Studying and removing effects of fixed topology in a quantum mechanical model

Arthur Dromard[1] & Marc Wagner[1]

[1]Department of Theoretical Physics
Goethe Universität, Frankfurt am Main

Lattice 2013

Outline

Outline

Goal of the study

Goal: Develop and test methods with which one can obtain physics results at unfixed topology from fixed topology simulations.

Frozen topology

Topology is expected to freeze for a small lattice spacings $a \lesssim 0.05 fm$[a]

[a] Lattice QCD without topology barriers - Luscher, Martin et al. JHEP 1107 (2011) 036 arXiv:1105.4749 [hep-lat] CERN-PH-TH-2011-116 and references there in.

Other motivations

Overlap fermions

With current algorithm topology is typically frozen, when using overlap fermions[a]

[a] Two-flavor QCD simulation with exact chiral symmetry - JLQCD Collaboration (Aoki, S. et al.) Phys.Rev. D78 (2008) 014508 arXiv:0803.3197 [hep-lat] KEK-CP-208, YITP-08-7

Mixed action with overlap valence and Wilson sea quarks

Avoiding problems due to the presence of zero-modes in the valence which are absent in the sea[a]

[a] Continuum Limit of Overlap Valence Quarks on a Twisted Mass Sea - Cichy, Krzysztof et al. Nucl.Phys. B847 (2011) 179-196 arXiv:1012.4412 [hep-lat] DESY-10-240, FTUAM-10-35, SFB-CPP-10-130

Fixed topology simulation

- For Simulation at fixed topology the partition function and the two points correlator function are defined by:

$$Z_Q = \int DAD\psi D\bar{\psi}\delta_{Q,Q[A]}e^{-S[A,\bar{\psi},\psi]}$$

$$C_Q(t) = \frac{1}{Z_Q}\int DAD\psi D\bar{\psi}\delta_{Q,Q[A]}O(t)O(0)e^{-S[A,\bar{\psi},\psi]}.$$

- Using Saddle point approximation for large volume :

$$C_Q(t) = A_Q e^{-M_Q t}$$

- With M_Q is defined as a mass at fixed Q and equals to[1]:

$$\boxed{M_Q \approx M(0) + \frac{M^{''}(0)}{2\chi_T V}\left(1 - \frac{Q^2}{\chi_T V}\right)}$$

$M(0)$ is the physical mass, M'' the second derivative of the mass by θ, V the space time volume and χ_T the topological susceptibility.

[1] Brower, R. et al. Phys.Lett. B560 (2003) 64-74 hep-lat/0302005 DUKE-TH-02-229, MIT-CTP-3201

Outline

- The method:
  1. Compute M_Q for different topological charges and different volumes.
  2. Extract the physical parameters $M(0)$, $M^{(2)}(0)$ and χ_T by fitting the data with suitable equation.

- Our work on the method:
  - Calculate explicit improved expression for $C_Q(t)$, $M_Q(t)$ from equations available in the literature. That expression should allow us to apply the previous method.

  - Testing the method

- One can derive the equation to the next orders for all observables[2].
- We have apply this result on the two points correlation function keeping in mind the idea of fitting the equation.

$$
\begin{aligned}
C_Q = & \frac{\alpha(0)}{\sqrt{(1+x_2)}} e^{-VTC_0(x_0)} \\
& \times exp\left(\frac{1}{C_2 VT}\left[-\frac{1}{8}\frac{C_4}{C_2}\left(\frac{(1+x_4)}{(1+x_2)^2}-1\right)+\frac{Q^2}{2}\frac{x_2}{(1+x_2)}\right]\right) \\
& \times exp\left(\frac{1}{(C_2 VT)^2}\left[\frac{1}{12}\frac{C_4^2}{C_2^2}\left(\frac{(1+x_4)^2}{(1+x_2)^4}-1\right)-\frac{1}{48}\left(\frac{C_6}{C_2}\frac{(1+x_6)}{(1+x_2)^3}-1\right)+\frac{Q^2}{4}\frac{C_4}{C_2}\left[\frac{1+x_4}{(1+x_2)^3}-1\right]\right]\right) \\
& \times exp\left(\frac{1}{C_2^3 V^3 T^3}\left[-\frac{1}{384}\frac{C_8}{C_2}\left(\frac{(1+x_8)}{(1+x_2)^4}-1\right)+\frac{19}{768}\frac{C_4 C_6}{C_2^2}\left(\frac{(1+x_4)(1+x_6)}{(1+x_2)^5}-1\right)\right]\right) \\
& \times exp\left(\frac{1}{C_2^3 V^3 T^3}\left[-\frac{5021}{9216}\frac{C_4^3}{C_2^3}\left(\frac{(1+x_4)^3}{(1+x_2)^6}-1\right)-\frac{1}{24}\frac{Q^4}{(C_2 VT)^3}\frac{C_4}{C_2}\left[\frac{1+x_4}{(1+x_2)^4}-1\right]\right]\right) \\
& \times exp\left(\frac{1}{C_2^3 V^3 T^3}\left[\frac{Q^2}{16}\frac{C_6}{C_2}\left(\frac{(1+x_6)}{(1+x_2)^4}-1\right)-\frac{Q^2}{3}\frac{C_4^2}{C_2^2}\left(\frac{(1+x_4)^2}{(1+x_2)^5}-1\right)\right]\right) \\
& + O\left(\frac{Q^4}{(D_2 TV)^4}, \frac{1}{(D_2 TV)^4}\right)
\end{aligned}
$$

- Here, the 8 parameters are: $M(0)$, $M^{(2)}$, χ_T, $M^{(4)}$, $E^{(4)}$, $M^{(6)}$, $E^{(6)}$ and $E^{(8)}$.

[2] Aoki, Sinya et al. Phys.Rev. D76 (2007) 054508 arXiv:0707.0396 [hep-lat]

Improved equation

- One can derive the equation to the next orders for all observables[2].
- We have apply this result on the two points correlation function keeping in mind the idea of fitting the equation.

$$
\begin{aligned}
C_Q =& \frac{\alpha(0)}{\sqrt{(1+x_2)}} e^{-VTC_0(x_0)} \\
&\times exp\left(\frac{1}{C_2 VT}\left[-\frac{1}{8}\frac{C_4}{C_2}\left(\frac{(1+x_4)}{(1+x_2)^2}-1\right)+\frac{Q^2}{2}\frac{x_2}{(1+x_2)}\right]\right) \\
&\times exp\left(\frac{1}{(C_2 VT)^2}\left[\frac{1}{12}\frac{C_4^2}{C_2^2}\left(\frac{(1+x_4)^2}{(1+x_2)^4}-1\right)-\frac{1}{48}\left(\frac{C_6}{C_2}\frac{(1+x_6)}{(1+x_2)^3}-1\right)+\frac{Q^2}{4}\frac{C_4}{C_2}\left[\frac{1+x_4}{(1+x_2)^3}-1\right]\right]\right) \\
&\times exp\left(\frac{1}{C_2^3 V^3 T^3}\left[-\frac{1}{384}\frac{C_8}{C_2}\left(\frac{(1+x_8)}{(1+x_2)^4}-1\right)+\frac{19}{768}\frac{C_4 C_6}{C_2^2}\left(\frac{(1+x_4)(1+x_6)}{(1+x_2)^5}-1\right)\right]\right) \\
&\times exp\left(\frac{1}{C_2^3 V^3 T^3}\left[-\frac{5021}{9216}\frac{C_4^3}{C_2^3}\left(\frac{(1+x_4)^3}{(1+x_2)^6}-1\right)-\frac{1}{24}\frac{Q^4}{(C_2 VT)^3}\frac{C_4}{C_2}\left[\frac{1+x_4}{(1+x_2)^4}-1\right]\right]\right) \\
&\times exp\left(\frac{1}{C_2^3 V^3 T^3}\left[\frac{Q^2}{16}\frac{C_6}{C_2}\left(\frac{(1+x_6)}{(1+x_2)^4}-1\right)-\frac{Q^2}{3}\frac{C_4^2}{C_2^2}\left(\frac{(1+x_4)^2}{(1+x_2)^5}-1\right)\right]\right) \\
&+ O\left(\frac{Q^4}{(D_2 TV)^4}, \frac{1}{(D_2 TV)^4}\right)
\end{aligned}
$$

- Here, the 8 parameters are:
  $M(0)$, $M^{(2)}$, χ_T, $M^{(4)}$, $E^{(4)}$, $M^{(6)}$, $E^{(6)}$ and $E^{(8)}$.

[2] Aoki, Sinya et al. Phys.Rev. D76 (2007) 054508 arXiv:0707.0396 [hep-lat]

Outline

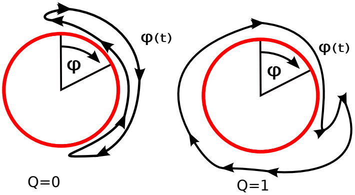
- A simple quantum mechanical model : particle on a circle with square well potential:

$$S_E(\theta) \equiv \int d\tau \;\; \frac{I}{2}\dot{\varphi}^2 - V_{Well} - \underbrace{i\theta \frac{1}{2\pi} \int_0^T dt\, \dot{\varphi}}_{=iQ\theta}$$

$$\Rightarrow H(\theta) \equiv \frac{1}{2I}\left(p - \frac{\theta}{2\pi}\right)^2 + \begin{cases} 0 & \varphi \in]-\pi + L, \pi - L[\\ -V_0 & \varphi \in [\pi - L, -\pi + L] \end{cases}$$

- $\theta \to -\theta$ symmetry as in QCD
- In contrast to QCD, one can calculate observable values analytically.

$\Rightarrow$Ideal test for the equation and the method



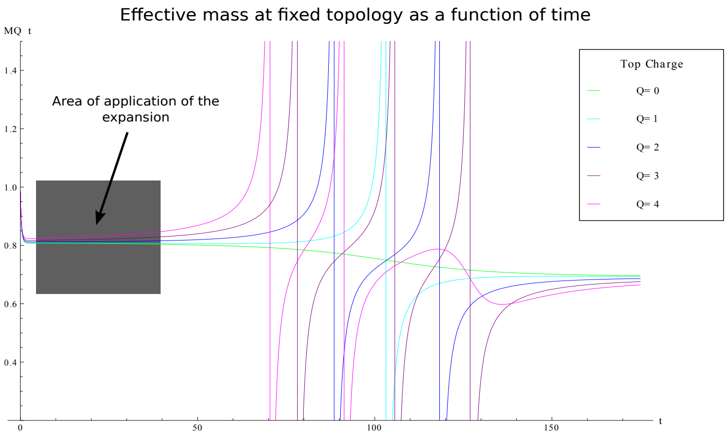
+ Testing equations

- Range of validity : the upper limits for $\frac{M^{(2)}t}{\chi_T V}$, $\frac{1}{\chi_T V}$ and for $\frac{Q^2}{\chi_T V}$
- Possibility to extract $M(\theta = 0), \chi_T$ and $M^{(2)}(\theta = 0)$ by fitting the curve. For an exploratory study in the Schwinger model cf. -Bietenholz[3] -Czaban poster

+ Method for testing :

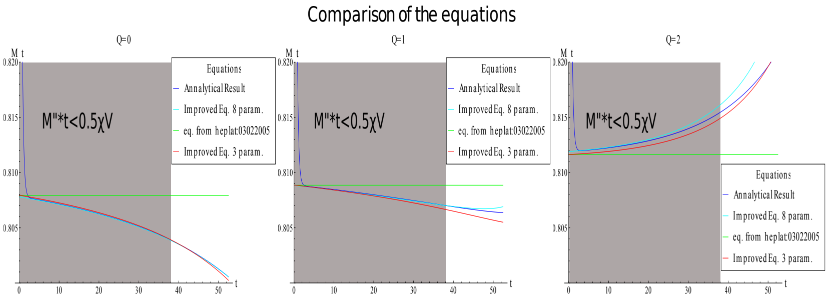
1. Calculate $E_n(\theta) \rightarrow M(\theta) = E_1(\theta) - E_0(\theta), \chi_T, \ldots$
2. Calculate Z_Q, C_Q and M_Q for different Q and different V
    1. Using exact equations
    2. Allow us to study the behavior beyond the validity of analytical expansion of $M(V, Q)$
3. Fit $M(V, Q)$ with the analytical expansions
    1. Get the parameters: $M(\theta = 0), \chi_T,$ and $M^{''}(0)$.
    2. Cross check with the result obtained in 1.

[3] Topological Summation in Lattice Gauge Theory - Bietenholz, Wolfgang et al. J.Phys.Conf.Ser. 378 (2012) 012041 arXiv:1201.6335 [hep-lat]

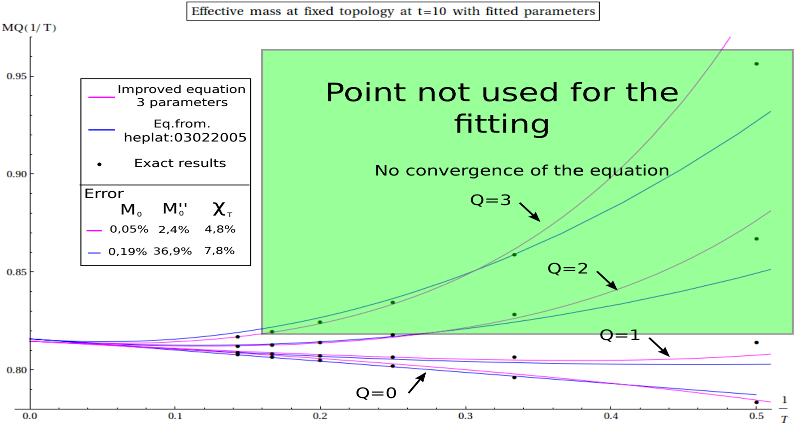


$$M_Q(t) := -\frac{d}{dt} ln\left(C_Q(t)\right) \tag{1}$$

- In contrast to QFT at unfixed Top.The effective mass is not a plateau.
- Expansion is valid in the plateau like region at not too big t



- Equations are good approximation for small t
- Improved equations reproduce the behavior of the mass with much better precision
  - The deviation from a plateau is reproduced
- Approximate range of validity of equations :
  - $M^{(2)}(\theta = 0)t < 0.5\chi_T V$ and $Q^2 < \chi_T V$



- Error : Difference between exact analytical result and results obtained by fitting
- Results are promising
- Using the improved formula reduced significantly the errors

Summary

1. The method for extracting the masses and the topological susceptibility from fixed top simulations is promising
2. The criteria to apply it are $\chi_T V > 2$, $\chi_T V > Q^2$ and $\chi_T V > 2M^{(2)} t$
   - For typical QCD simulations[4] $\chi_T V \approx O(10)$
3. The extraction of the parameters are improved by using improved equation.

- Outlook
  - Test the method on Schwinger Model with Wilson fermions (see C.Czaban poster)
  - Test the method on Full QCD.

[4] Topological susceptibility in Lattice QCD with unimproved Wilson fermions - Chowdhury, Abhishek et al. Phys.Lett. B707 (2012) 228-232 arXiv:1110.6013 [hep-lat]