# Two-flavor, scalar QED at finite density on the lattice

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# The sign problem of QCD

$$\langle A \rangle_\rho = \frac{\int D\sigma \; A[\sigma] \; \rho[\sigma]}{\int D\sigma \; \rho[\sigma]} \quad , \quad \rho \;=\; \det(M(\mu,\sigma))^2 \exp(-S[\sigma]) \quad .$$

- $\mu \; > \; 0$ favors quarks over antiquarks
- Charge conjugation corresponds to complex conjugation $\rightarrow$ $\det(M(\mu,\sigma)) \; \neq \; \det^*(M(\mu,\sigma))$

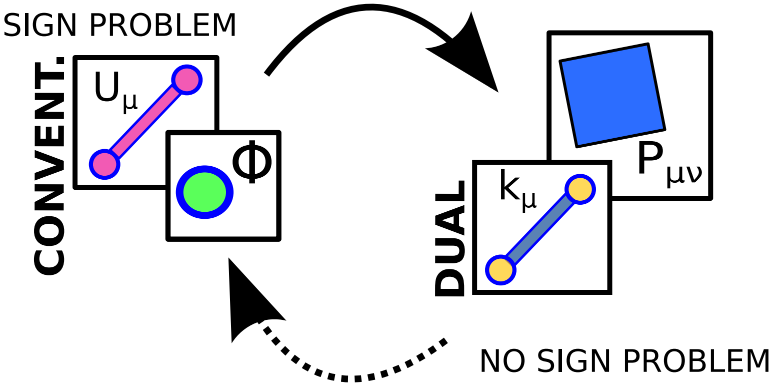
**$\rho$ can not be interpreted as probability-weight anymore. :-(**

## Some possible ways out

- Expansion in chemical potential around $\mu = 0$
- Reweighting
- **Rewriting to new degrees of freedom**

# What we are doing: Rewriting the action

We exactly map the system to a new equivalent representation where the sign problem is gone.



# Lattice action

$U_{\vec{n},\mu} \in U(1),\ \phi_{\vec{n}} \in \mathbb{C}$

$$
\begin{aligned}
S_G &= -\beta \sum_{\vec{n}} \sum_{\mu<\nu} Re\ U_{\vec{n},\mu}\, U_{\vec{n}+\hat{\mu},\nu}\, U^{\star}_{\vec{n}+\hat{\nu},\mu}\, U^{\star}_{\vec{n},\nu} \\
S_H &= \sum_{\vec{n}} \left[ \kappa^1\, |\phi^1_{\vec{n}}|^2 + \lambda^1\, |\phi^1_{\vec{n}}|^4 + \kappa^2\, |\phi^2_{\vec{n}}|^2 + \lambda^2\, |\phi^2_{\vec{n}}|^4 \right] \\
&- \sum_{\vec{n}} \left[ \sum_{\mu} \left( e^{\delta_{\mu 4}\mu^1} \phi^{1\star}_{\vec{n}}\, U_{\vec{n},\mu}\, \phi^1_{\vec{n}+\widehat{\mu}} + e^{-\delta_{\mu 4}\mu^1} \phi^{1\star}_{\vec{n}}\, U^{\star}_{\vec{n}-\widehat{\mu},\mu}\, \phi^1_{\vec{n}-\widehat{\mu}} \right) \right] \\
&- \sum_{\vec{n}} \left[ \sum_{\mu} \left( e^{\delta_{\mu 4}\mu^2} \phi^{2\star}_{\vec{n}}\, U^{\star}_{\vec{n},\mu}\, \phi^2_{\vec{n}+\widehat{\mu}} + e^{-\delta_{\mu 4}\mu^2} \phi^{2\star}_{\vec{n}}\, U_{\vec{n}-\widehat{\mu},\mu}\, \phi^2_{\vec{n}-\widehat{\mu}} \right) \right]
\end{aligned}
$$

Gauge field $U_{\vec{n},\mu}$ 1st flavor Higgs field $\phi^1_{\vec{n}}$ 2nd flavor Higgs field $\phi^2_{\vec{n}}$

# A sketch of rewriting the Higgs-part of the action

A single nearest neighbor term:

$$Z \;\propto\; e^{\phi_x^\star \, U_{x,\nu} \, \phi_{x+\widehat{\nu}}} \;=\; \sum_{k_{x,\mu}} \frac{1}{(k_{x,\mu})!} \left[ \phi_x^\star \, U_{x,\nu} \, \phi_{x+\widehat{\nu}} \right]^{k_{x,\mu}}$$

**Idea:** Use the expansion indices $k_{x,\mu}$ as new dual degrees-of-freedom and integrate out original d.o.f. $\phi_x$:

$$Z \;=\; \sum_{\{\phi\}} \sum_{\{U\}} e^{-S_G(U)-S_H(U,\phi)} \;=\; \sum_{\{\phi\}} \sum_{\{U\}} e^{-S_G(U)} \sum_{\{k,l\}} F(U,\phi,k,l)$$

$$\;=\; \sum_{\{k,l\}} \sum_{\{U\}} e^{-S_G(U)} \underbrace{\sum_{\{\phi\}} F(U,\phi,k,l)}_{\text{perform this summation } *}$$

* Summing up we obtain constraints for the dual flux variables $k_{x,\mu}$.

# A sketch of rewriting the gauge-part of the action

A single plaquette term from the gauge action:

$$e^{\beta\, U_{x,\mu}\, U_{\vec{n}+\hat{\mu},\nu}\, U^{\star}_{x+\hat{\nu},\mu}\, U^{\star}_{x,\nu}} = \sum_{p_{x,\mu\nu}} \frac{\beta^{\,p_{x,\mu\nu}}}{(p_{x,\mu\nu})!} \left[ U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^{\star}_{x+\hat{\nu},\mu}\, U^{\star}_{x,\nu} \right]^{p_{x,\mu\nu}}$$

**Same idea again:** Use the expansion indices $p_{k,\mu\nu}$ as new dual gauge-degrees-of-freedom and integrate out the original fields $U_{x,\mu}$:

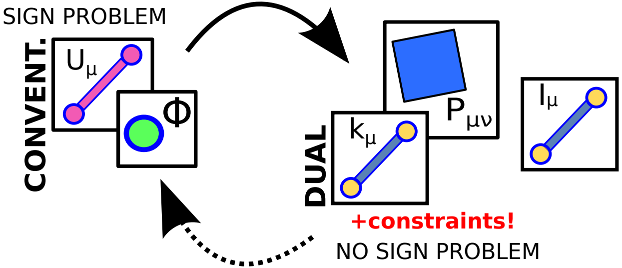
$$\begin{aligned} Z = \sum_{\{k,l\}} \sum_{\{U\}} F(U,k,l) e^{-S_G(U)} &= \sum_{\{k,l\}} \sum_{\{U\}} F(U,k,l) \sum_{\{p\}} B(U,p) \\ &= \sum_{\{k,l\}} \sum_{\{p\}} \underbrace{\sum_{\{U\}} F(U,k,l) B(U,p)}_{\text{perform this summation }*} \end{aligned}$$

* We again obtain a constraint, this time combining dual flux variables $k_{x,\mu}$ and dual plaquette variables $p_{x,\mu\nu}$.

# After rewriting the sign problem is just gone!

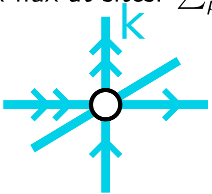
$$Z = \sum_{\{k,l\}} \sum_{\{p\}} FB(k,l,p) = \sum_{\{p,k^1,l^1,k^2,l^2\}} \mathcal{W}(p,k,l)\, \mathcal{C}_B(p,k^1,k^2)\, \mathcal{C}_F(k^i)$$

Even at finite mu the dual weight $\mathcal{W}[\mathbf{p},\mathbf{k},\mathbf{l}]$ is positive and real! :-)



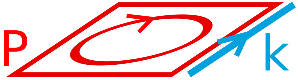
# The dual constraints

- **Constraint I:** Conserved k-flux at sites: $\sum_\mu \left[ k^i_{x,\mu} - k^i_{x-\widehat{\mu},\mu} \right] = 0$

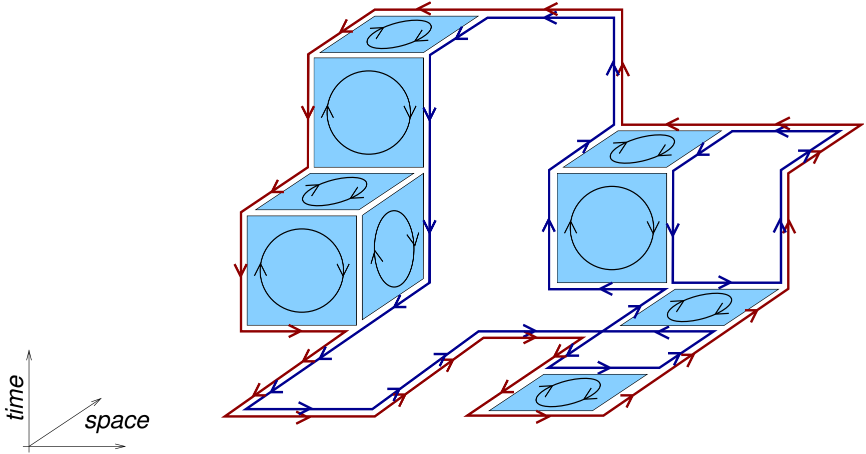



- **Constraint II:** Vanishing combined plaquette- and k-flux along links:

$$\prod_{x,\nu} \left( \sum_{\nu<\alpha} \left[ p_{x,\nu\alpha} - p_{x-\widehat{\alpha},\nu\alpha} \right] - \sum_{\alpha<\nu} \left[ p_{x,\alpha\nu} - p_{x-\widehat{\alpha},\alpha\nu} \right] + k^1_{x,\nu} - k^2_{x,\nu} \right) = 0$$

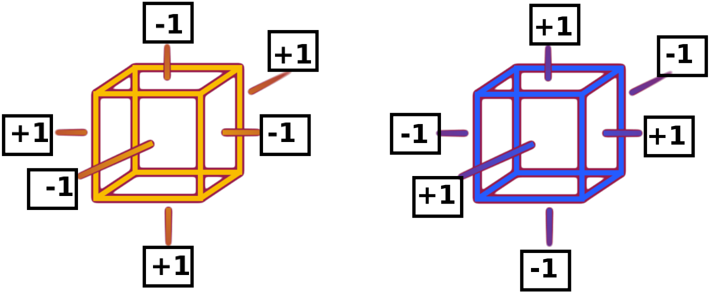



# Example configuration

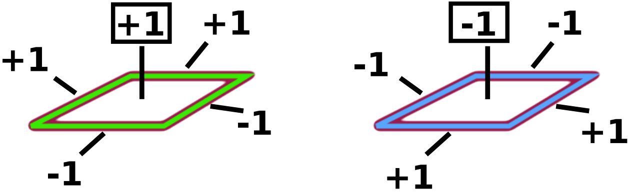


# Update algorithm: Cube and plaquette update

- **Cube update** (updating only gauge degrees-of-freedom)

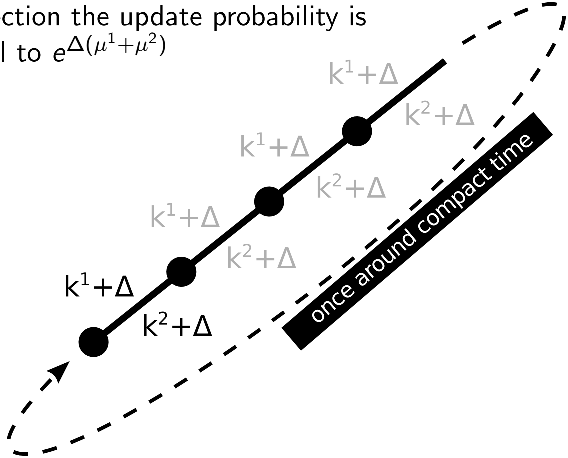



- **Plaquette update**



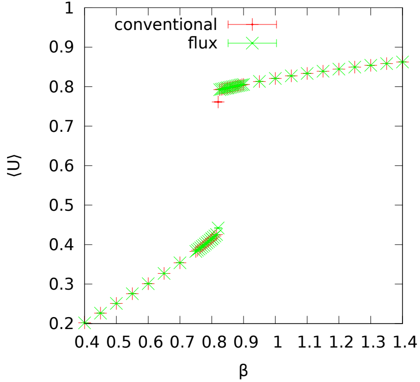
# Update algorithm: Non-trivial loops coupling to $\mu$

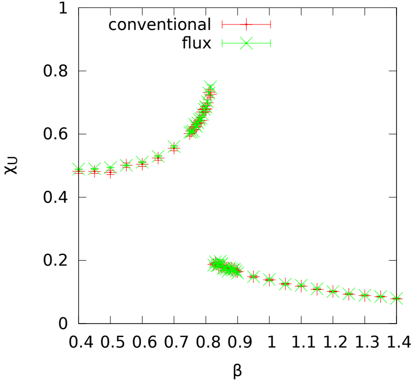
- **Loop update**
  In time-direction the update probability is proportional to $e^{\Delta(\mu^1+\mu^2)}$



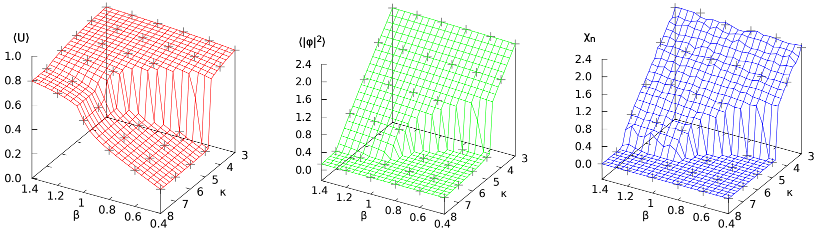
# Comparing to conventional results

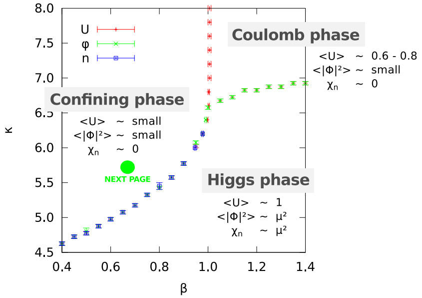
$$\lambda = 1.0 \ \kappa = 5.5$$



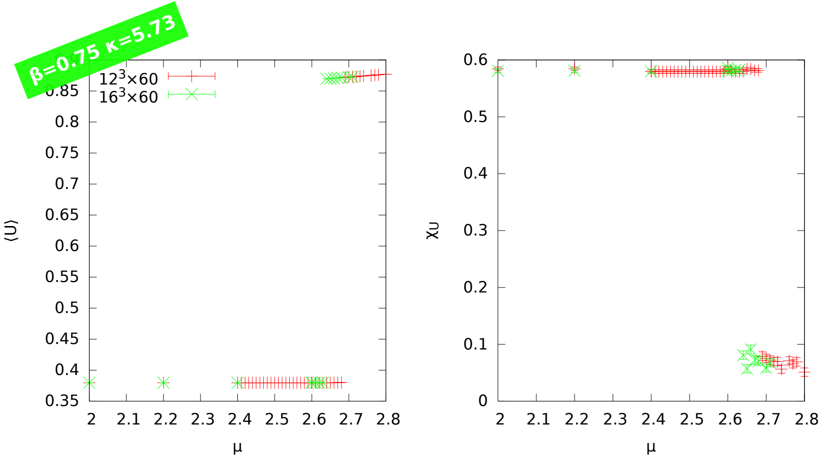



# The $\mu = 0$ phase diagram

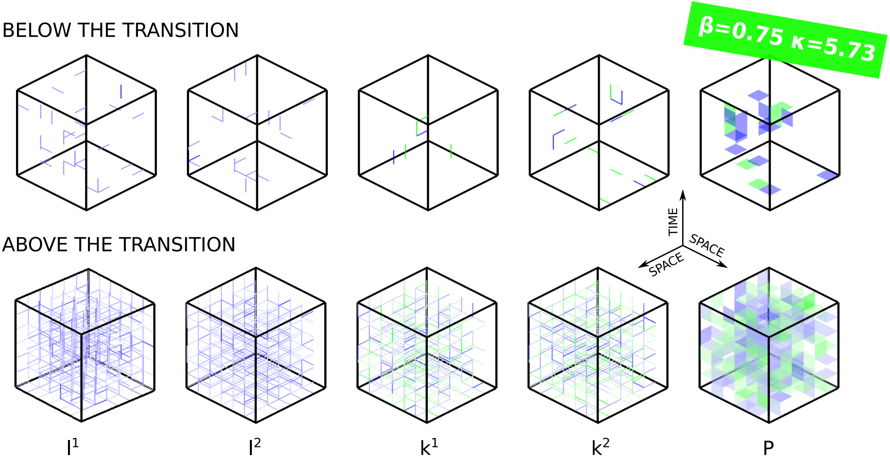


# A silver-blaze type phase transition



# Looking at occupation numbers



# Summary

- Complex action problems can be solved using dual representations of the action.
- The constraints for the dual variables can be handled both with local updates and with new techniques based on the worm algorithm concept.
- Interesting physics at finite $\mu$, e.g. Silver Blaze phenomenon.
- The studied models could serve as testbeds for other finite density lattice approaches.