Critical behaviour in the QCD Anderson transition

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Spectrum of the Dirac Operator

Dirac operator (in Euclidean space) $\slashed{D}$
- anti-Hermitian: purely imaginary spectrum
- $\{\slashed{D}, \gamma_5\} = 0$: symmetric w.r.t $\lambda = 0$

Chiral condensate in the chiral limit $\Leftrightarrow$ spectral density at the origin
[Banks, Casher (1980)]

\[
\langle \bar{\psi} \psi \rangle = \lim_{m \to 0} \lim_{V \to \infty} \frac{\pi \rho(0)}{V}, \quad \rho(\lambda) = \left\langle \sum_i \delta(\lambda - \lambda_i) \right\rangle
\]

Different localisation properties of the low-lying eigenmodes below and above the chiral-crossover temperature $T_c$
- $T < T_c$: extended
- $T > T_c$: localised
Localisation in the Dirac Spectrum

Low-lying modes are localised above the chiral-crossover temperature $T_c$


\[
\text{IPR} = \sum_x |\psi(x)|^4
\]

\[
\text{PR} = \text{IPR}^{-1} / V_4
\]

\[
NT = 4, \beta = 3.75
\]

\[
\ell = a \cdot \text{IPR}^{-1/4} \sim \text{loc. length}
\]

\[
\ell_{\text{localised}} \sim T^{-1}
\]

- Eigenmodes localised for $\lambda < \lambda_c(T)$
- No localised modes in the chirally broken phase: $\lambda_c(T_c) \sim 0$
- $\lambda_c \sim$ effective gap: low-lying modes do not contribute to hadronic correlators at large distance
Anderson Model in 3D

Tight-binding Hamiltonian for “dirty” conductors

\[ H = \sum_n \varepsilon_n |n\rangle \langle n| + \sum_{n,\mu} |n + \hat{\mu}\rangle \langle n| + |n\rangle \langle n + \hat{\mu}| \]

\( \varepsilon_n \): random on-site potential (width \( W \sim \) disorder), \( |n\rangle \): localised states

- No disorder (\( W = 0 \)): delocalised eigenstates
- Nonzero disorder: eigenstates at the band edge become localised due to destructive interference (Anderson localisation)

[Anderson (1958)]

As \( W \) increases, \( E_c \) moves towards the band center, for \( W > W_c \) all the states become localised: metal-insulator transition
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Analogs and Differences Between AM and QCD

Anderson Model

\[ E_{\text{c}}(W) \]

Extended localised

\[ W_{\text{c}} \]

QCD

\[ \lambda_{\text{c}}(T) \]

Extended localised

Localised modes $\leftrightarrow$ low spectral density
Modes not mixed by fluctuations $\rightarrow$ Poisson statistics

Extended modes $\leftrightarrow$ high spectral density
Modes mixed by fluctuations $\rightarrow$ Random Matrix Theory statistics

Most conveniently checked using the unfolded spectrum
Unfolding: local rescaling of eigenvalues $\lambda_i \rightarrow \frac{\lambda_i}{\langle \lambda_{i+1} - \lambda_i \rangle}$
Spectrum of the Dirac Operator above $T_c$

Symanzik improved gauge action, 2+1 stout smeared staggered fermions

[Budapest-Wuppertal collaboration]

$NT = 4, \beta = 3.75 \rightarrow T = 394 \text{ MeV} = 2.6 T_c, a = 0.125 \text{ fm}$

$\langle s^2 \rangle_\lambda = \int_0^\infty ds P_\lambda(s) s^2$

Curve becomes steeper as the volume is increased $\rightarrow$ true phase transition
3D Anderson model: metal-insulator second-order phase transition
Divergent correlation length at critical disorder $W_c$/at mobility edge $E_c$:

$$\xi_\infty(W) \propto |W - W_c|^{-\nu} \quad \xi_\infty(E) \propto |E - E_c|^{-\nu}$$

Finite volume $L^3$, take $Q(E, L)$ such that

$\lim_{L \to \infty} Q(E, L) = \begin{cases} 
Q_m & E < E_c \quad \text{(metallic side)} \\
Q_c & E = E_c \quad \text{(critical point)} \\
Q_i & E > E_c \quad \text{(insulator side)}
\end{cases}$

Finite-size scaling: $Q(E, L) = f(L/\xi_\infty(E)) = F(L^{1/\nu}(E - E_c))$

$$Q(E, L) \approx Q(E_c, L) + Q'(E_c, L)(E - E_c) = F(0) + F'(0)L^{1/\nu}(E - E_c)$$
Use one-parameter scaling to all orders to measure $\nu$ and $\lambda_c$

$$Q(\lambda, L) = f(L/\xi(\lambda)) = F(L^{1/\nu}(\lambda - \lambda_c)) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} L^{n/\nu} (\lambda - \lambda_c)^n$$

- Use several volumes in a two-variable fit
- Estimate the systematic error through constrained (Bayesian) fits including more and more terms in the expansion

$$Q = l_\lambda = \int_0^{s_0} ds P_\lambda(s)$$
Anderson model 3D

\[ \nu_{\text{chSE}} = 1.375 \pm 0.016 \]

\[ \nu_{\text{chUE}} = 1.43 \pm 0.04 \]

\[ \nu_{\text{chOE}} = 1.57 \pm 0.02 \]

QCD

\[ \nu = 1.425(65) \]

\[ (L_{\text{min}} = 36) \]

[Slevin, Ohtsuki (1997), (1999), Asada et al., (2005)]
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Shape Analysis

Plot two observables against each other, if points collapse on a single curve → universal path in the space of probability distribution [Varga et al. (1995)]

Family of RM models connecting Poisson ↔ RMT [talk by S.M. Nishigaki]

Points flow towards the Poisson and RMT “fixed points” as $L \to \infty$

Unstable fixed point $\approx$ critical point, different universality class

Universal path also changing $T$ or $a$?
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![Graph showing points flow towards the Poisson and RMT "fixed points" as \( L \to \infty \)]

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Summary and Outlook

- Dirac spectrum above $T_c$ shows a localisation/delocalisation transition analogous to the Anderson transition in condensed matter
- Critical exponent consistent with Anderson model: same universality class?

Open issues:

- Inclusion of corrections to scaling
- Study of the multifractal structure of eigenmodes near the transition
References

Check possible systematic effects due to the choice of the fitting range and of the width of the bins

$$I_\lambda \rightarrow \int_{B_{\Delta \lambda}(\lambda)} d\lambda' \; I_{\chi'} \rho(\lambda') / \int_{B_{\Delta \lambda}(\lambda)} d\lambda' \; \rho(\lambda')$$
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\[ I_\lambda \rightarrow \int_{B_{\Delta\lambda}(\lambda)} d\lambda' I_{\lambda'} \rho(\lambda') \bigg/ \int_{B_{\Delta\lambda}(\lambda)} d\lambda' \rho(\lambda') \]