Extended Mean Field Study of Complex ϕ^4-Theory at Finite Density and Temperature

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Motivation

- More matter than antimatter $\rightarrow$ chemical potential

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- Toy models give valuable insight and experience before applying new methods to QCD

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Leads to conserved Noether current and charge,

$$j_\nu = i(\varphi^*(x)\partial_\nu \varphi(x) - \partial_\nu \varphi^*(x)\varphi(x)), \quad Q = \int \mathrm{d}^3\vec{x}\, j_0(\vec{x})$$

Chemical potential, μ couples to conserved charge, Q. After discretizing, $x = a(n_x, n_y, n_z, n_t)$, we obtain the standard lattice action:

$$S = \sum_x \left(\eta |\varphi_x|^2 + \lambda |\varphi_x|^4 - \sum_{\nu=1}^{4} \left[e^{-\mu\delta_{\nu,t}} \varphi_x^* \varphi_{x+\hat{\nu}} + e^{\mu\delta_{\nu,t}} \varphi_x^* \varphi_{x-\hat{\nu}} \right] \right),$$

with $\eta = m_0^2 + 8$.

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Sign problem!

Let us analyze the model using mean field theory.

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$$S_{\mathrm{MF}} = \eta|\varphi_0|^2 + \lambda|\varphi_0|^4 - \sum_{\nu=1}^{4}\left[e^{\pm\mu\delta_{\nu,t}}\varphi_0^*\phi + e^{\pm\mu\delta_{\nu,t}}\varphi_0\phi\right]$$

- $\varphi_{x\neq 0} = \phi \in \mathbb{R}$

Let us analyze the model using mean field theory.

$$S_{\mathrm{MF}} = \eta|\varphi_0|^2 + \lambda|\varphi_0|^4 - 4\mathrm{Re}[\varphi_0]\phi(3 + \cosh(\mu))$$

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- $\varphi_{x\neq 0} = \phi \in \mathbb{R}$
- Action is purely real.
- Second order phase transition at critical chemical potential,

$$3 + \cosh \mu_c(\eta, \lambda) = \frac{\sqrt{\lambda}}{2\frac{\mathrm{Exp}(-K^2)}{\sqrt{\pi}\mathrm{Erfc}(K)} - 2K},$$

$$K = \eta/\left(2\sqrt{\lambda}\right).$$

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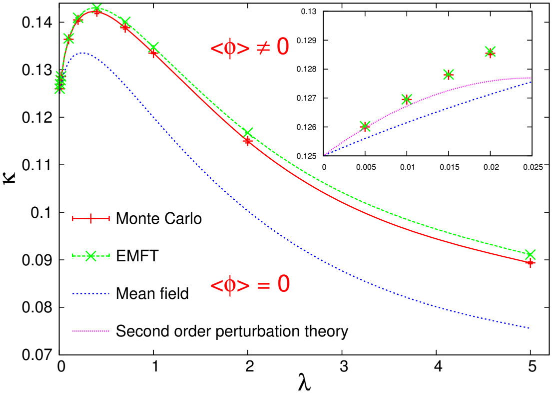
- Only local variables are accessible, we get no information on for example the Green's function.
- No dependence on system size, i.e. we are restricted to zero temperature.
- We can do better!

Extended Mean Field Theory (EMFT)



(κ, λ)-phase diagram of real φ^4-theory in four dimensions[1].

[1] O. Akerlund, P. de Forcrand, A. Georges and P. Werner, hep-lat/1305.7136

EMFT equations

Again focus on a single lattice site ($x = 0$),

$$S = \eta|\varphi_0|^2 + \lambda|\varphi_0|^4 - \sum_{\nu=1}^{4}\left[e^{\mp\mu\delta_{\nu,t}}\varphi_0^*\varphi_{\pm\hat{\nu}} + e^{\pm\mu\delta_{\nu,t}}\varphi_0^*\varphi_{\mp\hat{\nu}}\right] + S_{\text{ext}}$$

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$$S = \frac{\eta}{2}\vec{\varphi}_0^\dagger\vec{\varphi}_0 + \frac{\lambda}{4}(\vec{\varphi}_0^\dagger\vec{\varphi}_0)^2 \underbrace{- \sum_{\nu=1}^{4} \vec{\varphi}_{\pm\hat{\nu}}^\dagger E(\pm\mu\delta_{\nu,t})\vec{\varphi}_0}_{-\Delta S} + S_{\text{ext}}$$

Introducing,

$$\vec{\varphi}^\dagger = (\varphi^*, \varphi), \quad E(x) = \begin{pmatrix} e^{-x} & 0 \\ 0 & e^{x} \end{pmatrix}, \quad G = \begin{pmatrix} \langle\varphi^*\varphi\rangle_c & \langle\varphi\varphi\rangle_c \\ \langle\varphi^*\varphi^*\rangle_c & \langle\varphi\varphi^*\rangle_c \end{pmatrix}$$

Expanding $\vec{\varphi}$ around its (real) mean,

$$\vec{\varphi}^\dagger = \delta\vec{\varphi}^\dagger + (\phi, \phi)$$

yields (up to a constant),

$$S = \frac{\eta}{2}\vec{\varphi}_0^\dagger\vec{\varphi}_0 + \frac{\lambda}{4}(\vec{\varphi}_0^\dagger\vec{\varphi}_0)^2 - 2\vec{\phi}^\dagger\vec{\varphi}_0(3+\cosh(\mu)) - \underbrace{\sum_{\pm\nu}\delta\vec{\varphi}_{\hat{\nu}}^\dagger E(\pm\mu\delta_{\nu,t})\vec{\varphi}_0}_{-\delta S} + S_{\text{ext}}$$

Formal integration over the field at all lattice sites except the origin results in replacing $-\delta S$ by its cumulant expansion w.r.t. S_{ext}.

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The lowest nontrivial contribution is,

$$\frac{1}{2}\left\langle \sum_{\pm\nu} \delta\vec{\varphi}^{\dagger}_{\hat{\nu}} E(\pm\mu\delta_{\nu,t})\delta\vec{\varphi}_0 \sum_{\pm\rho} \delta\vec{\varphi}^{\dagger}_{\hat{\rho}} E(\pm\mu\delta_{\rho,t})\delta\vec{\varphi}_0 \right\rangle_{S_{\text{ext}}} = \frac{1}{2}\delta\vec{\varphi}^{\dagger}_0 \Delta \delta\vec{\varphi}_0,$$

where $\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12} & \Delta_{11} \end{pmatrix}$ is a real, symmetric, matrix.

Putting all together we obtain the EMFT action,

$$S_{\text{EMFT}} = \frac{1}{2}\vec{\varphi}^{\dagger}\left(\eta \mathbf{I} - \Delta\right)\vec{\varphi} + \frac{\lambda}{4}(\vec{\varphi}_0^{\dagger}\vec{\varphi}_0)^2 - \vec{\phi}^{\dagger}\vec{\varphi}_0(2(3+\cosh(\mu)) - \Delta_{11} - \Delta_{12}).$$

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ϕ and Δ are to be self-consistently determined.

Self-consistency conditions

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  - $Z_{\text{EMFT}} = \int \mathrm{d}\varphi \exp\left(-S_{\text{EMFT}}\right)$ is the generator of all local diagrams (involving φ_0).
  - $G(\vec{r} = \vec{0}, t = 0) = G_{\text{EMFT}}$.

Green's functions

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- Crucial approximation: Neglect k-dependence of $\widetilde{\Sigma}(k)$ and set it equal to Σ_{EMFT}
- $\widetilde{G}^{-1}(k) \approx G_{\text{EMFT}}^{-1} + \Delta - 2\sum_\nu \cos(k_\nu - i\mu\delta_{\nu,t})\mathbb{I}$

Self-consistency equations

- $$\phi = \langle \varphi \rangle_{S_{\text{EMFT}}}$$
- $$G_{\text{EMFT}} = \int \mathrm{d}^4 k [G_{\text{EMFT}}^{-1} + \Delta - 2 \sum_{\nu} \cos(k_\nu - i\mu\delta_{\nu,t}) \mathbb{I}]^{-1}$$

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$$G_{\mathrm{EMFT}} = \begin{pmatrix} \langle\varphi^*\varphi\rangle_{S_{\mathrm{EMFT}}} & \langle\varphi\varphi\rangle_{S_{\mathrm{EMFT}}} \\ \langle\varphi^*\varphi^*\rangle_{S_{\mathrm{EMFT}}} & \langle\varphi\varphi^*\rangle_{S_{\mathrm{EMFT}}} \end{pmatrix} - \vec{\phi}\vec{\phi}^\dagger$$

$$S_{\mathrm{EMFT}} = \frac{1}{2}\vec{\varphi}^\dagger (\eta\mathbb{I} - \Delta)\vec{\varphi} + \frac{\lambda}{4}(\vec{\varphi}_0^\dagger\vec{\varphi}_0)^2 - \vec{\phi}^\dagger\vec{\varphi}_0(2(3+\cosh(\mu)) - \Delta_{11} - \Delta_{12}).$$

Finite temperature

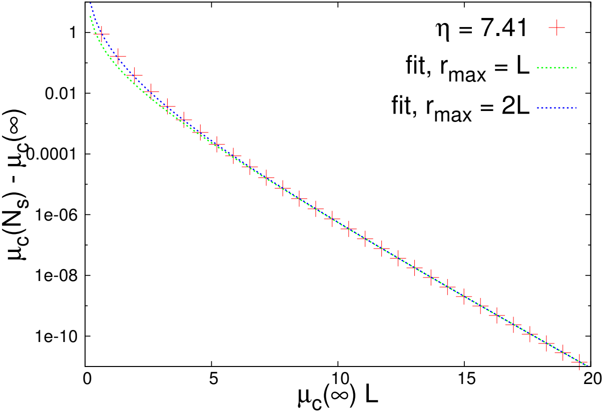
Finite temperature (volume) is conceptually trivial:

$$k_\nu \in [-\pi, \pi] \to \frac{2\pi}{N_\nu} n_\nu, \;\; n_\nu \in \{-N_\nu/2, \dots, N_\nu/2 - 1\}$$

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}k_\nu}{2\pi} \to \frac{1}{N_\nu} \sum_{n_\nu = -N_\nu/2}^{N_\nu/2-1}$$

Finite volume effects

$$V(r) \propto \frac{\exp(-mr)}{r^{3/2}}$$



Critical chemical potential at $T = 0$ and $\lambda = 1$

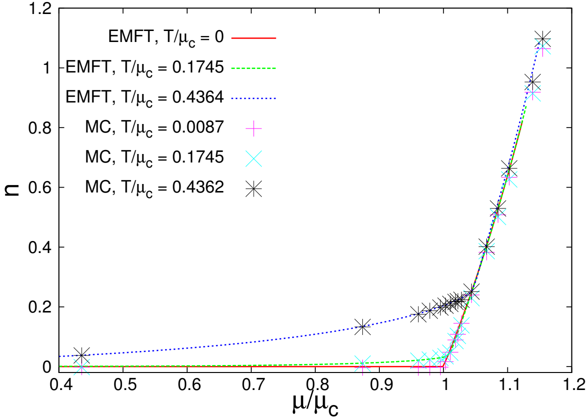
η	9.00	7.44
MF	1.12908	-
EMFT	1.14582	0.17202
Monte Carlo[2]	1.146(1)	0.170(1)
Complex Langevin[3]	1.15(?)	-

[2] C. Gattringer and T. Kloiber, Nucl.Phys. B869, 56-73, (2013), hep-lat/1206.2954

[3] G. Aarts, PoS, LAT2009, hep-lat/0910.3772

Density[4]

$$n = \frac{T}{V}\frac{\partial Z}{\partial \mu}$$



$$\eta = 9$$

[4]Monte Carlo data from: C. Gattringer and T. Kloiber, Nucl.Phys. B869, 56-73, (2013), hep-lat/1206.2954

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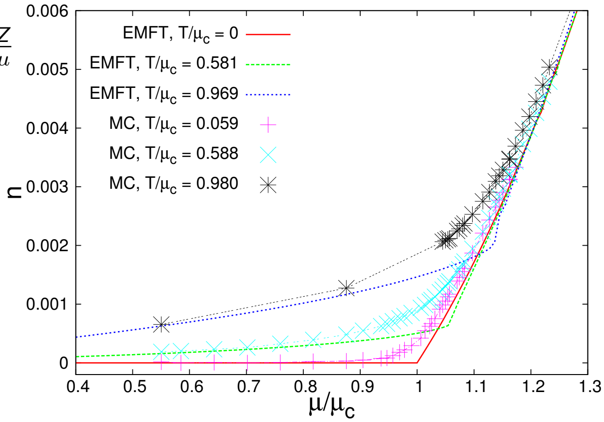

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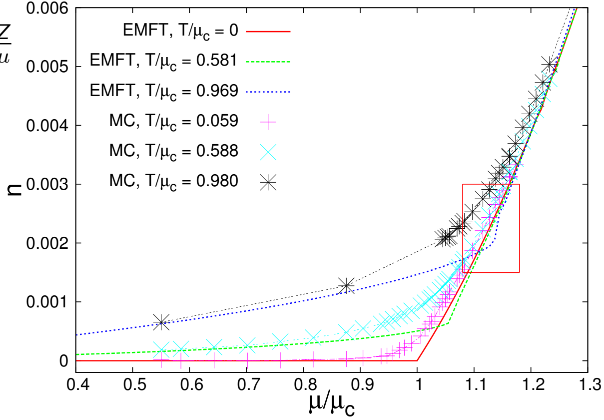

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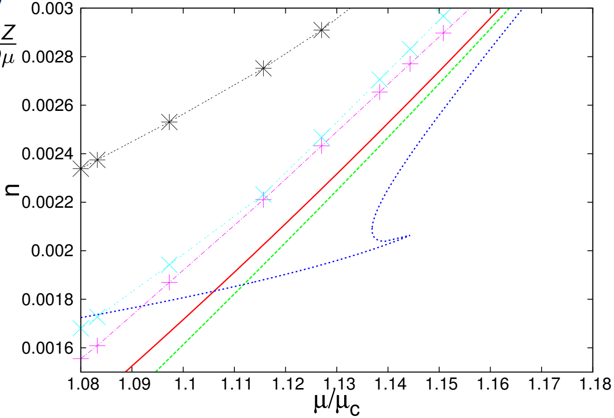

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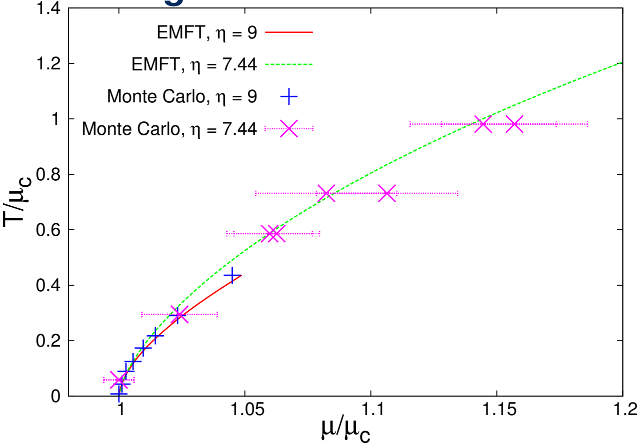
(μ/T)-Phase diagram[5]



[5]Monte Carlo data from: C. Gattringer and T. Kloiber, Nucl.Phys. B869, 56-73, (2013), hep-lat/1206.2954

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- QCD?

Thank you for your attention!

Efficient k-integration

$$G(0) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \widetilde{G}^{-1}(k).$$

$$\widetilde{G}(k) = \left[G_{\text{EMFT}}^{-1} + \Delta - 2 \sum_{\nu=1}^{d} \cos\left(k_\nu - i\mu\delta_{\nu,t}\right) \mathbb{I}_{N\times N} \right]^{-1}$$
$$= \left[A - \epsilon(k,\mu)\mathbb{I}_{N\times N}\right]^{-1},$$

$$\frac{1}{a-\epsilon(k,\mu)} = \int_0^\infty \mathrm{d}\tau \exp\left(-\tau a\right) \prod_{\nu=1}^{d} \exp\left(2\tau \cos(k_\nu - i\mu\delta_{\nu,t})\right)$$

$$I_0(x) = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi} \exp(x\cos(k+z))$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{a-\epsilon(k,\mu)} = \int_0^\infty \mathrm{d}\tau \exp\left(-\tau a\right) I_0(2\tau)^d.$$

$$\widetilde{G}^{-1}(k) = \frac{\mathrm{Adj}(A - \epsilon(k,\mu)\mathbb{I})}{\det\left(A - \epsilon(k,\mu)\mathbb{I}\right)},$$

Partial fractions decomposition gives,

$$\widetilde{G}^{-1}(k)_{ij} = \sum_{n=1}^{N} \frac{B_n^{ij}}{\lambda_n - \epsilon(k,\mu)},$$

λ are eigenvalues of A.

$$G(0)_{ij} = \sum_{n=1}^{N} B_n^{ij} C_n, \quad C_n = \int_0^\infty \mathrm{d}\tau \exp\left(-\tau\lambda_n\right) I_0(2\tau)^d$$

Density

$$n = \frac{\partial Z}{\partial \mu}$$
$$= 2\sinh\mu\,\langle\varphi\rangle^2 + 2\left(\sinh\mu\int\frac{\mathrm{d}^4 k}{(2\pi)^4}\mathrm{Re}[\,\langle\varphi^*(k)\varphi(k)\rangle_c]\cos(k_4)\right.$$
$$\left. + \cosh\mu\int\frac{\mathrm{d}^4 k}{(2\pi)^4}\mathrm{Im}[\,\langle\varphi^*(k)\varphi(k)\rangle_c]\sin(k_4)\right)$$

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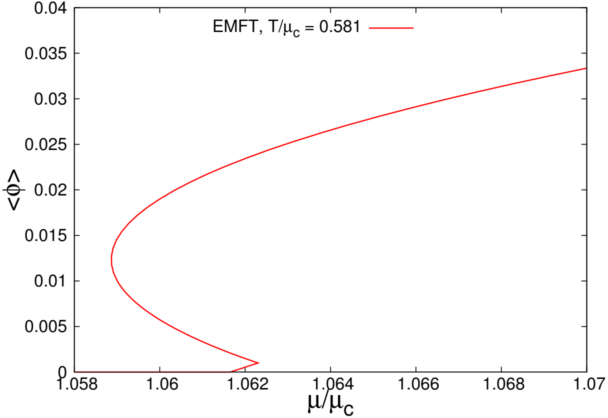
First order jump



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