Partially twisted boundary conditions for scalar mesons

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Plan

- Introduction: scalar mesons on the lattice
- Partially twisted boundary conditions in the EFT framework
- Symmetry relations
- Derivation of the partially twisted Lüscher equation
- Mixing in the neutral meson sector
- Conclusions, outlook

The scalar resonances

- What is the nature of a resonance? are these resonances:
  - a “standard” bound state of quarks $q\bar{q}$?
  - an “exotic” state: a hadronic molecule, glueball, tetraquark, etc?
- What is a correct formulation of the question within QFT?
- What are the experimental signatures?
  - The observed structure of the resonance spectrum
  - Compositeness condition or pole counting criterion

    S. Weinberg, D. Morgan and M.R. Pennington, N.A. Törnqvist, . . .
- How does one extract the resonance pole position in the coupled channel $\pi\pi - K\bar{K}$ and $\pi\eta - K\bar{K}$ systems on the lattice?
- Which tools can be applied on the lattice?
  - Study of the volume dependence
  - Twisted boundary conditions
  - Quark mass dependence

Lüscher equation with coupled channels

V. Bernard, M. Lage, U.-G. Meißner and AR, JHEP 1101 (2011) 019

$f_0(980)$: Two-channel equation with $\boxed{1} = K\bar{K}$, $\boxed{2} = \pi\pi$

$$
\begin{aligned}
T_{11} &= V_{11} + V_{11} i p_1 T_{11} + V_{12} i p_2 T_{21} \\
T_{21} &= V_{21} + V_{21} i p_1 T_{11} + V_{22} i p_2 T_{21}
\end{aligned}
$$

Resonance pole(s) are determined from the secular equation:

$$1 - ip_1 V_{11} - ip_2 V_{22} - p_1 p_2 (V_{11} V_{22} - V_{12}^2) = 0$$

Finite volume: multi-channel Lüscher equation: $ip_i \to \dfrac{2}{\sqrt{\pi} L} Z_{00}(1, q_i^2)$

⇒ Find $V_{ij}(s)$ from the lattice data: one equation for V_{11}, V_{12}, V_{22}

⇒ Find the position of the pole(s) from the secular equation

⟶ Parameterize potentials: eff. range expansion, unitary ChPT

Twisted boundary conditions

P.F. Bedaque, PLB 593 (2004) 82
G.M. de Diviitis, R. Petronzio and N. Tantalo, PLB 595 (2004) 408
G.M. de Diviitis and N. Tantalo, hep-lat/0409154
C.T. Sachrajda and G. Villadoro, PLB 609 (2005) 73

The twisting angle: $\boldsymbol{\theta}_i \in (0, 2\pi)$

$$\psi_i(\mathbf{x} + \mathbf{n}L) = e^{i\mathbf{n}\boldsymbol{\theta}_i}\psi_i(\mathbf{x})\,, \qquad i = u, d, s$$

Modification of the zeta-function (take CM frame for simplicity)

$$Z_{00}(1; q^2) \to Z_{00}^{\theta}(1, q^2) = \frac{1}{\sqrt{4\pi}} \lim_{s\to 1} \sum_{\mathbf{n}\in Z^3} \frac{1}{((\mathbf{n} - \boldsymbol{\theta}/2\pi)^2 - q^2)^s}$$

$\Rightarrow$ Take data for every single $\boldsymbol{\theta}_i$

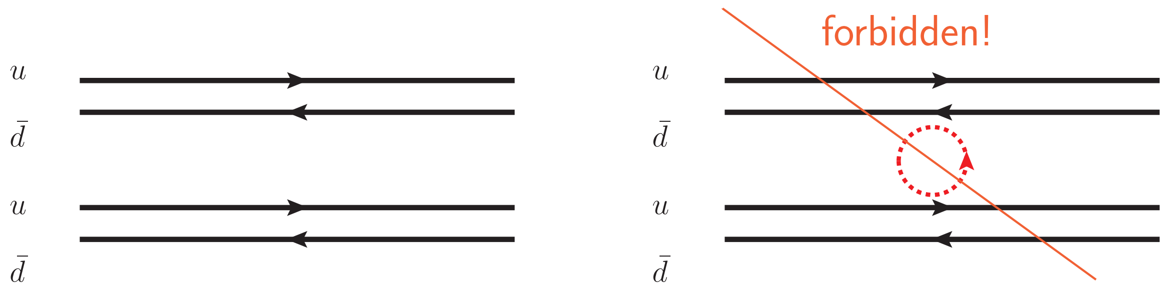
$\Rightarrow$ Adjust L and $\boldsymbol{\theta}_i$ to get the same CM energy

$\Rightarrow$ Accurate extraction of the resonance parameters from data!

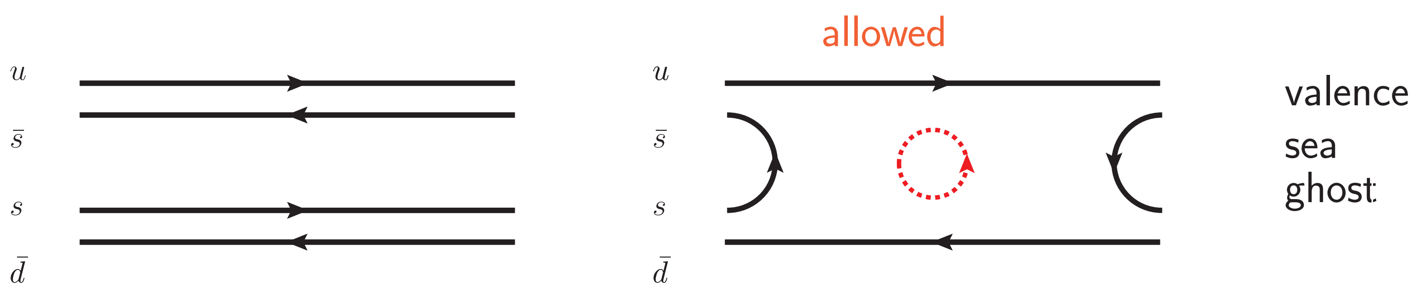
Full vs partial twisting

C.T. Sachrajda and G. Villadoro, PLB 609 (2005) 73

P.F. Bedaque and J.-W. Chen, PLB 616 (2005) 208



$I = 2,\ I_3 = 2:$ partial twisting = full twisting



$I = 1,\ I_3 = 1:$ partial twisting $\neq$ full twisting

EFT framework for partial twisting

S.R. Sharpe and N. Shoresh, PRD 64 (2001) 114510

$$S_F = \int d^4x \left[\bar\psi_{\mathsf{v}}(\not{D} + m_{\mathsf{val}})\psi_{\mathsf{v}} + \bar\psi_{\mathsf{s}}(\not{D} + m_{\mathsf{sea}})\psi_{\mathsf{s}} + \bar\psi_{\mathsf{g}}(\not{D} + m_{\mathsf{gh}})\psi_{\mathsf{g}} \right]$$

$$m_{\mathsf{val}} = m_{\mathsf{sea}} = m_{\mathsf{gh}}\,, \qquad m_u = m_d = \hat{m} \neq m_s$$

$$U = \exp\{i\sqrt{2}\Phi/F\}\ : \qquad U \to LUR^\dagger\,, \quad L, R \in SU(2N|N)$$

$$\Phi = \begin{pmatrix} M_{\mathsf{vv}} & M^\dagger_{\mathsf{sv}} & M^\dagger_{\mathsf{gv}} \\ M_{\mathsf{sv}} & M_{\mathsf{ss}} & M^\dagger_{\mathsf{gs}} \\ M_{\mathsf{gv}} & M_{\mathsf{gs}} & M_{\mathsf{gg}} \end{pmatrix}, \qquad \mathrm{str}\,\Phi = 0$$

$$\mathcal{L} = \frac{F_0^2}{4}\mathrm{str}\,(\partial_\mu U \partial^\mu U^\dagger) - \frac{F_0^2}{4}\mathrm{str}(\chi U + U\chi^\dagger) + \text{higher-order terms}$$

The physical basis

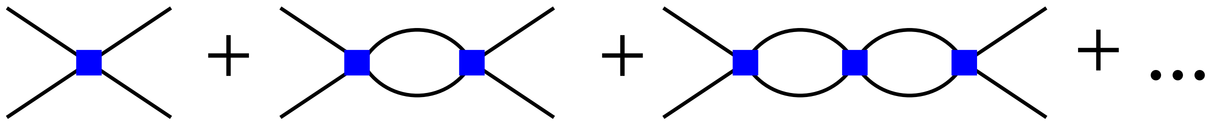
- The fields, describing η,η' in various sectors, mix due to str $\Phi=0$
- Diagonalization of the kinetic term at $O(p^2)$ is achieved by:

$$
\begin{aligned}
\phi_1 &= \omega_1/\sqrt{2}-\omega_2/\sqrt{6}-\omega_5/2+\omega_8/2\\
\phi_2 &= -\omega_1/\sqrt{2}-\omega_2/\sqrt{6}-\omega_5/2+\omega_8/2\\
\phi_3 &= \sqrt{6}\,\omega_2/3+\omega_3/\sqrt{2}-\omega_6\sqrt{2}\\
\phi_4 &= -\omega_2/\sqrt{6}+\omega_4/\sqrt{2}+\omega_5/2+\omega_8/2\\
\phi_5 &= -\omega_2/\sqrt{6}-\omega_4/\sqrt{2}+\omega_5/2+\omega_8/2\\
\phi_6 &= \sqrt{6}\,\omega_2/3+\omega_3/\sqrt{2}+\omega_6/\sqrt{2}\\
\phi_7 &= -\omega_2/\sqrt{6}+\omega_7/\sqrt{2}+\omega_8\\
\phi_8 &= -\omega_2/\sqrt{6}-\omega_7/\sqrt{2}+\omega_8\\
\phi_9 &= \sqrt{6}\,\omega_2/3+\sqrt{2}\,\omega_3
\end{aligned}
$$

Propagator: $D_{\alpha\beta}=\text{diag}\,(D_\pi, D_\eta, -D_s, D_\pi, D_\pi, D_s, -D_\pi, -D_\pi, 0)$

Matching to the non-relativistic EFT

T =



$$T_{\alpha\beta} = V_{\alpha\beta} + \sum_{\gamma} V_{\alpha\gamma} G_\gamma T_{\gamma\beta}$$

Loops for different intermediate states (S denotes $(\bar{s}s)$ meson):

$$K\bar{K} \quad : \quad G_K = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(2w_K(\mathbf{k}))^2} \frac{1}{2w_K(\mathbf{k}) - P_0}$$

$$\pi\eta \quad : \quad G_\eta = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2w_\pi(\mathbf{k}) 2w_\eta(\mathbf{k})} \frac{1}{w_\pi(\mathbf{k}) + w_\eta(\mathbf{k}) - P_0}$$

$$\pi\pi \quad : \quad G_\pi = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(2w_\pi(\mathbf{k}))^2} \frac{1}{2w_\pi(\mathbf{k}) - P_0}$$

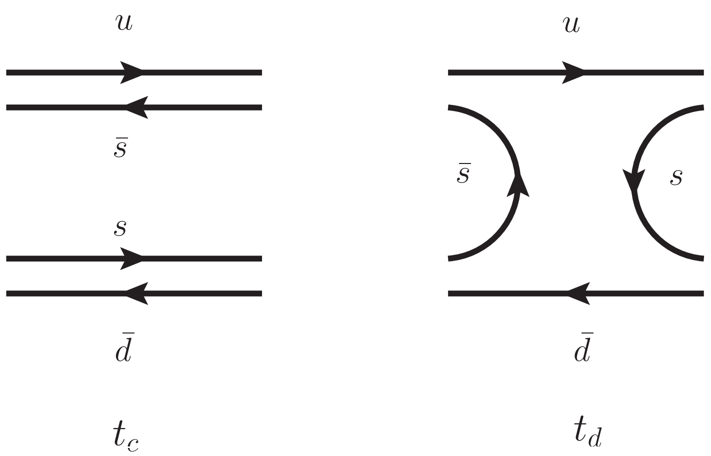
$$\pi S \quad : \quad G_s = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2w_\pi(\mathbf{k}) 2w_s(\mathbf{k})} \frac{1}{w_\pi(\mathbf{k}) + w_s(\mathbf{k}) - P_0}$$

Choice of the basis

Index	Channel	Quark content
1	$\vert\pi^+_{\mathrm{vv}}\eta_{\mathrm{vv}}\rangle$	$-\frac{1}{\sqrt{6}}\,\vert(u_{\mathrm{v}}\bar d_{\mathrm{v}})(u_{\mathrm{v}}\bar u_{\mathrm{v}}+d_{\mathrm{v}}\bar d_{\mathrm{v}}-2s_{\mathrm{v}}\bar s_{\mathrm{v}})\rangle$
2	$\vert\pi^+_{\mathrm{vv}}\eta'_{\mathrm{vv}}\rangle$	$-\frac{1}{\sqrt{3}}\,\vert(u_{\mathrm{v}}\bar d_{\mathrm{v}})(u_{\mathrm{v}}\bar u_{\mathrm{v}}+d_{\mathrm{v}}\bar d_{\mathrm{v}}+s_{\mathrm{v}}\bar s_{\mathrm{v}})\rangle$
3	$\vert\pi^+_{\mathrm{vv}}\eta_{\mathrm{ss}}\rangle$	$-\frac{1}{\sqrt{6}}\,\vert(u_{\mathrm{v}}\bar d_{\mathrm{v}})(u_{\mathrm{s}}\bar u_{\mathrm{s}}+d_{\mathrm{s}}\bar d_{\mathrm{s}}-2s_{\mathrm{s}}\bar s_{\mathrm{s}})\rangle$
4	$\vert\pi^+_{\mathrm{vv}}\eta'_{\mathrm{ss}}\rangle$	$-\frac{1}{\sqrt{3}}\,\vert(u_{\mathrm{v}}\bar d_{\mathrm{v}})(u_{\mathrm{s}}\bar u_{\mathrm{s}}+d_{\mathrm{s}}\bar d_{\mathrm{s}}+s_{\mathrm{s}}\bar s_{\mathrm{s}})\rangle$
5	$\vert\pi^+_{\mathrm{vv}}\eta_{\mathrm{gg}}\rangle$	$-\frac{1}{\sqrt{6}}\,\vert(u_{\mathrm{v}}\bar d_{\mathrm{v}})(u_{\mathrm{g}}\bar u_{\mathrm{g}}+d_{\mathrm{g}}\bar d_{\mathrm{g}}-2s_{\mathrm{g}}\bar s_{\mathrm{g}})\rangle$
6	$\vert\pi^+_{\mathrm{vv}}\eta'_{\mathrm{gg}}\rangle$	$-\frac{1}{\sqrt{3}}\,\vert(u_{\mathrm{v}}\bar d_{\mathrm{v}})(u_{\mathrm{g}}\bar u_{\mathrm{g}}+d_{\mathrm{g}}\bar d_{\mathrm{g}}+s_{\mathrm{g}}\bar s_{\mathrm{g}})\rangle$
7	$\vert K^+_{\mathrm{vv}}\bar K^0_{\mathrm{vv}}\rangle$	$\vert(u_{\mathrm{v}}\bar s_{\mathrm{v}})(s_{\mathrm{v}}\bar d_{\mathrm{v}})\rangle$
8	$\vert K^+_{\mathrm{vs}}\bar K^0_{\mathrm{vs}}\rangle$	$\vert(u_{\mathrm{v}}\bar s_{\mathrm{s}})(s_{\mathrm{s}}\bar d_{\mathrm{v}})\rangle$
9	$\vert K^+_{\mathrm{vg}}\bar K^0_{\mathrm{vg}}\rangle$	$\vert(u_{\mathrm{v}}\bar s_{\mathrm{g}})(s_{\mathrm{g}}\bar d_{\mathrm{v}})\rangle$
10	$\vert\pi^+_{\mathrm{vs}}\pi^0_{\mathrm{vs}}\rangle$	$\frac{1}{2}(-(u_{\mathrm{v}}\bar d_{\mathrm{s}})(u_{\mathrm{s}}\bar u_{\mathrm{v}}-d_{\mathrm{s}}\bar d_{\mathrm{v}})+(u_{\mathrm{v}}\bar u_{\mathrm{s}}-d_{\mathrm{v}}\bar d_{\mathrm{s}})(u_{\mathrm{s}}\bar d_{\mathrm{v}})\rangle$
11	$\vert\pi^+_{\mathrm{vg}}\pi^0_{\mathrm{vg}}\rangle$	$\frac{1}{2}(-(u_{\mathrm{v}}\bar d_{\mathrm{g}})(u_{\mathrm{g}}\bar u_{\mathrm{v}}-d_{\mathrm{g}}\bar d_{\mathrm{v}})+(u_{\mathrm{v}}\bar u_{\mathrm{g}}-d_{\mathrm{v}}\bar d_{\mathrm{g}})(u_{\mathrm{g}}\bar d_{\mathrm{v}})\rangle$

- Symmetry relations less complicated in this basis
- Free propagator no more diagonal

Symmetry relations for the T-matrix (example)



$$
\begin{aligned}
G_{77} &= \langle \bar K^0_{\text{vv}} K^+_{\text{vv}} | G | K^+_{\text{vv}} \bar K^0_{\text{vv}} \rangle = t_c - t_d \\
G_{88} &= \langle \bar K^0_{\text{vs}} K^+_{\text{vs}} | G | K^+_{\text{vs}} \bar K^0_{\text{vs}} \rangle = t_c - t_d \\
G_{99} &= \langle \bar K^0_{\text{vg}} K^+_{\text{vg}} | G | K^+_{\text{vg}} \bar K^0_{\text{vg}} \rangle = -t_c - t_d \\
G_{78} &= G_{79} = G_{89} = G_{87} = G_{97} = G_{98} = -t_d
\end{aligned}
$$

$\hat{m} \neq m_s$!

$\hookrightarrow$ Relations for the scattering matrix elements:

$$T_{88} = a, \quad T_{99} = -a + 2y\,, \quad T_{78} = T_{79} = T_{89} = y$$

+ many more relations. . .

Reproducing ChPT without sea & ghost sectors

- The symmetries of T = the symmetries of the potential
- $\pi\pi$ and πS loops do not contribute to the matrix elements of T in the valence sector

Valence sector decouples from unphysical sector:

$$\alpha = V_{K\bar{K}\to K\bar{K}}\,, \quad \beta = V_{K\bar{K}\to\pi\eta}\,, \quad \gamma = V_{\pi\eta\to\pi\eta}$$

$$T_{K\bar{K}\to K\bar{K}} = \frac{\alpha - G_\eta(\alpha\gamma - \beta^2)}{D}$$

$$T_{K\bar{K}\to\pi\eta} = \frac{\beta}{D}$$

$$T_{\pi\eta\to\pi\eta} = \frac{\gamma - G_K(\alpha\gamma - \beta^2)}{D}$$

$$D = (1 - G_K\alpha)(1 - G_\eta\gamma) - G_K G_\eta \beta^2$$

$\hookrightarrow$ Known result in the valence sector is reproduced!

Derivation of the Lüscher equation: Scenario 1

Twisting only s-quark: $\boldsymbol{\theta}_u = \boldsymbol{\theta}_d = 0$, $\boldsymbol{\theta}_s \neq 0$

$$G_\eta \to G_\eta^L\,, \qquad G_\pi \to G_\pi^L\,, \qquad G_s \to G_s^L$$

$$G_K \to G_K^L \quad (\text{sea})\,, \qquad G_K \to G_K^{L,\theta} \quad (\text{valence, ghost})$$

In the valence sector:

$$1 - \alpha G_K^L - \gamma G_\eta^L + (\alpha\gamma - \beta^2) G_K^L G_\eta^L = 0$$

$$1 - (\alpha - \delta^L) G_K^{L,\theta} = 0$$

- Equations are still in the CM frame
- δ^L couples valence and sea, ghost sectors
- The first equation does not depend on the parameter $\boldsymbol{\theta}$

$\hookrightarrow$ Not interesting . . .

Derivation of the Lüscher equation: Scenario 2

Twisting only u-quark: $\boldsymbol{\theta}_d = \boldsymbol{\theta}_s = 0$, $\boldsymbol{\theta}_u \neq 0$

$$G_\eta \to G_\eta^{L,\theta}\,, \qquad G_\pi \to G_\pi^{L.\theta}\,, \qquad G_s \to G_s^{L,\theta}\,, \qquad G_K \to G_K^{L,\theta}$$

In the valence sector:

$$1 - \alpha G_K^{L,\theta} - \gamma G_\eta^{L,\theta} + (\alpha\gamma - \beta^2) G_K^{L,\theta} G_\eta^{L,\theta} = 0$$

- Equations are in the moving frame
- Partially twisted Lüscher equation is equivalent to the fully twisted one, despite the presence of the annihilation diagrams

$\hookrightarrow$ Interesting!

- Other scenarios possible, can be investigated in a similar fashion

Effect of the neutral meson mixing

If $\hat{m} \neq m_s$, then η and η' are not exactly octet and singlet fields!

Transformation matrix between $SU(3)$ and physical bases:

$$\Lambda_{i\alpha} = \sum_{m=1}^{9} \tilde{\Lambda}_{im} \overset{0}{\Lambda}_{m\alpha}$$

Linear relations between 4-point functions:

$$\sum d_{ji} G_{ij} = 0\,, \qquad \hat{m} = m_s \quad \text{not assumed!}$$

T matrix obeys same relations as 4-point function even for $\hat{m} \neq m_s$

$$T_{ij}^{\text{on-shell}} = \sum_{\alpha\beta} \overset{0}{\Lambda}_{i\alpha}\, T_{\alpha\beta}^{\text{on-shell}}\, \overset{0}{\Lambda}{}^{T}_{\beta j} \quad \longrightarrow \quad \sum_{ij} d_{ji} T_{ij}^{\text{on-shell}} = 0$$

Same Lippmann-Schwinger equation for $\hat{m} \neq m_s$

$$T_{ij}^{\text{on-shell}} = V_{ij} + \sum_{nm} V_{in} G_{nm} T_{mj}^{\text{on-shell}}\,, \qquad G_{ij} = \sum_{\gamma} (\overset{0}{\Lambda}{}^{T})^{-1}_{i\gamma} G_{\gamma} (\overset{0}{\Lambda})^{-1}_{\gamma j}$$

Conclusions

- Lattice possesses unique tools for a detailed study of the properties of the scalar resonances
- Twisted boundary conditions in the multi-channel scattering problem: extraction of the near-threshold poles, studying molecular vs non-molecular nature of the $f_0(980), a_0(980)$
- Applying EFT methods in a finite volume, it is possible to derive the partially twisted Lüscher equation even in the presence of the annihilation diagrams
- In the channel with total isospin $I = 1$, partial twisting of the *light* quarks is equivalent to the full twisting. Partial twisting of the s-quark has no effect.
- Outlook: apply the method to study $I = 0$ sector, DK-molecules, etc