Controlling errors in measurements for QCD at finite μ

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Lattice 2013, Mainz

Reminder of the notation

The baryon number susceptibility:

$$\frac{1}{T^2}\,\chi_B(T,\mu_B) = \frac{1}{T^4}\,\frac{\partial^2 P(T,\mu_B)}{\partial \mu_B^2}.$$

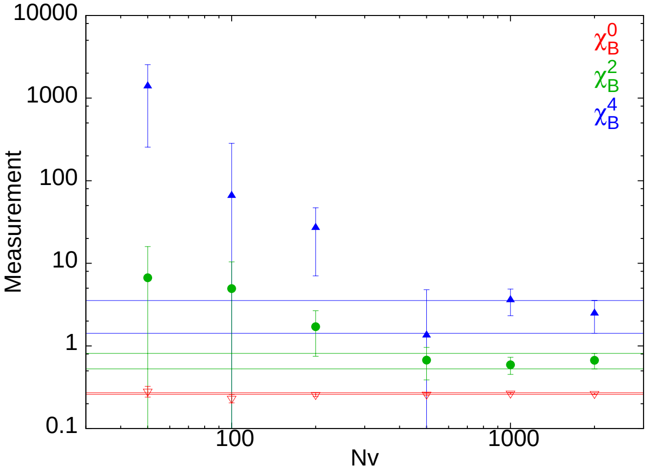
Series expansion of the susceptibility:

$$\frac{1}{T^2}\,\chi_B(T,\mu_B) = \frac{1}{T^2}\,\chi_B^0 + \frac{1}{2!}\chi_B^2\left(\frac{\mu_B}{T}\right)^2 + \frac{1}{4!}\,T^2\chi_B^4\left(\frac{\mu_B}{T}\right)^4 + \cdots$$

Expansion in powers of $z = \mu_B/T$. Coefficients depend on T but not on μ_B.

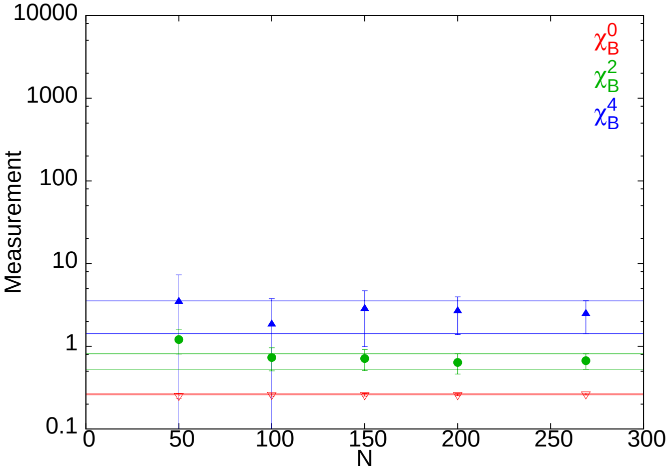
Results obtained with $N_f = 2$ staggered quarks with $m_\pi/m_\rho \simeq 0.26$. Lattice sizes are 8×32^3. 50K+ configurations at each β, 2000 fermion sources for measurements.

The number of fermion sources



Multi-loop (multi-trace) operators dominate errors. Higher orders very hungry for sources.

The number of configurations



The number of configurations: $250N$; errors are the usual $1/\sqrt{N}$.
Ratios are fat-tailed; errors by bootstrap.

Susceptibility for $\mu \neq 0$

Resum a series into a Padé approximant. For example,

$$[0,1]: \qquad S(z) = \frac{c}{z_* - z}$$

$$[1,1]: \qquad S(z) = \frac{a + bz}{z_* - z}$$

Width of the critical region? If we define it by

$$\left| \frac{S(z)}{S(0)} \right| > \Lambda,$$

then for $[0,1]$ Padé: $|z - z_*| \leq z_*/\Lambda$.
Errors in extrapolation? Simple case of $[0,1]$ Padé, we have

$$\left| \frac{\Delta S}{S} \right| > \frac{1}{1 - \Lambda\delta},$$

where δ is fractional error in z_*. General case similar.

A divergence

Want to evaluate the [0, 1] Padé approximant

$$P(z; z_*) = \frac{1}{z_* - z},$$

at various $z = \mu_B/T$ for z_* determined from lattice measurement.
Distribution of z_*; so for any z, there is a probability that $z = z_*$.
$\langle P \rangle$ and $\sigma(P)$ both diverge.

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$\langle P \rangle$ and $\sigma(P)$ both diverge.
See this another way. Assume that the distribution of z_* is Gaussian with mean 1 and variance σ^2. Then the distribution of P at fixed z is given by

$$p(P;z) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\frac{1}{P^2}\mathrm{e}^{-(z-1-1/P)^2/(2\sigma^2)}.$$

The distribution is normalizable but none of the moments exist.

Bootstrap is a regularization

Due to finite statistics (N), the maximum and minimum values of the Padé approximant are bounded: $|P| \leq \Lambda(N)$.

If one estimates $P(z; z_*)$ by a bootstrap, then one should take the number of bootstrap samples to be $\mathcal{O}(N)$. By accounting for the restricted range $|P| \leq \Lambda$, all the integrals are regularized. If the measurements are made with statistics of N, then $\sigma^2 \propto 1/N$. In generic samples

$$\epsilon(\Lambda) = 1 - \int_{-\Lambda}^{\Lambda} dPp(P; z),$$

where Λ is such that $N\epsilon(\Lambda) \ll 1$.

Now, $\sigma^2 \propto 1/N$. In the limit $N \to \infty$ can one have finite $\langle P \rangle$ and $\langle P^2 \rangle$?

Finite results: renormalization!

The condition $N\epsilon \ll 1$ is satisfied if the growth of Λ is bounded by $\Lambda \propto N^{3/2}$. Extreme value statistics: this happens. So cutoff can removed as $N \to \infty$. Then for Gaussian distributed z_*,

$$\delta\langle P\rangle \simeq \mathrm{e}^{-K(1-z)^2 N} \log(N)$$

$$\delta\langle P^2\rangle \simeq \mathrm{e}^{-K(1-z)^2 N} N\sigma$$

As a result a bootstrap estimation will lead to good estimates of mean and error except for $|z-1| < \mathcal{O}(1/\sqrt{N})$.

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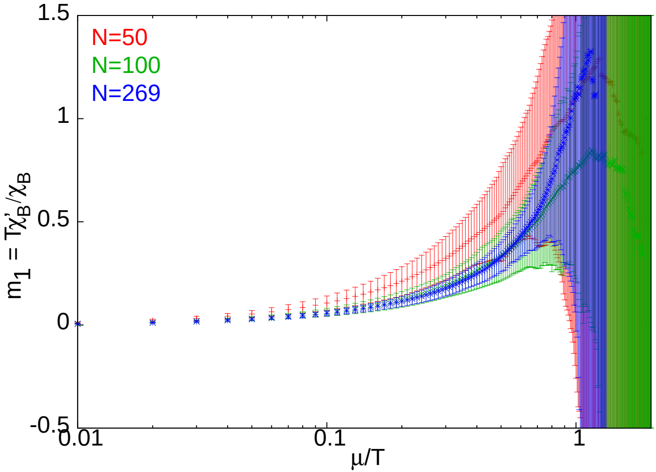
Same strategy beyond the Gaussian approximation: bound the growth of $\langle P\rangle$ and $\langle P^2\rangle$ by verifying that the estimate of the error in the pole narrows faster than the growth of the probability in the tail of the distribution of the value of $P(z;a)$. Works when z_* is the ratio of two Gaussian distributed variates (each with variance going as $1/N$).

The theory works



If $\chi_B \simeq (z_*^2 - z^2)^{-\psi}$ then χ'_B/χ_B has a simple pole.

Many faces of m_1

Define:

$$m_1(z) = \frac{d \log \chi_B}{dz} = \frac{\chi_B'(z)}{\chi_B(z)}.$$

Can be continued to finite chemical potential, and hence can be compared to measurements. **SG: arXiv:0909.4630 (CPOD)**

1. Constructed the Padé approximants and noted that comparison to experiment can be used in two ways. **Gavai, SG: arXiv:1001.3796**
   - If freezeout point is assumed to be known, then can be used to set a temperature scale from experiment. **SG etal: Science 332 (2011) 1525**
   - If temperature scale is measured on lattice, then can be used to extract freezeout point from experiment. **Bazavov etal: arXiv:1208.1220; Borsanyi etal: 1305.5161**
2. Padé resummation gives ODE which can be integrated to continue measurements to finite μ.

The DLOG Pade

At a critical point $\chi_B \simeq (z_*^2 - z^2)^{-\psi}$. Since

$$\chi_B = \frac{\partial^2 (P/T^4)}{\partial z^2},$$

the continuity and finiteness of P at the CEP forces $\psi \leq 1$.

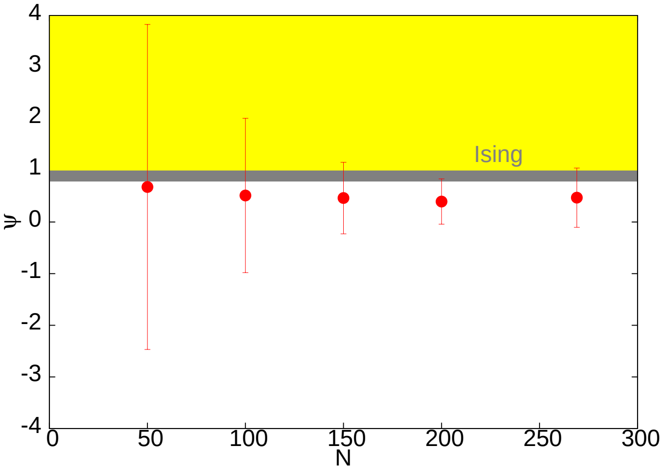
Branch cut: hard to do a simple Padé analysis. Well-known technique, convert to a problem with a pole:

$$m_1(z) = \frac{d \log \chi_B}{dz} \simeq \frac{2\psi z}{z_*^2 - z^2}.$$

Use the series to estimate the critical exponent.

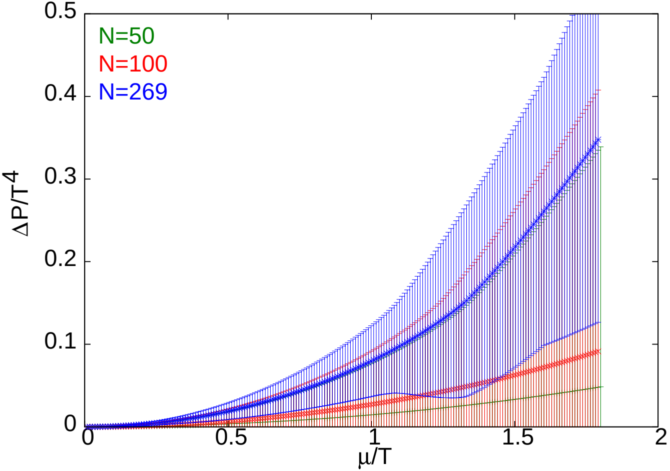
Accurate results require fine statistical control of at least 3 series coefficients of χ_B. Preliminary results now.

Critical exponent



Large errors in ψ, but $\psi < 1$ as expected from continuity of pressure. 3d Ising: $\psi \geq 0.79$; mean field theory: $\psi \geq 0.66$.

Statistical uncertainty in pressure



Integrate to get pressure: more in Datta's talk (Session 8A)

Critical point and critical region

1. Study of the errors in χ_B^n and various operators which go into it began a decade ago, and is now mature. Errors can be controlled, and radii of convergence can be estimated with confidence. (Gavai, Lattice 2013).
2. Study of errors in Padé approximants is more recent: now understood completely. Indicates one of the ways in which critical slowing down may be manifested.
3. The detailed study of Padé approximants has now begun. First results on the critical exponent given here: consistent with 3d Ising, but cannot distinguish between models. More on susceptibilities and pressure in another talk. (Datta, Lattice 2013).
4. Noise reduction techniques? Important open question.