NPR of bilinear operators with improved staggered quarks

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We present matching factors for the bilinear operators obtained using the non-perturbative renormalization method (NPR) for improved staggered fermions on the MILC asqtad lattices ($N_f = 2 + 1$).

We obtain the wave function renormalization factor $Z_q$ from the conserved vector and axial currents. Also we obtain the mass renormalization factor $Z_m$ from scalar and pseudo-scalar bilinear operators.

We also calculate the renormalization factor of bilinear operators whose taste are scalar or pseudo-scalar.
Bilinear Operator Renormalization

- $\tilde{p}$ is the momentum in reduced Brillouin zone.

\[ p \in \left( -\frac{\pi}{a}, \frac{\pi}{a} \right]^4, \quad \tilde{p} \in \left( -\frac{\pi}{2a}, \frac{\pi}{2a} \right]^4, \quad p = \tilde{p} + \pi_B \]

where $\pi_B(\equiv \frac{\pi}{a} B)$ is cut-off momentum in hypercube.

- $a$ : lattice spacing.
- $B$ : vector in hypercube. Each element is 0 or 1
  ex) $B = (0, 0, 1, 1)$
Figure: The Green’s functions of bilinear operator: The diagrams that contribute to bilinear operator.
Bilinear Operator Renormalization

Red Circle : 1PI Diagram.

Unamputated Green's function
\( \tilde{\mathcal{H}}^\alpha \)

Amputated Green's function
\( \tilde{\Lambda}^\alpha \)

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Bilinear Operator Renormalization

\[ \alpha, \beta : \text{the indices to represent different operators.} \]
\[ \text{ex) } \alpha = (\gamma_\mu \otimes 1), \beta = (1 \otimes 1) \]

\[ \text{M.C. : momentum conservation condition. } \tilde{p} = \tilde{q} + \tilde{k} \]
The projection operator is

\[ \hat{P}_{BA; c_2c_1}^\beta = \frac{1}{48} (\gamma_S^\dagger \otimes \xi_F^\dagger)_{BA} \delta_{c_2c_1} \]

The renormalization of \( \Gamma(\tilde{p}, \tilde{q}) \) is

\[ \Gamma_{R(\tilde{p}, \tilde{q})}^{\alpha\sigma} = \sum_{\beta} Z_q^{-1} Z_O^{\alpha\beta} \Gamma_0^{\beta\sigma}(\tilde{p}, \tilde{q}) \]

- \( A, B \): hypercube index
- \( c \): color index
- \( \alpha, \beta, \sigma \): the indices to represent different operators.
- \( \Gamma_0 \): bare projected amputated Green’s function
- \( \Gamma_R \): renormalized projected amputated Green’s function
- \( Z_q \): the wave function renormalization factor for quark fields
- \( Z_O^{\alpha\beta} \): the renormalization factor of operator.
The RI-MOM scheme prescription is

$$\Gamma_{R}^{\alpha\sigma}(\tilde{p}, \tilde{p}) = \Gamma_{\text{tree}}^{\alpha\sigma}(\tilde{p}, \tilde{p}) = \delta^{\alpha\sigma},$$

where $\Gamma_{\text{tree}}^{\alpha\sigma}(\tilde{p}, \tilde{p})$ is tree level projected amputated Green’s function.

Therefore

$$Z_{q} \cdot (Z_{O}^{-1})^{\alpha\beta} = \Gamma_{0}^{\alpha\beta}(\tilde{p}, \tilde{p})$$
Simulation Detail

- $20^3 \times 64$ MILC asqtad lattice ($a \approx 0.12fm$, $am_{\ell}/am_s = 0.01/0.05$).
- HYP smearing
- The number of configurations is 30.
- 5 valence quark masses (0.01, 0.02, 0.03, 0.04, 0.05)
- 14 external momenta in the units of $(\frac{2\pi}{L_s}, \frac{2\pi}{L_s}, \frac{2\pi}{L_s}, \frac{2\pi}{L_t})$.

<table>
<thead>
<tr>
<th>$n(x, y, z, t)$</th>
<th>$a\tilde{p}$</th>
<th>GeV</th>
<th>$n(x, y, z, t)$</th>
<th>$a\tilde{p}$</th>
<th>GeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 0, 1, 3)</td>
<td>0.5330</td>
<td>0.8835</td>
<td>(1, 2, 2, 4)</td>
<td>1.0210</td>
<td>1.6922</td>
</tr>
<tr>
<td>(1, 1, 1, 2)</td>
<td>0.5785</td>
<td>0.9588</td>
<td>(2, 1, 2, 6)</td>
<td>1.1114</td>
<td>1.8420</td>
</tr>
<tr>
<td>(1, 1, 1, 3)</td>
<td>0.6187</td>
<td>1.0254</td>
<td>(2, 2, 2, 7)</td>
<td>1.2871</td>
<td>2.1332</td>
</tr>
<tr>
<td>(1, 1, 1, 4)</td>
<td>0.6710</td>
<td>1.1122</td>
<td>(2, 2, 2, 8)</td>
<td>1.3421</td>
<td>2.2243</td>
</tr>
<tr>
<td>(1, 1, 1, 5)</td>
<td>0.7328</td>
<td>1.2146</td>
<td>(2, 2, 2, 9)</td>
<td>1.4018</td>
<td>2.3233</td>
</tr>
<tr>
<td>(1, 1, 1, 6)</td>
<td>0.8019</td>
<td>1.3291</td>
<td>(2, 3, 2, 7)</td>
<td>1.4663</td>
<td>2.4302</td>
</tr>
<tr>
<td>(1, 2, 1, 5)</td>
<td>0.9128</td>
<td>1.5128</td>
<td>(3, 3, 3, 9)</td>
<td>1.8562</td>
<td>3.0764</td>
</tr>
</tbody>
</table>
For the conserved vector current, the renormalization factor $Z_{O}^{\alpha \beta} = 1$.

Therefore

$$Z_{q} = \Gamma_{0}^{\alpha \beta} (\tilde{p}, \tilde{p}),$$

where $\alpha = \beta = (\gamma_{\mu} \otimes 1)$. 
RI-MOM scheme to SI scheme

We divide the RI-MOM scheme data by the RG running factor to calculate the scale-invariant (SI) quantity.

\[ Z_{q}^{SI} = \frac{c(\alpha_s(\mu_0))}{c(\alpha_s(\mu))} Z_{q}^{RI-MOM}(\mu), \quad (\mu_0 = 2\text{GeV}, \quad \mu^2 = \tilde{p}^2) \]

This Wilson coefficient \( c(x) \) is calculated using four-loop running formula.

The data point is \( y = \frac{Z_q}{Z_{V \otimes S}} \).

![Graphs showing RI-MOM and SI schemes](image-url)
Mass Fitting \( (y = \frac{Z_q}{Z_{V \otimes S}}) \)

We use linear fit for mass fitting and take chiral limit value for each momentum.

\[
\begin{align*}
\chi^2/\text{d.o.f} & = 0.0024(62) \\
\end{align*}
\]

\[
\begin{array}{c|c|c}
\hline
\text{am} & 0.76016(15) & -0.0049(21) \\
\hline
\end{array}
\]

\[
f(m, a, \tilde{p}) = c_1 + c_2(\text{am})
\]
Momentum Fitting \( y = \frac{Z_q}{Z_{V \otimes S}} \)

We fit using \((a\tilde{p})^2 > 1\) data to avoid non-perturbative effects at small \((a\tilde{p})^2\). The fitting function is a power series of \((a\tilde{p})^2\).

\[
f(m, a, \tilde{p}) = c_1 + c_2 (a\tilde{p})^2 + c_3 ((a\tilde{p})^2)^2 + c_4 ((a\tilde{p})^2)^3
\]

To remove the discretization error, we take the \((a\tilde{p})^2 \to 0\) limit after fitting.

\[
\begin{array}{cc}
c_1 & c_2 \\
1.0764(44) & -0.1908(69) \\
c_3 & c_4 \\
0.0279(33) & -0.00350(49) \\
\chi^2/d.o.f & \\
0.06(16)
\end{array}
\]
Scalar Bilinear Operator \((1 \otimes 1)\) Analysis

By the Ward-Takahashi identity,

\[ Z_m = \frac{1}{Z_S} \]

Therefore

\[ \frac{Z_q}{Z^{\alpha\beta}_O} = \Gamma_0^{\alpha\beta}(\tilde{p}, \tilde{p}) = Z_q \cdot Z_m, \]

where \(\alpha = \beta = (1 \otimes 1)\).
RI-MOM scheme to SI scheme

We divide the RI-MOM scheme data by the RG running factor to calculate the scale-invariant (SI) quantity.

\[
Z_{q}^{SI} \cdot Z_{m}^{SI} = \frac{c(\alpha_s(\mu_0))}{c(\alpha_s(\mu))} Z_{q}^{RI-MOM}(\mu) \cdot \frac{c'(\alpha_s(\mu_0))}{c'(\alpha_s(\mu))} Z_{m}^{RI-MOM}(\mu)
\]

\[
(\mu_0 = 2\text{GeV}, \quad \mu^2 = \bar{\rho}^2)
\]
Mass Fitting

From the Ward-Takahashi identity, the amputated Green’s function is

$$\Lambda_S = \frac{1}{48} \frac{\partial Tr[S^{-1}(\tilde{p})]}{\partial m}$$

where $S(\tilde{p})$ is quark propagator. and the propagator is

$$\frac{1}{48} Tr[S^{-1}(\tilde{p})] = C_1 \frac{Z_q \langle \bar{\chi}\chi \rangle}{\tilde{p}^2} + Z_q Z_m m + \ldots$$

When we consider the effect of zero-modes on chiral condensate term,

$$\langle \bar{\chi}\chi \rangle \sim \frac{n_0}{m V}$$

where $n_0$ is the number of zero-modes, $V$ is the four dimensional space-time volume.

So we take derivative about mass on chiral condensate term, we can obtain $\frac{1}{(am)^2}$ term.
The fitting function is

\[ f(m, a, \bar{p}) = c_1 + c_2 (am) + c_3 (am)^2 + c_4 \frac{1}{(am)^2}, \]

We take \( c_1 \) as a chiral limit value.

The data point is \( y = Z_q \cdot Z_m = \frac{Z_q}{Z_{S \otimes S}} \).

\[ \begin{array}{c c}
   c_1 & c_2 \\
   1.4036(22) & -0.573(72) \\
   c_3 & c_4 \\
   0.28(67) & 0.000001 \\
   \chi^2/\text{d.o.f} & \\
   0.00008(51) \\
\end{array} \]
Momentum Fitting \( (y = Z_q \cdot Z_m = \frac{Z_q}{Z_{S\otimes S}}) \)

We fit using \((a\tilde{p})^2 > 1\) data to avoid non-perturbative effects at small \((a\tilde{p})^2\). The fitting function is a power series of \((a\tilde{p})^2\).

\[
f(m, a, \tilde{p}) = c_1 + c_2(a\tilde{p})^2 + c_3((a\tilde{p})^2)^2
\]

To remove the discretization error, we take the \((a\tilde{p})^2 \rightarrow 0\) limit after fitting.

\[
\begin{array}{c|c|c}
  & c_1 & c_2 \\
\hline
  c_1 & 1.342(16) & 0.041(12) \\
  c_3 & -0.0060(21) & 0.18(28) \\
\end{array}
\]
## Preliminary Result

<table>
<thead>
<tr>
<th>$Z_q$</th>
<th>NPR</th>
<th>$\chi^2$/dof ($m_q$ fit)</th>
<th>$\chi^2$/dof ($\tilde{p}$ fit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>($\gamma_\mu \otimes 1$)</td>
<td>1.0764(44)</td>
<td>0.0024(62)</td>
<td>0.06(16)</td>
</tr>
<tr>
<td>($\gamma_\mu 5 \otimes \gamma_5$)</td>
<td>1.075(32)</td>
<td>0.0003(27)</td>
<td>0.12(28)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$Z_m$</th>
<th>NPR</th>
<th>$\chi^2$/dof ($m_q$ fit)</th>
<th>$\chi^2$/dof ($\tilde{p}$ fit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1 $\otimes$ 1)</td>
<td>1.246(15)</td>
<td>0.00008(51)</td>
<td>0.18(28)</td>
</tr>
<tr>
<td>($\gamma_5 \otimes \gamma_5$)</td>
<td>1.255(18)</td>
<td>0.0000008(36)</td>
<td>0.06(19)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$Z^{\alpha\alpha}_O$</th>
<th>NPR</th>
<th>$\chi^2$/dof ($m_q$ fit)</th>
<th>$\chi^2$/dof ($\tilde{p}$ fit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>($\gamma_5 \otimes 1$)</td>
<td>1.113(20)</td>
<td>0.0397(22)</td>
<td>0.40(35)</td>
</tr>
<tr>
<td>($\gamma_\mu 5 \otimes 1$)</td>
<td>1.131(23)</td>
<td>0.0001(12)</td>
<td>0.11(41)</td>
</tr>
<tr>
<td>($\gamma_\mu \gamma_\nu \otimes 1$)</td>
<td>1.125(21)</td>
<td>0.0003(24)</td>
<td>0.76(51)</td>
</tr>
<tr>
<td>(1 $\otimes \gamma_5$)</td>
<td>1.079(18)</td>
<td>0.000004(23)</td>
<td>0.19(48)</td>
</tr>
<tr>
<td>($\gamma_\mu \otimes \gamma_5$)</td>
<td>1.157(10)</td>
<td>0.000004(23)</td>
<td>0.40(60)</td>
</tr>
<tr>
<td>($\gamma_\mu \gamma_\nu \otimes \gamma_5$)</td>
<td>1.154(23)</td>
<td>0.0008(34)</td>
<td>0.56(47)</td>
</tr>
</tbody>
</table>
We obtain the wave function renormalization factor $Z_q$ from conserved vector and axial current and mass renormalization factor $Z_m$ from scalar and pseudo-scalar bilinear operators.

Also we calculate the renormalization factor of bilinear operators which has scalar or pseudo-scalar taste.

We plan to analyse other bilinear operators which have V, A, T tastes in near future.