Lattice QCD with overlap fermions

At the example of non-leptonic kaon decays

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Introduction

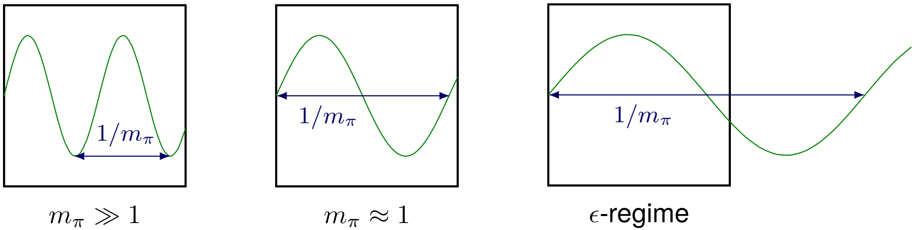
Experiment

Transition amplitudes for $K \to \pi\pi$ of the two isospin final states are significantly enhanced

$$|A_0|/|A_2| \sim 22.1, \qquad \text{"}\Delta I = 1/2\text{-rule"}$$

- Direct computation of the decay amplitudes is a formidable challenge
- Somewhat easier to determine LECs of the effective chiral weak Hamiltonian via lattice simulations
  - Matching does not necessitate physical kinematics nor physical quark masses
  - However, large volumes and sufficient small quark masses are required
- Our goal is to understand the rôle of the charm quark
  - Compute correlation functions outside the GIM limit
  - Because of the increased computational costs use GPU-based simulation programs

QCD in the ϵ-regime



- There are several possibilities for the order of the limits of volume and quark masses
- Approach the chiral limit first by decreasing the quark masses
  - The so-called ϵ-regime of χPT: $m_\pi L \ll 1$, $F_\pi L \gg 1$
  - Possible to work out NLO corrections without introducing additional LECs
- Lattice simulations in ϵ-regime are quite demanding
- Remark: Observables in the ϵ-regime depend on the topology of the gauge field
  - Classify correlation functions by the topological charge ν

Low-energy constants (LECs)

P. Hernandez, M. Laine, C. Pena, E. Torro, J. Wennekers and H. Wittig, JHEP **0805** *(2008) 043*

- Weak Hamiltonian in the SU(4)-symmetric case (GIM limit)

$$H_{\mathrm{w}} = \frac{g_{\mathrm{w}}^2}{4M_{\mathrm{W}}^2} V_{\mathrm{us}}^* V_{\mathrm{ud}} \{ k_1^+ Z_{11}^+ Q_1^+ + k_1^- Z_{11}^- Q_1^- \}$$

- Operators are $Q_1^{\pm} = [O_1]_{\mathrm{rsuv}} \pm [O_1]_{\mathrm{rsvu}}$ and transform in the $\mathbf{84}$ and $\mathbf{20}$ representation
    - The generic four-quark operator is $[O_1]_{\mathrm{rsuv}} = (\bar{\psi}_{\mathrm{r}} \gamma_\mu \psi_{\mathrm{u}})(\bar{\psi}_{\mathrm{s}} \gamma_\mu \psi_{\mathrm{v}})$
- Can be expanded in the non-perturbative regime in terms of the Goldstone boson fields

$$\mathcal{H}_{\mathrm{W}} = \frac{g_{\mathrm{W}}^2}{4M_{\mathrm{W}}^2} V_{\mathrm{us}}^* V_{\mathrm{ud}} \{ g_1^+ \mathcal{Q}_1^+ + g_1^- \mathcal{Q}_1^- \}$$

- At leading-order the transition amplitudes in terms of the LECs are

$$\frac{|A_0|}{|A_2|} = \frac{1}{\sqrt{2}} \left(\frac{1}{2} + \frac{3}{2} \frac{g_1^-}{g_1^+} \right)$$

The overlap operator

- Chiral fermions are required to perform a straightforward matching of lattice QCD and chiral effective theory

$$D_{\mathsf{N}} = \frac{1}{\bar{a}}\big(1 + \gamma_5 \, \mathrm{sign}(Q)\big) \qquad Q = \gamma_5(aD_{\mathsf{W}} - 1 - s), \quad \bar{a} = \frac{a}{1+s}$$

- $\mathrm{sign}(Q)$ has to be evaluated by a polynomial approximation
- In the ϵ-regime, Q^2 can develop exceptionally low eigenvalues
- Final expression for the sign-function

$$\mathrm{sign}(Q) \simeq \mathbb{P}_+ - \mathbb{P}_- + (1 - \mathbb{P}_+ - \mathbb{P}_-) X P_{n,\varepsilon}(X^2), \qquad X = Q/\|Q\|$$

L. Giusti, C. Hoelbling, M. Lüscher and H. Wittig, Comput. Phys. Commun. **153** *(2003) 31*

Correlation functions

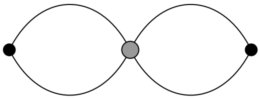
- Two types of three-point correlators can be utilized for the matching between lattice QCD and chiral effective theory
  - We consider a correlation function of two pseudo-scalar densities and a weak operator
  - Also possible is a correlation function of two left-handed currents and a weak operator
- Pseudo-scalar correlators develop poles in $1/(mV)^n$

L. Giusti, P. Hernandez, M. Laine, P. Weisz and H. Wittig, JHEP **0401** *(2004) 003*

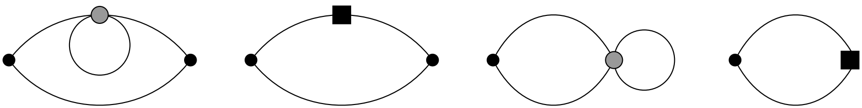
- Some propagators can be substituted by projectors to zero-mode wave function

$$S_{\mathsf{m}}(x,y) = \sum_{v_i \in \mathcal{K}} \frac{v_i(x) v_i^\dagger(y)}{mV} + \dots$$

- Residues are easier to compute numerically since they require fewer quark propagators



“figure-8”-diagram



“eye”-diagram subtraction diagram $K \to$ vac-diagrams

- ■ $Q_2^{\pm} = (m_{\mathsf{u}}^2 - m_{\mathsf{c}}^2)\big\{m_{\mathsf{d}}(\bar{s}P_+d) + m_{\mathsf{s}}(\bar{s}P_-d)\big\}$
- “Figure-8” has been computed on conventional hardware
  - Gives us a good cross-check on our GPU-based implementation

Correlation functions

- "Figure-8" diagram is then given by

$$A_\nu^\pm(x_0 - z_0, y_0 - z_0) = -\lim_{m\to 0}(mV)^2 \int_{\boldsymbol{x}}\int_{\boldsymbol{y}} \langle \partial_{x_0} P(x) O_1^\pm(z) \partial_{y_0} P(y) \rangle_\nu$$

  - The pseudo-scalar density is $P = i\bar{\psi}\gamma_5\psi$
  - Compute derivatives to avoid contaminations from higher order LECs
- It is convenient to normalize the three-point functions with bare two-point functions of the form

$$B_\nu(x_0 - z_0) = \lim_{m\to 0}(mV) \int_{\boldsymbol{x}} \langle \partial_{x_0} P(x) L_0(z) \rangle_\nu$$

  - The left-handed current is $L_0 = \bar{\psi}\gamma_0 P_- \psi$
  - Note: This two-point function can be related to the two-point function of two pseudoscalar densities through the non-singlet axial Ward identity
- The ratios are directly related to the LECs

$$R_\nu^\pm \equiv \frac{A_\nu^\pm(x_0 - z_0, y_0 - z_0)}{B_\nu(x_0 - z_0) B_\nu(y_0 - z_0)} = [g_1^\pm]^{\text{bare}} \left(1 \mp \frac{1}{|\nu|}\right)$$

Zero-mode expansion of the correlators

- Expansion of the three-point function $A_\nu^\pm = \bar{A}_\nu \pm \tilde{A}_\nu$

$$\bar{A}_\nu \equiv \lim_{m\to 0} \frac{1}{L^3} \int_{\boldsymbol{z}} \langle \sum_{i=1}^{|\nu|} v_i^\dagger(z)\gamma_\mu \eta_i(z;x_0) \sum_{j=1}^{|\nu|} v_j^\dagger(z)\gamma_\mu \eta_j(z;y_0)\rangle_\nu$$

$$\tilde{A}_\nu \equiv -\lim_{m\to 0} \frac{1}{L^3} \int_{\boldsymbol{z}} \langle \sum_{i,j=1}^{|\nu|} v_i^\dagger(z)\gamma_\mu \eta_j(z;y_0) v_j^\dagger(z)\gamma_\mu \eta_i(z;x_0)\rangle_\nu$$

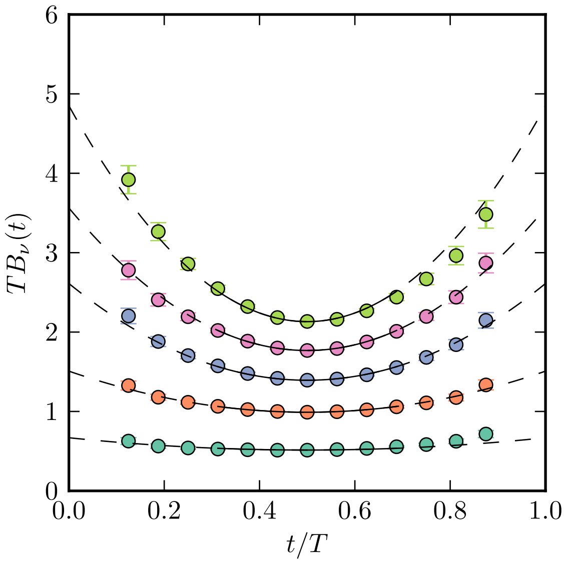
  - The extended propagator is given by $\eta_i(z;x_0) = \partial_{x_0} \int_{\boldsymbol{x}} P_{-\chi} S_m(z,x) P_\chi v_i(x)$
- Expansion of the two-point function

$$B_\nu(x_0 - z_0) = \lim_{m\to 0} \frac{1}{L^3} \int_{\boldsymbol{z}} \langle \sum_{i=1}^{|\nu|} v_i^\dagger(z)\gamma_0 \eta_i(z;x_0)\rangle_\nu$$

Simulation parameters

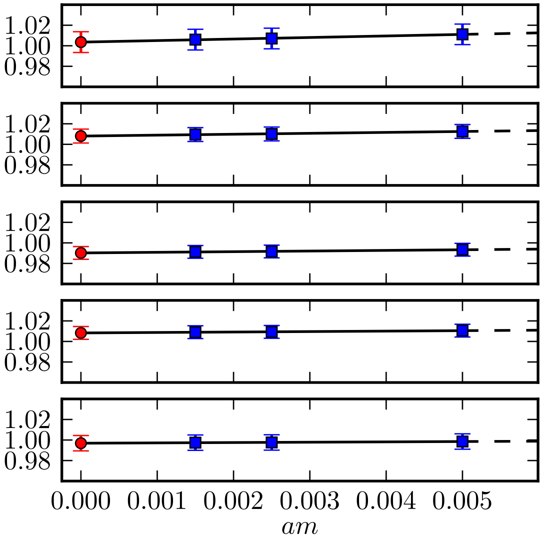
Lattice	β	V	$\|\nu\|$	$N_{\text{cfg}}^{\|\nu\|}$	x_0/a, y_0/a	am
A1	5.8458	16^4	1–5	$180, 157, 169, 126, 94$	$5, 11$	$0.0015, 0.0025, 0.005$

Two-point function $B_\nu(x_0 - z_0)$

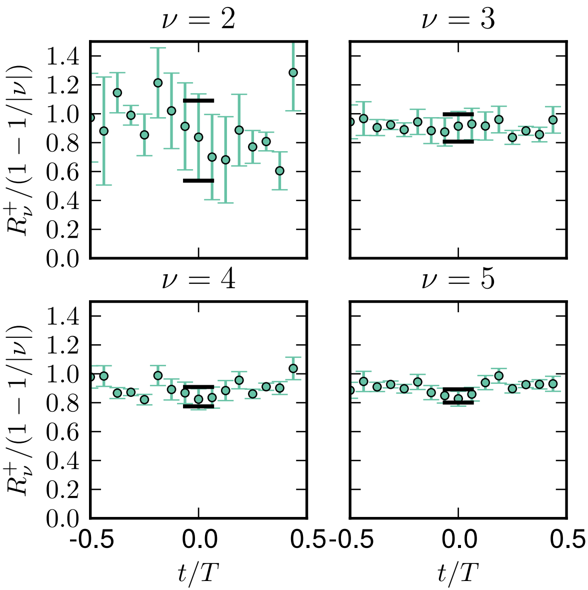


From χPT: $T\mathcal{B}_\nu(x_0 - z_0) = |\nu|\left\{1 + \frac{2|\nu|}{(FL)^2} h_1(\tau_x)\right\}$

Ward identity D_ν/B_ν normalized to $Z_A = 1.710$

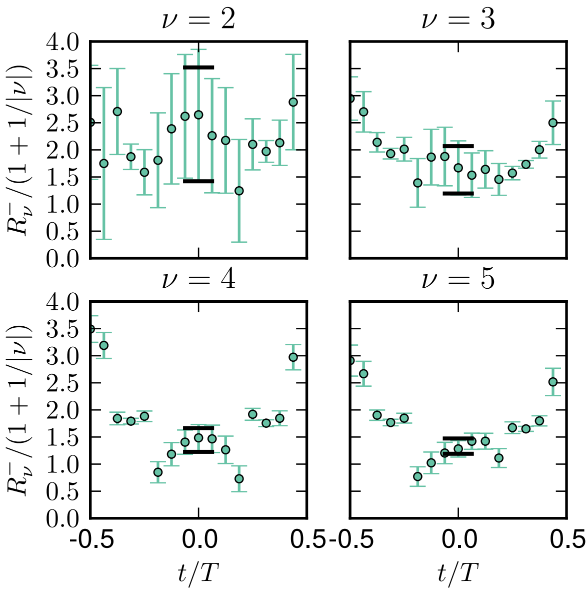


$$Z_A B_\nu(x_0 - z_0) = D_\nu(x_0 - z_0) \equiv \frac{1}{V} \int_{\boldsymbol{x}} \langle \sum_{i,j=1}^{|\nu|} v_j^\dagger(x) v_i(x) v_i^\dagger(z) v_j(z) \rangle_\nu$$



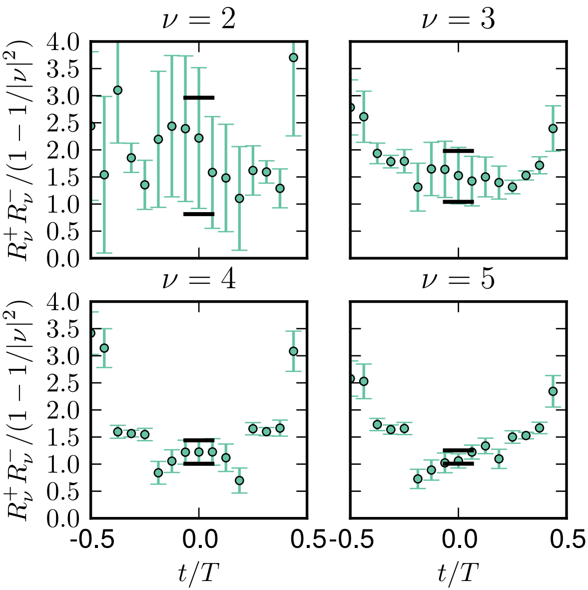
At LO we have $R_\nu^+ = [g_1^+]^{\text{bare}}(1 - \frac{1}{|\nu|})$

$R_\nu^-/(1+1/|\nu|)$



At LO we have $R_\nu^- = [g_1^-]^{\text{bare}}\left(1+\frac{1}{|\nu|}\right)$

$R_\nu^+ R_\nu^- / (1 - 1/\nu^2)$



We expect $R_\nu^+ R_\nu^- = [g_1^+ g_1^-]^{\text{bare}} (1 - \frac{1}{|\nu|^2})$ even at NLO

Results on $g_1^{\pm}$

$\lvert\nu\rvert$	$[g_1^+]^{\text{bare}}$	$[g_1^-]^{\text{bare}}$	$[g_1^+ g_1^-]^{\text{bare}}$
2	$0.81(28)$	$2.47(1.05)$	$1.88(1.07)$
3	$0.90(9)$	$1.63(44)$	$1.51(47)$
4	$0.84(7)$	$1.45(22)$	$1.22(22)$
5	$0.84(5)$	$1.33(14)$	$1.13(12)$
w.a.	$0.85(4)$	$1.40(11)$	$1.17(10)$

- $[g_1^+]^{\text{bare}}$ from $[g_1^+ g_1^-]^{\text{bare}}$ gives $0.83(5)$ which is consistent

Low-mode averaging (LMA)

- The chiral propagator can be split into a “low” and a “high” part

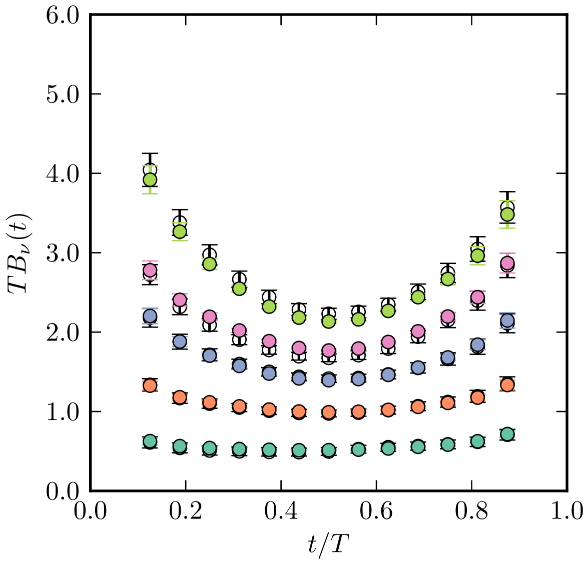
$$P_{-\chi} S_m(z,x) P_\chi = \sum_{k=1}^{N_{\mathsf{low}}} \Psi_k(z) \otimes \Psi_k(x) + P_{-\chi} S_m^{\mathsf{sub}}(z,x) P_\chi$$

- The two-point function separates into two parts as well: $B_\nu = B_\nu^{\mathsf{l}} + B_\nu^{\mathsf{h}}$
- The high part is formally the same but with the subspace propagator plugged in
- The low part now allows for an additional averaging over time translations

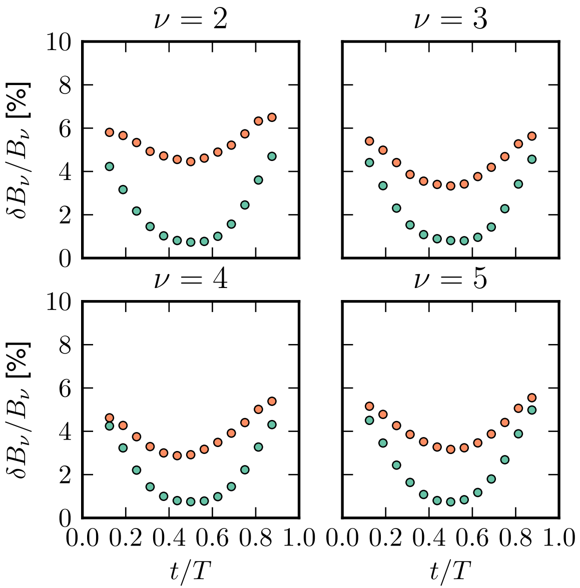
$$B_\nu^{\mathsf{l}}(t) = \lim_{m\to 0} \sum_{i=1}^{|\nu|} \sum_{k=1}^{N_{\mathsf{low}}} \frac{1}{V} \int_{x,z} \delta(x_0 - z_0 - t) \langle v_i^\dagger(z) \gamma_0 P_- \Psi_k(z) \partial_{x_0} \left[\Psi_k^\dagger(x) P_+ v_i(x) \right] \rangle_\nu$$

- LMA for the three-point function is performed analogously

Comparison of conventional computation and LMA



Relative error $\delta B_\nu(t)/B_\nu(t)$



Summary and outlook

Summary

- Estimation of the weak low-energy constants $g_1^{\pm}$ in the SU(4) limit
- Contributions of topological zero-modes to the three-point function of pseudo-scalar
- Good signal on the two-point function and agreement on the non-singlet axial Ward identity
- Final result on the LECs at LO: $g_1^+ = 0.85(4)$, $g_1^- = 1.40(11)$
- An enhancement is already recognizable

Todo

- NLO corrections for $R_\nu^{\pm}$
- LMA for the three-point function
- Computation of the additional diagrams

Thank you for your attention.