Large-N volume independence vs. Hagedorn (in)stability

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Based on work done in collaborations with
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Main result

Two widely accepted property of large-N QCD with adjoint fermions seems to be in conflict.

1)Volume independence (absence of any phase transition)

2)Hagedorn growth of density of states (forcing the presence of phase tr.)

Resolution implies an exact spectral degeneracy/cancellation between the bosonic and fermionic Hilbert spaces at $N = \infty$.

The most plausible explanation is emergent fermionic symmetry.

I will tell you the background story for this controversial claim.

Why this claim is not obviously wrong?

Coleman-Mandula + Haag, J. T. Lopuszanski, and M. Sohnius thm:

Supersymmetry algebra is the ONLY graded Lie algebra of symmetries of the non-trivial S-matrix consistent with relativistic QFT.

$N_f > 1$ massless adjoint rep. fermions is a non-supersymmetric relativistic QFT. ➡ Obviously, incorrect?

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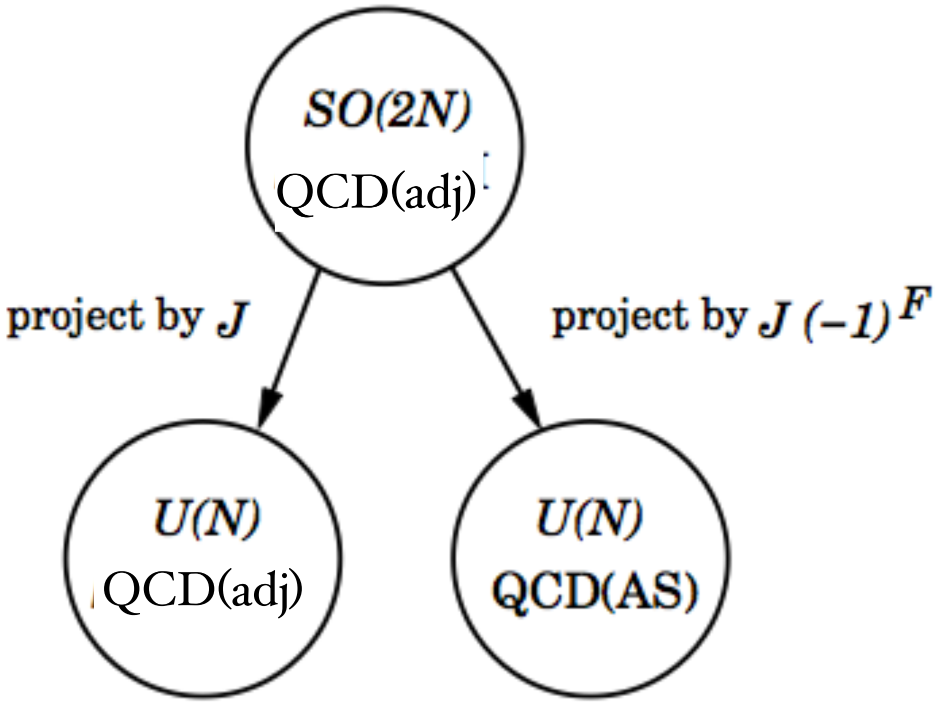
No, the S-matrix of $N = \infty$ theory is trivial!
An emergent fermionic symmetry is not ruled out by HLS.

Why care about QCD(adj)?

Large-N orbifold/orientifold equivalences

Armoni-Shifman-Veneziano 2003-2007
Kovtun, Unsal, Yaffe, 2003-2007



$\mathcal{H} \equiv \mathcal{B} \oplus \mathcal{F}$ $\quad$ $\mathcal{H} \approx \mathcal{B}$

QCD(adj): Hilbert space has both bosonic and fermionic subspaces with O(1) masses at large-N limit.

QCD(AS): The Hilbert space of states with O(1) masses is purely bosonic.

The C-even Hilbert space of QCD(AS) is inherited from the bosonic - Hilbert space of orientifold partner, QCD(adj) i.e., they are isomorphic.

Why care about QCD(AS/adj)?

QCD(AS) is as natural generalization as QCD(F) to large-N of SU(3) QCD!

SU(3) QCD
QCD(F)=QCD(AS)
Nf Dirac Fermions
$\mathbb{R}^4$

$N_c \to \infty$

SU(N) QCD(AS)
Nf Dirac Fermions
$\mathbb{R}^4$

up to 1/N,
matrix model for QCD.

Orientifold equivalence
Armoni-Shifman-Veneziano,
MU-Yaffe

SU(N) QCD(Adj)
Nf Weyl Fermions
1^4 or $\mathbb{R} \times 1^3$

Orbifold
equivalence
or volume
independence
Kovtun-Ünsal-Yaffe

SU(N) QCD(Adj)
Nf Weyl Fermions
$\mathbb{R}^4$

Large N volume independence

"Eguchi-Kawai reduction" or "large-N reduction"

$SU(N)$ gauge theory on toroidal compactifications of $\mathbb{R}^4$

down to four-manifold $\mathbb{R}^{4-d} \times (S^1)^d$

No volume dependence in leading large N behavior of topologically trivial single-trace observables (or their connected correlators)

provided

no spontaneous breaking of center symmetry or translation invariance.

Proof: Comparison of large N loop equaions (Eguchi-Kawai 82) or
N=∞ classical dynamics (Yaffe 82).

Volume independence: example of an orbifold equivalence.

Birth of the idea is in lattice gauge theory.

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Reduction of Dynamical Degrees of Freedom in the Large-N Gauge Theory

Tohru Eguchi and Hikaru Kawai
Department of Physics, Faculty of Science, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan
(Received 19 January 1982)

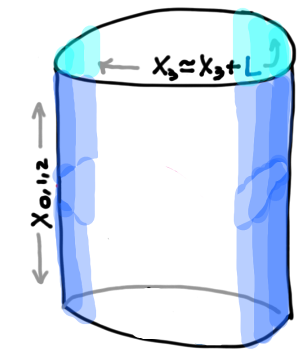
However, single-site Eguchi-Kawai proposal fails, Bhanot, Heller, Neuberger,82
Proposals for cures in early 80's:

Quenched EK Bhanot, Heller, Neuberger, 82, fails, Bringoltz-Sharpe 2008,

OldTwisted EK Gonzales-Arroyo, Okawa, 83, fails, Teper, Vairinhos 2007
Azeyanagi, Hanada, Hirata, Ishikawa 2008,
Bietenholz, Nishimura et.al. 2007

NewTwisted EK Gonzales-Arroyo, Okawa, 2010, works.

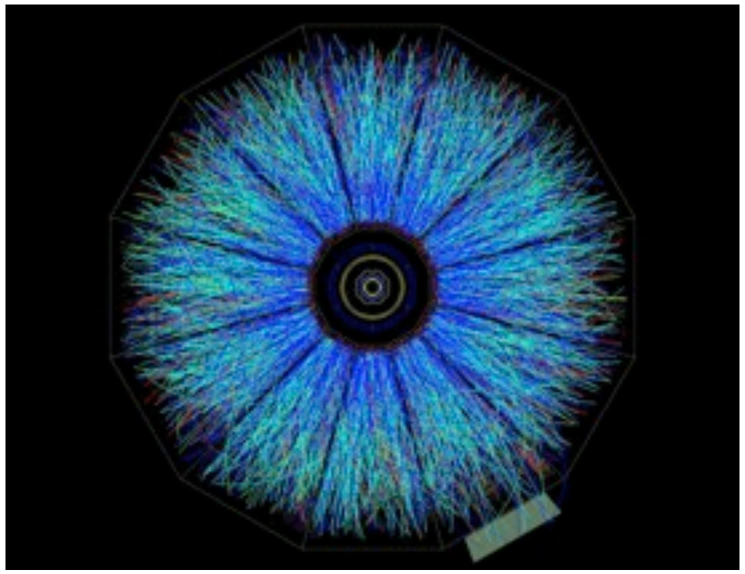
What is the physical reason for EK to fail?
Phase transitions/rapid crossovers



high $- T$ | low $- T$

$\mathbb{R}^3$ $\quad$ $\mathbb{R}^3 \times S^1$ $\quad$ $\mathbb{R}^4$

Phase transition



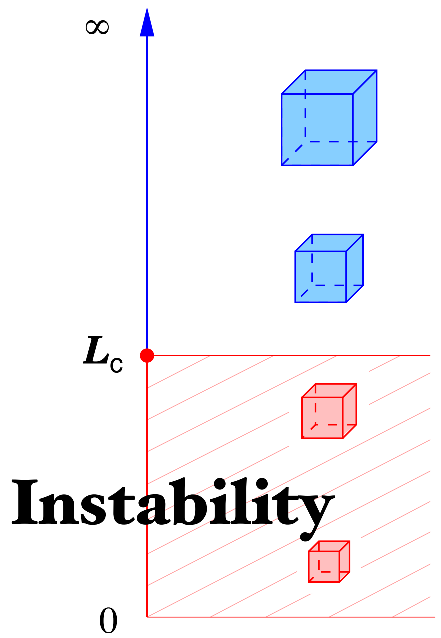
$$V_{1-\mathrm{loop}}[\Omega] = (-1)\frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\,|\mathrm{tr}\,\Omega^n|^2$$

Gross, Pisarski, Yaffe 1980

Quark-Gluon Plasma phase (sQGP)

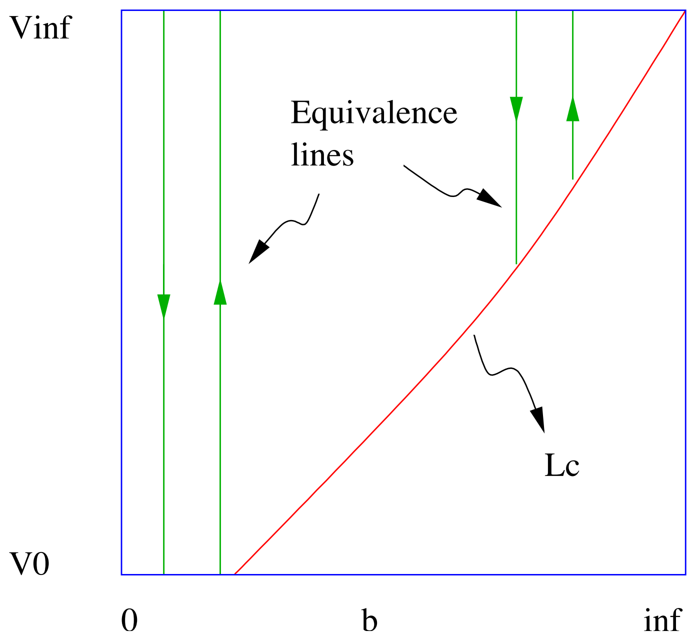
Thermal fluctuations at high-T are **impossible** to overcome.

The hope of using large-N reduction as a tool box faded away.



80's: EK, QEK, TEK.
Eguchi, Kawai, EK,
Gonzalez-Arroyo, Okawa, TEK,
Bhanot, Heller, Neuberger, QEK,
Gross, Kitazawa,
Yaffe,
Migdal, Kazakov,
Parisi et.al.
Das, Wadia, Kogut, ,....

Notable exception: Partial reduction of Kiskis, Narayanan, Neuberger, 2003, 2007



Keep the physical volume of the lattice theory larger than the hadronic scale: $L_c = a\, N_{1,2,3,4}$ This requires (towards continuum) taking larger $N_{1,2,3,4}$.

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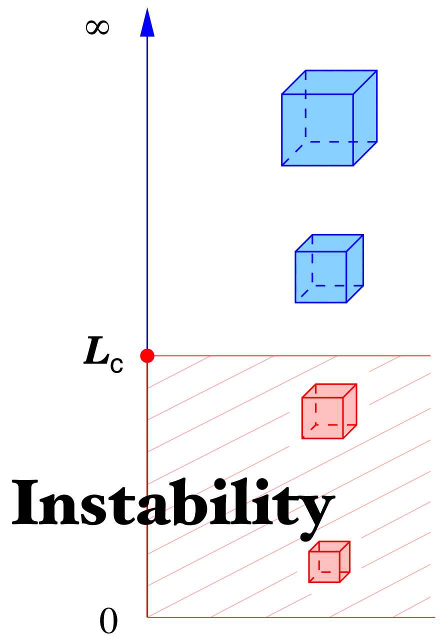

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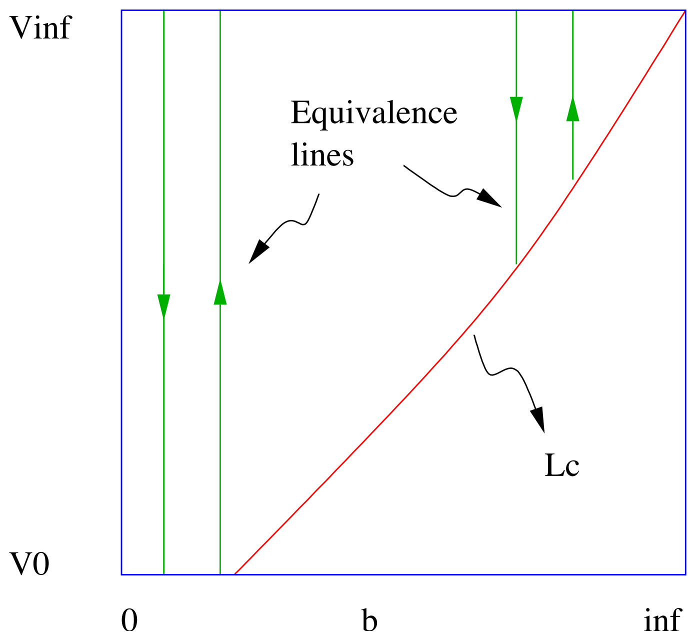

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If one wishes EK to work, one **must** abandon thermal compactification!

QCD(adj) on $\mathbb{R}^3 \times S^1$

Periodic boundary conditions for fermions is not merely a b.c., it is an idea!

In order to determine the phase of small S1 theory, borrow the potential for Wilson-line from Gross, Pisarski, Yaffe 1980, MU Yaffe 2006. All such calculations up to 2007 gave a negative sign or zero in front of the order parameter. Our simple "calculation", given below in full detail, is the first positive sign, i.e., stability of center symmetry, i.e. evidence in favor of working EK.

$$V_{1-\text{loop}}[\Omega] = \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} (-1 + N_f) \, |\text{tr}\, \Omega^n|^2 \qquad \Omega = e^{\int_{S^1} A}$$

$-1 < 0$ instability, "calculations between 1980-2007"

$-1 + 1 = 0$ Supersymmetric case, $N_f = 1$, marginal

$-1 + 2 = 1 > 0$ QCD(adj), $N_f > 1$, **stability.**

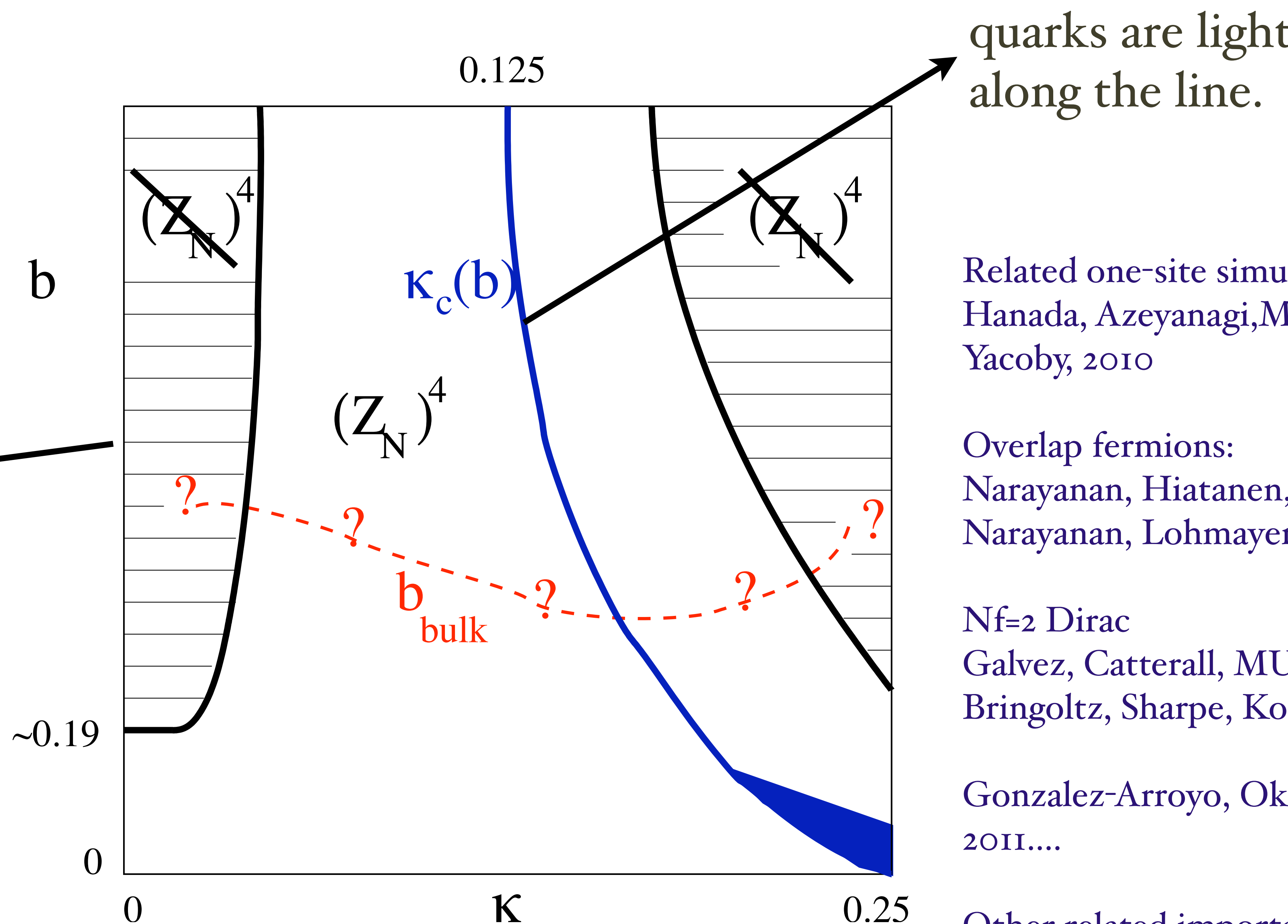
with Kovtun and Yaffe,07

Phase diagram for one site-model

The first one-site simulations of QCD(adj) by Bringoltz+S.Sharpe,0906.3538.

Continuum physics

$b = 1/\lambda_{\text{lat}}$



Original EK

strong-coupling/
lutticy physics

Related one-site simulations:
Hanada, Azeyanagi,MU,
Yacoby, 2010

Overlap fermions:
Narayanan, Hiatanen, 2009...
Narayanan, Lohmayer

Nf=2 Dirac
Galvez, Catterall, MU
Bringoltz, Sharpe, Koren 2011..

Gonzalez-Arroyo, Okawa,
2011....

Other related important works:
Ogilvie, Meisinger, Myers,
D'elia, Cossu,
Veneziano, Armoni, Dorigoni,
Bedaque, Buchoff, Cherman,..
Wosiek,
Cohen,........

the historical evolution of the field is brilliantly coined
by A. Gonzalez-Arroyo in his recent talk at Pierre-fest:
Birth, death and resurrection of large N reduced models

Beware: Unbroken center means two things!

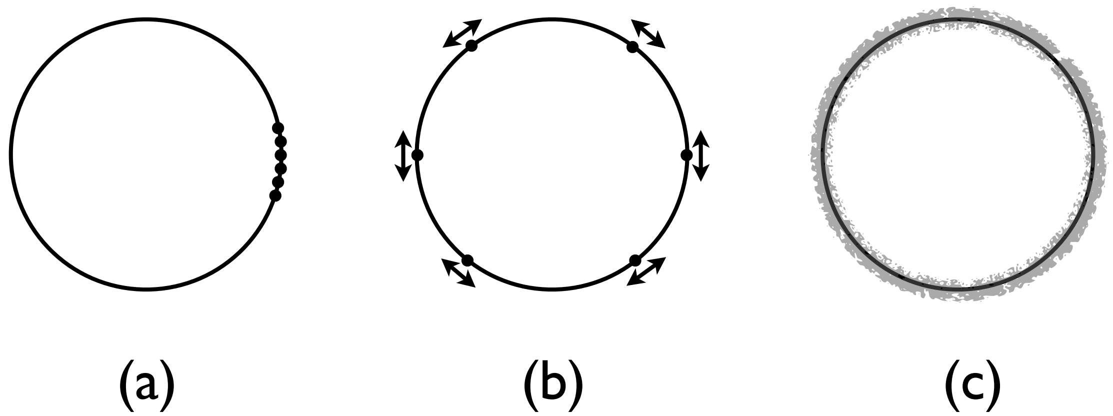
- broken center symmetry

  $\langle \mathrm{tr}\, \Omega \rangle \neq 0 \Leftrightarrow$ eigenvalues clump (a)

  $m_{KK} = 1/L, 2/L, \ldots,$

  perturbative control when $L\Lambda << 1$

  integrate out $\Rightarrow$ 3*d* effective theory, L-dependent

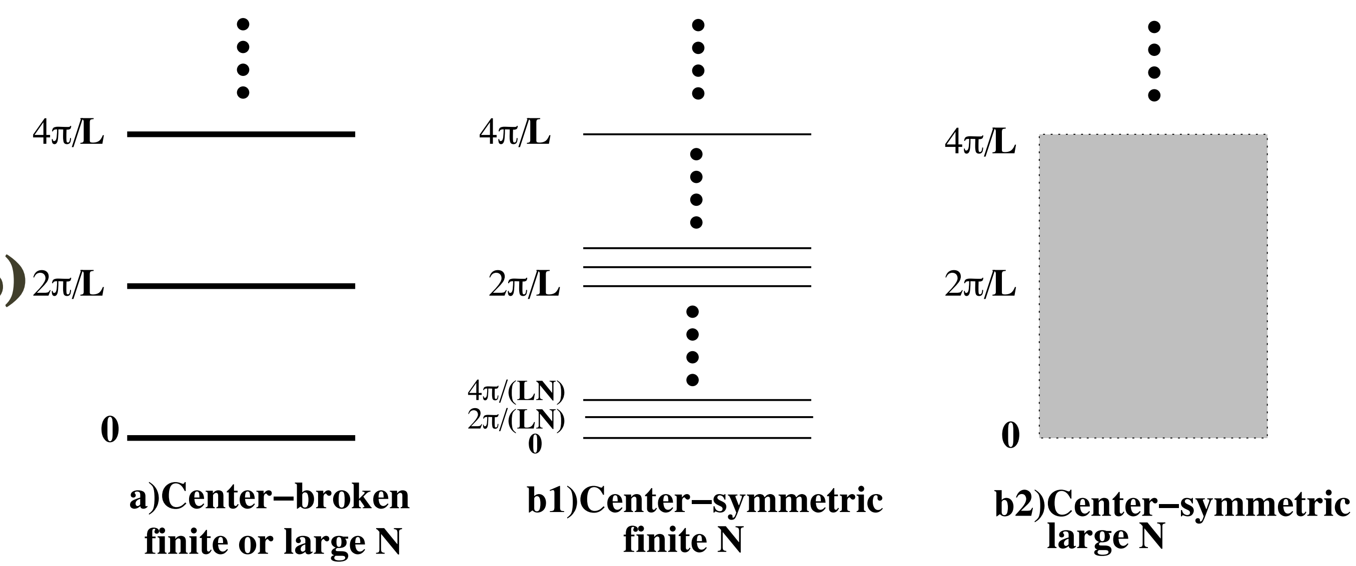


- center-symmetric case:

  $\langle \mathrm{tr}\, \Omega \rangle = 0 \Leftrightarrow$ eigenvalues repel

  $m_{KK} = 1/NL, 2/NL, \ldots,$

  semi-classical control when $NL\Lambda < 1$ (b)



Volume independence:

$N \to \infty$, L fixed : $NL\Lambda >> 1$ **(c)**

WARNING: B & C are continuously connected, but B is semi-classically calculable (~volume dependent), while C is not. In literature, the distinction is barely emphasized!

Phyiscal Intuition behind pbc

In susy-theory, index: Witten, 1980 (predates EK-proposal)

$$\widetilde{Z}(L) = \mathrm{tr}[e^{-LH}(-1)^F] = \text{Invariant}$$

$$Z = Z_B + Z_F$$
$$\widetilde{Z} = Z_B - Z_F$$

Non-susy theory: **Twisted partition function** (not an index, even more useful!)

$$\begin{aligned}\tilde{Z}(L) &\equiv \mathrm{Tr}(-1)^F e^{-LH} = Z_{\mathcal{B}} - Z_{\mathcal{F}} \\ &= \int dM[\rho_{\mathcal{B}}(M) - \rho_{\mathcal{F}}(M)]e^{-LM}\end{aligned}$$

compare with partition function

$$Z(\beta) = \mathrm{Tr} e^{-\beta H} = \int dM[\rho_{\mathcal{B}}(M) + \rho_{\mathcal{F}}(M)]e^{-\beta M}$$

Hagedorn scaling & main puzzle

$$\rho(M) \to \frac{1}{M}\left(\frac{T_H}{M}\right)^a e^{M/T_H}.$$

Hagedorn, 65, Veneziano-Fubini 69,...

Bringoltz-Teper 05,
Meyer 05,
......

$$\tilde{\rho} \equiv \rho_{\mathcal{B}} - \rho_{\mathcal{F}}$$ relative density of states

No symmetry reason (for $N_f > 1$) for the **leading and all N-independent sub-leading growth** of fermionic and bosonic d.o.s. to cancel. However,

Volume independence imply:

$$\tilde{\rho}(M) \lesssim e^{ML_H/N^p}, \quad p > 0: \quad ➡ \; L_c \sim \Lambda^{-1}/N^p.$$ p=1, on S1

No exponential growth of the relative d.o.s. with p=0 is permitted by volume independence (such would imply $L_c \sim 1/\Lambda$ in contradiction to vol. ind.)

First, standard Hagedorn growth

Take a stringy toy model. This is to demonstrate the point of principle that Hagedorn growth may be cancelled even in the absence of susy, and it happens quite beautifully in the mathematical sense. Watch carefully.....

$$M^2 = N/\alpha'$$

$$N = \sum_{n\in\mathbb{N}} n\, a_n^\dagger a_n + \sum_{i=1}^{N_f} \sum_{n\in\mathbb{N}} n\, f_{i\,n}^\dagger f_{i\,n}$$

Generating function:

$$\mathrm{Tr} q^N = \prod_{n=1}^{\infty} \frac{(1+q^n)^{N_f}}{1-q^n} = \sum_{n=0}^{\infty} d(n)\, q^n$$

$$d(n) \sim \exp\left(\sqrt{2\pi^2(1+N_f/2)n/3}\right), \quad n \gg 1,$$

Indeed, the **expected Hagedorn growth**. No surprise!

How could this even be possible without susy?

Now, consider the graded/twisted generating function for $N_f = 1$ (susy).

$$\mathrm{Tr}\left[(-1)^F q^N\right] = 1$$

and $N_f = 2$ (no susy).

$$\mathrm{Tr}\left[(-1)^F q^N\right] = 1 + \sum_{n=1}^{\infty}\left[(-1)^n q^{\frac{3n^2-n}{2}} + (-1)^n q^{\frac{3n^2+n}{2}}\right]$$

$$= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + \ldots.$$

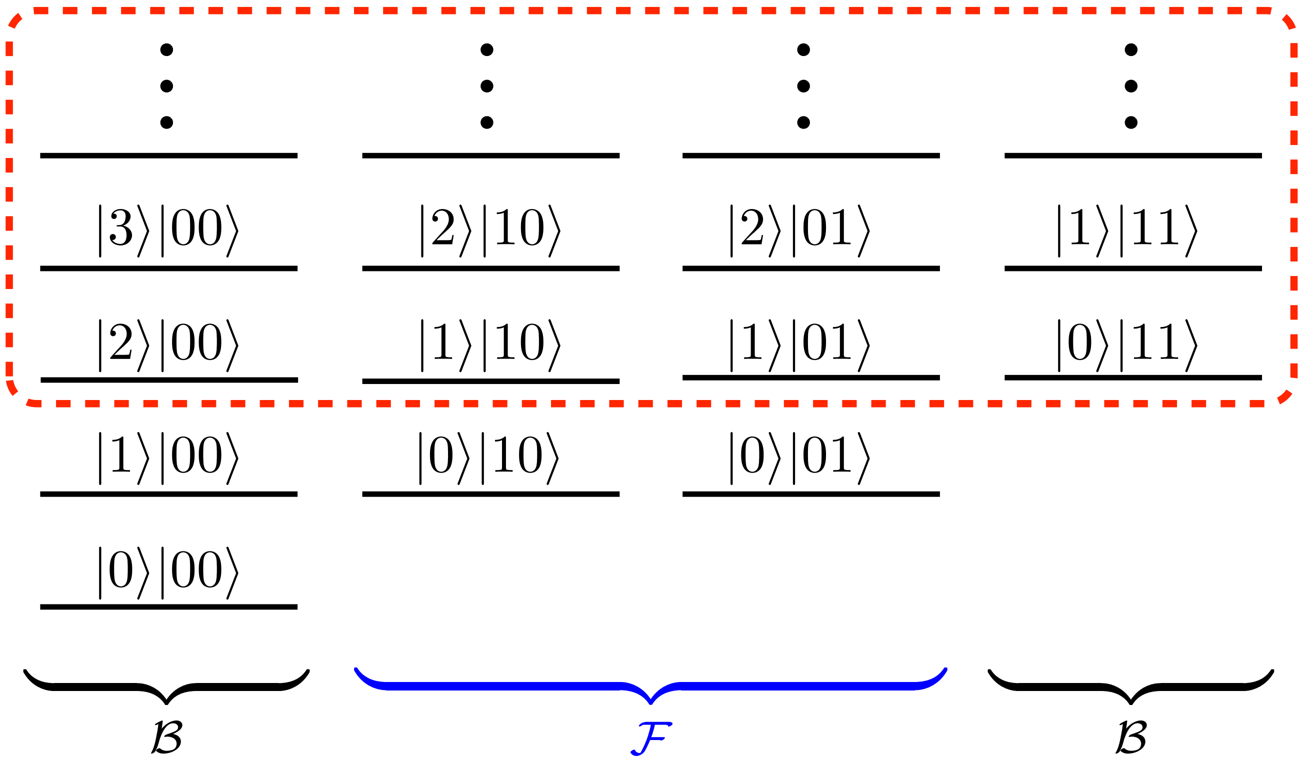
No Hagedorn growth without supersymmetry in relative d.o.s.

Euler's pentagonal number thm: The difference in populations is $0, \pm 1$!

The twisted generating function counts the partitions of the integers, but in a "graded" way. The partitions with an even number of terms are counted positively, whereas partitions with an odd number are counted negatively. The levels at which the $(-1)^n$-mismatch occurs are called generalized pentagonal numbers, $p_n^{\pm} = (3n^2 \pm n)/2$.

Why does it happen? Again.....

Stringy toy model: Collection of infinitely many SHO. Take a single SHO and take N_f =2



Partition function: all states contribute.

Twisted partition function: for N_f=1, only the ground state contributes. (Cohomology group is the ground state).
for N_f=2, only the first two level contribute for each SHO! (Coh is lowest two levels.)
Reason: Despite lack of supersymmetry, we still have fermionic charges. Not in contradiction to HLS, because the theory is free. [Recall, so is large-N QCD(adj)].

$$Q_i = \sum_{n \in \mathbb{N}} \sqrt{n}\, a_n^\dagger f_{i\,n},$$

Concluding remarks

- Volume independence is an exact property of $N = \infty$ center symmetric quantum gauge theories.

- Confronted with Hagedorn growth of density of states, it leads to the conclusion of exact spectral degeneracy, with a most likely explanation of emergent fermionic symmetry.

- To my mind, QCD(adj) is gold, both for reliable semi-classics and volume independence! At the same time, it is not detached from the real world QCD, thanks to orientifold equivalence. I hope more people show interest to this theory.

- The emergent fermionic symmetry is a rather non-trivial prediction coming out of volume independence, answering the criticism "what the volume independence is good for?"

Sherlock Holmes to Watson

It is an old maxim of mine that when you have excluded the impossible, whatever remains, however improbable, must be the truth.