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Effective lattice theory for finite density QCD: derivation

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Introduction

- Derive the fermionic contributions to $\mathcal{O}(\kappa^4)$ in our 3d effective theory approach
- Instead of a graphical expansion, we employ only manipulations on block matrices
- Include gauge corrections
- Resum classes of terms in order to improve convergence

Definition of the effective action

- Partition function with Wilson fermions after Grassmann integration

$$Z = \int [dU_\mu] \det[Q_W] \exp[S_g]$$

- Integrate out spatial link variables

$$e^{S_{\text{eff}}} \equiv \int [dU_k] \det[Q_W] \exp[S_g]$$

- Effective action depends only on Polyakov loops

$$W_i = \prod_{\tau=1}^{N_\tau} U_0(\vec{x}_i, \tau)$$

- $\rightarrow$ dimensionally reduced theory expanded in β and κ

Pure gauge theory

- Expand Wilson action in characters
- Leading order corresponds to a chain of N_τ plaquettes

$$e^{S^g_{\mathrm{eff}}} = u^{N_\tau} \sum_{<ij>} \mathrm{Tr}\, W_i\, \mathrm{Tr}\, W_j$$

- For small T ($=$ large N_τ) this is exponentially small, since $u < 1$
- Corrections increase the effective coupling, but it remains vanishingly small
- Next-to-nearest neighbour interactions are even stronger suppressed
- For now, neglect gauge corrections $=$ Strong coupling limit

Quark determinant

- Well-known analytical tool: Hopping parameter expansion

$$\begin{aligned} Q_W &= 1 - \kappa \sum_{\pm\nu} (1 + \gamma_\nu) U_\nu(x) \\ &= 1 - T^+ - T^- - S^+ - S^- = 1 - T - S \end{aligned}$$

- At finite μ temporal hops are modified

$$\kappa \quad \rightarrow \quad \kappa e^{a\mu} \equiv \frac{y}{2}; \quad \kappa e^{-a\mu} \equiv \frac{\bar{y}}{2}$$

- Polyakov loops thus have a fugacity factor

$$c \equiv y^{N_\tau} = (2\kappa e^{a\mu})^{N_\tau} = (2\kappa)^{N_\tau} e^{\frac{\mu}{T}} = \exp\left[\frac{\mu - m_{\mathrm{stat}}}{T}\right]$$

- Expansion strategy: Retain all powers of the fugacity c and expand in κ (i.e. those coming without additional μ factor)

Static quark determinant

- With $Q_W = 1 - T - S$ expand around $S = 0$, i.e. the static limit (and use that $(1+\gamma_0)(1-\gamma_0) = 0$)

$$\begin{aligned} \det[Q_{\rm stat}] &= \det\left[1 - T^+\right]\left[1 - T^-\right] \\ &= \prod_i \det\left[1 + cW_i\right]^{2N_f}\left[1 + \bar{c}W_i^\dagger\right]^{2N_f} \end{aligned}$$

- Partition function after temporal link integration ($\bar{c} = 0$)

$$Z = \left[1 + 4c^{N_c} + c^{2N_c}\right]^{N_s^3}$$

- Baryon number density

$$a^3 n = \frac{1}{N_\tau N_s^3}\frac{\partial}{\partial\mu}\ln Z = \frac{4N_c c^{N_c} + 2N_c c^{2N_c}}{1 + 4c^{N_c} + c^{2N_c}}$$

Static quark determinant

- The zero T number density in this approximation reads

$$\lim_{T\to 0} a^3 n \;=\; \begin{cases} 0, & \mu < m_{\mathrm{stat}} \\ 2N_c, & \mu > m_{\mathrm{stat}} \end{cases}$$

- With a truncation of $\det[Q_{\mathrm{stat}}]$ in c, like

$$\det[Q_{\mathrm{stat}}] \approx \exp\left[2c\sum_i \mathrm{Tr}\, W_i\right] \;,$$

  we do not get saturation on the lattice, i.e. violate Pauli's principle
- Lesson: Need to sum up all windings in each term

Leading correction

- Introduce some definitions and perform some algebra

$$\begin{aligned}\det[Q] &= \det[1-T-S] = \det[1-T][1-(1-T)^{-1}S] \\ &= \det[Q_{\mathrm{stat}}][Q_{\mathrm{kin}}]\end{aligned}$$

- Split S in positive and negative directions

$$\begin{aligned}\det[Q_{\mathrm{kin}}] &= \det[1-(1-T)^{-1}(S^+ + S^-)] = \det[1-P-M] \\ &= \exp\left[\mathrm{Tr}\ln(1-P-M)\right]\end{aligned}$$

- Trace in coordinate space (=closed loops), i.e. only terms with an equal number of P and M survive
- To $\mathcal{O}(\kappa^2)$

$$\det[Q_{\mathrm{kin}}] = \det[1-PM][1+\mathcal{O}(\kappa^4)]$$

- Higher corrections can be systematically included in this way

Static propagator

- Essential ingredient for computing corrections:

$$D = (1 - T)^{-1} = (1 - T^{+})^{-1} + (1 - T^{-})^{-1} - 1$$

- Only temporal hops are involved: closed form expression available (but it is a quite lengthy expression). Splitting it in spin space:

$$D_{t_1,t_2} = A_{t_1,t_2} + \gamma_0 B_{t_1,t_2}$$

- For the leading correction we only need the part of B_{t_1,t_2} diagonal in time

$$B_{t_1,t_1} = -\frac{1}{2}\frac{cW}{1+cW} + \frac{1}{2}\frac{\bar{c}W^\dagger}{1+\bar{c}W^\dagger}$$

Static propagator

$$
\begin{aligned}
A &= \delta_{\tau_1\tau_2}\left(1-\frac{1}{2}\frac{cW}{1+cW}-\frac{1}{2}\frac{\bar{c}W^\dagger}{1+\bar{c}W^\dagger}\right)\\
&\quad+\Theta(\tau_2-\tau_1)\frac{1}{2}\left(\frac{y^{\tau_2-\tau_1}W(\tau_1,\tau_2)}{1+cW}-\frac{\bar{y}^{N_\tau+\tau_1-\tau_2}W^\dagger(\tau_1,\tau_2)}{1+\bar{c}W^\dagger}\right)\\
&\quad+\Theta(\tau_1-\tau_2)\frac{1}{2}\left(-\frac{y^{N_\tau+\tau_2-\tau_1}W(\tau_1,\tau_2)}{1+cW}+\frac{\bar{y}^{\tau_1-\tau_2}W^\dagger(\tau_1,\tau_2)}{1+\bar{c}W^\dagger}\right)\\
B &= \delta_{\tau_1\tau_2}\frac{1}{2}\left(-\frac{cW}{1+cW}+\frac{\bar{c}W^\dagger}{1+\bar{c}W^\dagger}\right)\\
&\quad+\Theta(\tau_2-\tau_1)\frac{1}{2}\left(\frac{y^{\tau_2-\tau_1}W(\tau_1,\tau_2)}{1+cW}+\frac{\bar{y}^{N_\tau+\tau_1-\tau_2}W^\dagger(\tau_1,\tau_2)}{1+\bar{c}W^\dagger}\right)\\
&\quad+\Theta(\tau_1-\tau_2)\frac{1}{2}\left(-\frac{y^{N_\tau+\tau_2-\tau_1}W(\tau_1,\tau_2)}{1+cW}-\frac{\bar{y}^{\tau_1-\tau_2}W^\dagger(\tau_1,\tau_2)}{1+\bar{c}W^\dagger}\right)
\end{aligned}
$$

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Result for leading correction

- Now the spatial link integration is nontrivial

$$\int [dU_k] \det[1 - PM] = \int [dU_k][1 - \mathrm{Tr} PM + \mathcal{O}(\kappa^4)]$$

- For every spatial link there are now exactly one U and one $U^\dagger$ and with

$$\int dU \; U_{ij} U^\dagger_{kl} = \frac{1}{N_c} \delta_{il} \delta_{jk}$$

we have (omitting $\bar{c}$ contributions)

$$\int [dU_k][1 - \mathrm{Tr} PM] = \prod_{<ij>} \left[1 - \frac{\kappa^2 N_\tau}{N_c} \mathrm{Tr} \frac{cW_i}{1 + cW_i} \mathrm{Tr} \frac{cW_j}{1 + cW_j} \right]$$

Higher order corrections

- Nearest-neighbour interaction also contains higher orders

$$\det[1 - PM] = \exp\left[-\mathrm{Tr}\sum_n \frac{1}{n}(PM)^n\right]$$

- Full kinetic determinant contains couplings between all points

$$\begin{aligned}\det[Q_{\mathrm{kin}}] &= \det[1 - PM - P^2M^2 - \ldots] \\ &= \det[1 - PM][1 - P^2M^2][1 + \mathcal{O}(\kappa^6)]\end{aligned}$$

- Note that $P = \sum_{i=1}^{3} P_i$, i.e. P^2 mixes different directions

Gauge corrections

- Leaving the strong coupling limit we have to compute

$$\int [dU_k] \det[Q_{\mathrm{kin}}] \prod_p \left[1 + \sum_{r \neq 0} d_r a_r(\beta) \chi_r(U_p) \right]$$

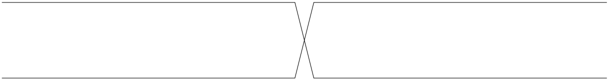
- Now: double series expansion in β and κ
- No conceptual difficulties, only a larger number of terms to be computed
- So far: results up to and including $\mathcal{O}(\kappa^n \beta^m)$ with $n + m = 4$

Resummations

- Why resummations:
  - Include higher order graphs at minimal cost
  - Does not compromise correctness of originial series
  - But: At some point, imperative to include them
- Leading κ correction

$$\int [dU_k] \det[1 - PM] = \prod_{<ij>} \left[1 - \frac{\kappa^2 N_\tau}{N_c} \mathrm{Tr} \frac{cW_i}{1 + cW_i} \mathrm{Tr} \frac{cW_j}{1 + cW_j} \right]$$

- Small T region: coefficient diverges for fixed κ



Resummations

- These terms exponentiate

- After resumming these terms

$$\int [dU_k] \det[1 - PM] = \prod_{<ij>} \exp\left[-\frac{\kappa^2 N_\tau}{N_c} \mathrm{Tr} \frac{cW_i}{1 + cW_i} \mathrm{Tr} \frac{cW_j}{1 + cW_j} \right]$$

- Similar results hold also for other terms, e.g. the single Polyakov line coupling receives renormalization

Conclusions

- Hopping expansion straightforward, but correct physics requires some attention (Pauli principle)
- Resummations necessary to improve convergence (or even to get meaningful results)
- A few more orders are still in reach (Resummations; Nonperturbative determination of the effective couplings;. . .)
- Probe thermal lattice QCD with (heavy) quarks at small temperatures