[iF]^{-1} - [1 - i\mathcal{M}iG]^{-1}\,i\mathcal{M}}\right]$$

$$iF_{k,k'} = \delta_{k,k'}\frac{1}{2}\left[\frac{1}{L^3}\sum_{\vec a} - \int_{\vec a}\right]\frac{i\,4\pi Y(\hat{a}^*)Y^*(\hat{a}^*)}{2\omega_a 2\omega_{P-k-a}(E-\omega_k-\omega_a-\omega_{P-k-a}+i\epsilon)}$$

- Matrix "indices" are [Spectator mom. **k**=2π**n**/L] x [CM angular mom. *l*,*m*]
- For fixed E & **P**, as |**k**| increases, remaining two-particle system drops below threshold, so F becomes exponentially suppressed (since sum and integral do not hit pole)
- Thus **k** index is naturally truncated (with, say, N terms required)
- *l* is truncated if both $\mathcal{M}$ and $\mathcal{M}_{df,3\to3}$ vanish for $l > l_{max}$

Truncation in 3 particle case

$$\det[F_{\rm three}^{-1} + i\mathcal{M}_{df,3\to3}] = 0$$

- Thus can truncate the quantization condition to that for an $[N(2l_{max}+1)]^2$ block
- Given prior knowledge of $\mathcal{M}$ (from 2 particle analysis) each energy level E_i of the 3 particle system gives information on $\mathcal{M}_{df,3\to3}$ at the corresponding 3-particle CM energy E_i^*
- Could proceed by parameterizing $\mathcal{M}_{df,3\to3}$ by a number of parameters (e.g. one!), in which case one would need at least that many levels at given energy to determine parameters
- Given $\mathcal{M}$ and $\mathcal{M}_{df,3\to3}$ one can reconstruct $\mathcal{M}_{3\to3}$

$$i\mathcal{M}_{df,3\to3} \equiv i\mathcal{M}_{3\to3} - \left[i\mathcal{M}\frac{i}{2\omega(E-3\omega)}i\mathcal{M} + \int i\mathcal{M}\frac{i}{2\omega(E-3\omega)}\frac{1}{2\omega}i\mathcal{M}\frac{i}{2\omega(E-3\omega)}i\mathcal{M} + \cdots\right]$$

Important check: threshold expansion

Threshold expansion

- Given complexity of derivation & new features of result, it is clearly important to check it to the extent possible
- Can do so for **P**=0 and near threshold: $E=3m+\Delta E$, with $\Delta E \sim 1/L^3+\ldots$
- In other words, study energy shift of three particles (almost) at rest
- Dominant effects (L^{-3}, L^{-4}, L^{-5}) involve 2-particle interactions, but 3-particle interaction enters at L^{-6}
- For large L, particles are non-relativistic ($\Delta E \ll m$) and can use NREFT methods
- This has been done previously by [Beane, Detmold & Savage, 0707.1670] and [Tan, 0709.2530]

NR EFT results

[Beane, Detmold & Savage, 0707.1670]

2 particles

$$E_0(2,L) = \frac{4\pi a}{ML^3}\left\{1 - \left(\frac{a}{\pi L}\right)\mathcal{I} + \left(\frac{a}{\pi L}\right)^2[\mathcal{I}^2 - \mathcal{J}] + \left(\frac{a}{\pi L}\right)^3[-\mathcal{I}^3 + 3\mathcal{I}\mathcal{J} - \mathcal{K}]\right\} + \frac{8\pi^2 a^3}{ML^6} r + \mathcal{O}(L^{-7}), \tag{11}$$

- 2-particle result agrees with [Luscher]
- Scattering length *a* is in nuclear physics convention
- *r* is effective range
- $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ are zeta-functions

3 particles

$$E_0(3,L) = \frac{12\pi a}{ML^3}\left\{1 - \left(\frac{a}{\pi L}\right)\mathcal{I} + \left(\frac{a}{\pi L}\right)^2[\mathcal{I}^2 + \mathcal{J}] + \left(\frac{a}{\pi L}\right)^3[-\mathcal{I}^3 + \mathcal{I}\mathcal{J} + 15\mathcal{K} - 8(2\mathcal{Q} + \mathcal{R})]\right\} + \frac{64\pi a^4}{ML^6}(3\sqrt{3} - 4\pi)\log(\mu L) + \frac{24\pi^2 a^3}{ML^6} r + \frac{1}{L^6}\eta_3(\mu) + \mathcal{O}(L^{-7}), \tag{12}$$

- 3 particle result through L^{-4} is 3x(2-particle result) from number of pairs
- Not true at L^{-5},L^{-6} where additional finite-volume functions $\mathcal{Q}$, $\mathcal{R}$ enter
- $\eta_3(\mu)$ is 3-particle contact potential, which requires renormalization

NR EFT results

[Beane, Detmold & Savage, 0707.1670]

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Tan has 36 instead of 24, but a different definition of η_3

NR EFT results

[Beane, Detmold & Savage, 0707.1670]

$$E_0(3,L) = \frac{12\pi a}{ML^3}\left\{1 - \left(\frac{a}{\pi L}\right)\mathcal{I} + \left(\frac{a}{\pi L}\right)^2[\mathcal{I}^2 + \mathcal{J}] + \left(\frac{a}{\pi L}\right)^3[-\mathcal{I}^3 + \mathcal{I}\mathcal{J} + 15\mathcal{K} - 8(2\mathcal{Q} + \mathcal{R})]\right\} + \frac{64\pi a^4}{ML^6}(3\sqrt{3} - 4\pi)\log(\mu L) + \frac{24\pi^2 a^3}{ML^6} r + \frac{1}{L^6}\eta_3(\mu) + \mathcal{O}(L^{-7}), \qquad (12)$$

zeta-functions

$$\mathcal{I} = Z_{00}(1,0) = \sum_{\vec{n}\neq 0}^{\Lambda} \frac{1}{\vec{n}^2} - 4\pi\Lambda\,, \quad \mathcal{J} = Z_{00}(2,0) = \sum_{\vec{n}\neq 0} \frac{1}{(\vec{n}^2)^2}\,, \quad \mathcal{K} = Z_{00}(3,0) = \sum_{\vec{n}\neq 0} \frac{1}{(\vec{n}^2)^3}$$

additional finite-volume quantities

dim. reg.

$$\hat{\mathcal{Q}} = \sum_{\mathbf{i}\neq \mathbf{0}} \sum_{\mathbf{j}\neq \mathbf{0}} \frac{1}{|\mathbf{i}|^2|\mathbf{j}|^2(|\mathbf{i}|^2 + |\mathbf{j}|^2 + |\mathbf{i}+\mathbf{j}|^2)} \rightarrow \mathcal{Q} + \frac{4}{3}\pi^4\log(\mu L) - \frac{2\pi^4}{3(d-3)}$$

$$\hat{\mathcal{R}} = \sum_{\mathbf{j}\neq \mathbf{0}} \frac{1}{|\mathbf{j}|^4}\left[\sum_{\mathbf{i}} \frac{1}{|\mathbf{i}|^2 + |\mathbf{j}|^2 + |\mathbf{i}+\mathbf{j}|^2} - \frac{1}{2}\int d^d\mathbf{i}\,\frac{1}{|\mathbf{i}|^2}\right] \rightarrow \mathcal{R} - 2\sqrt{3}\pi^3\log(\mu L) + \frac{\sqrt{3}\pi^3}{d-3}$$

Expanding our result

$$\det[1 + F_{\rm three} i\mathcal{M}_{\rm df,3\to3}] = 0$$

$$F_{\rm three} \quad = -\frac{iF}{2\omega L^3}\left[\frac{1}{3} + \frac{1}{\mathcal{M}^{-1}+F+G}F\right]$$

- Take $\mathcal{M}$ to be purely s-wave and $\mathcal{M}_{df,3\to3}$ to be a constant (i.e. l_{max}=0)
- F_{three}, F, G are then truncated to matrices in spectator momentum space
- Can show that $[F_{three}]_{\mathbf{0},\mathbf{0}}$ dominates other matrix elements by at least L^2, so quantization condition becomes

$$[F_{\rm three}]_{\mathbf{0},\mathbf{0}} = -i\mathcal{M}_{\rm df,3\to3}$$

- F is $O(L^0)$, so to cancel the $1/L^3$ in F_{three} need $[\mathcal{M}^{-1}+F+G]^{-1}\sim L^3$
- Roughly speaking this requires the cancellation of L^0, L^{-1} & L^{-2} terms in $[\mathcal{M}^{-1}+F+G]$, which requires tuning E and determines the L^{-3}, L^{-4} & L^{-5} in ΔE
- The L^{-6} term in ΔE is then determined by the quantization condition

Examples of expansions

$$(F_R)_{k,k} = \frac{1}{2}\left[\frac{1}{L^3}\sum_{\vec a} - \mathcal{P}\int_{\vec a}\right]\frac{1}{2\omega_a 2\omega_{-k-a}(E-\omega_k-\omega_a-\omega_{-k-a})}$$

Real part (Imag. part cancels with $\mathcal{M}$)

UV finite quantity (though, strictly speaking, need to regulate the sum & integral separately before taking difference---which we do)

- NR expansion:

$$[F_R]_{0,0} = \frac{1}{8\omega_q}\left[\frac{1}{q^2L^3} - \frac{\mathcal{I}}{4\pi^2 L} - \frac{q^2L^3\mathcal{J}}{(4\pi^2L)^2} - \frac{(q^2L^3)^2\mathcal{K}}{(4\pi^2L)^3} + \dots\right]$$

q is momentum of each of non-spectator pair

$$[F_R]_{k,k} = -\frac{1}{16\pi^2 mL}\sum_{\vec n_a}^{\text{reg}}\frac{1}{\vec n_k^2 + \vec n_a^2 + (\vec n_k + \vec n_a)^2}$$

contributes to $\mathcal{R}$

$$G_{k,p} = \frac{1}{2\omega_p L^3 2\omega_{p+k}(E-\omega_p-\omega_k-\omega_{p+k})}$$

- NR expansion:

$$G_{0,0} = \frac{1}{4m^2\Delta E L^3}\,,$$

~ L^0

$$G_{0,k} = -\frac{1}{16\pi^2 mL}\frac{1}{\vec n}^2\left[1 + \frac{m\Delta E L^3}{4\pi^2 L\vec n^2} + \dots\right],$$

~ L^{-1}

$$G_{k,p} = -\frac{1}{16\pi^2 mL}\frac{2}{\vec n_k^2 + \vec n_p^2 + (\vec n_k + \vec n_p)^2}$$

~ L^{-1}

Need spectator-momentum matrix structure of F & G to evaluate $[F_{\text{three}}]_{0,0}$

Our threshold expansion

$$E = 3m + \frac{12\pi a}{mL^3}\left[1 - \left(\frac{a}{\pi L}\right)\mathcal{I} + \left(\frac{a}{\pi L}\right)^2 [\mathcal{I}^2 + \mathcal{J}] + \left(\frac{a}{\pi L}\right)^3 [-\mathcal{I}^3 + \mathcal{I}\mathcal{J} + 15\mathcal{K} - 16\mathcal{Q} - 8\mathcal{R}]\right]$$

$$+ \frac{72a^3\pi^2 r}{mL^6} + \frac{36a^2\pi^2}{m^3L^6} + \frac{\tilde{a}_6}{L^6} + \mathcal{O}(1/L^7)$$

Our threshold expansion

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agrees with [Beane et al.] and [Tan] modulo definitions of $\mathcal{Q}$ & $\mathcal{R}$

$$+ \frac{72a^3\pi^2 r}{mL^6} + \frac{36a^2\pi^2}{m^3L^6} + \frac{\tilde{a}_6}{L^6} + \mathcal{O}(1/L^7)$$

$$\mathcal{Q} \equiv -2048 L^3 m^3 \pi^6 \sum_{\vec{k}\neq 0, \vec{p}\neq 0} G_{0,k} G_{k,p} G_{p,0}$$

$$= \underbrace{\sum_{\vec{n}_k\neq 0, \vec{n}_p\neq 0} \frac{1}{\vec{n}_k^2 \vec{n}_p^2 \left[\vec{n}_k^2 + \vec{n}_p^2 + (\vec{n}_k + \vec{n}_p)^2\right]}}_{\hat{\mathcal{Q}} \text{ of [Beane et al.]}} + \mathcal{O}(1/L)$$

UV convergent!

Log divergent after NR expansion, so requires regulation as in [Beane et al.]

Similar situation for $\mathcal{R}$

Our threshold expansion

$$E = 3m + \frac{12\pi a}{mL^3}\left[1 - \left(\frac{a}{\pi L}\right)\mathcal{I} + \left(\frac{a}{\pi L}\right)^2[\mathcal{I}^2 + \mathcal{J}] + \left(\frac{a}{\pi L}\right)^3[-\mathcal{I}^3 + \mathcal{I}\mathcal{J} + 15\mathcal{K} - 16\mathcal{Q} - 8\mathcal{R}]\right]$$

G required to get correct factors
in these terms

$$+ \frac{72a^3\pi^2 r}{mL^6} + \frac{36a^2\pi^2}{m^3L^6} + \frac{\tilde{a}_6}{L^6} + \mathcal{O}(1/L^7)$$

Our threshold expansion

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$$+ \frac{72a^3\pi^2 r}{mL^6} + \frac{36a^2\pi^2}{m^3L^6} + \frac{\tilde{a}_6}{L^6} + \mathcal{O}(1/L^7)$$

[Beane et al] have 24, [Tan] has 36, we have 72

[Beane et al.] and [Tan] do not have this term

$$\tilde{a}_6 = -\frac{\mathcal{M}_{df,3\to 3,\Lambda,00}(E = 3m, \vec{P} = 0)}{48m^3}$$

Physical, finite quantity, with no μ dependence
Directly related to scattering amplitudes
In [Beane et al.] this term is

$$\frac{1}{L^6}\eta_3(\mu)$$

Interpretation of "differences"

$$+\frac{64\pi a^4}{ML^6}(3\sqrt{3}-4\pi)\log(\mu L)+\frac{24\pi^2 a^3}{ML^6}r+\frac{1}{L^6}\eta_3(\mu) \qquad \textbf{vs.} \qquad +\frac{72a^3\pi^2 r}{mL^6}+\frac{36a^2\pi^2}{m^3L^6}+\frac{\tilde{a}_6}{L^6}$$

[Beane et al.] [Hansen & SRS]

- We do not know *a priori* the relation between $\mathcal{M}_{\mathrm{df},3\to3}$ and η_3
- $\mathcal{M}_{\mathrm{df},3\to3}$ is physical, while η_3 is a short-distance parameter, indirectly related to physical quantities
- We can view this comparison as providing the relation between $\mathcal{M}_{\mathrm{df},3\to3}$ and η_3 if we equate the two expressions
- As far as we can see, there is nothing forbidding this relation to include the finite a^2 and a^3r terms
  - Indeed, a similar finite difference is required to match [Beane et al.] with [Tan]
- It would clearly be good to check this purported relation in another context

Closing comments

- Having a formalism is only the first step, especially as it is complicated
- Threshold expansion check gives us confidence in the expression & shows how it can be used in practice
- We plan further studies of its practical utility using simple forms for the scattering amplitudes
- We also plan to compare in more detail with [Polejaeva & Rusetsky], [Briceno & Davoudi] & [HAL QCD] (see following talk by Sinya Aoki)

Closing comments

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Thank you!
Any questions?

Backup Slides

Comment on derivation

- Our derivation is rather involved (40+ pages)
- Including 3→3 Bethe-Salpeter kernel is easy
- Difficulty comes from multiple 2→2 interactions: analyze according to number of "switches"
- Leads to unsymmetrized, divergent contributions to 3→3 amplitudes
- Symmetrization occurs only after combining terms with different numbers of switches---i.e. all orders summation
- Removing divergences leads to switch factors *G*
- There is probably a better approach....

Comment on derivation