The hadronic vacuum polarization with twisted boundary conditions

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Outline

Muon g-2

Finite Volume Issues:
- Need for twisted boundary conditions

Ward-Takahashi Identity:
- Vacuum Polarization no longer transverse

Numerical checks:
- Perturbation Theory
- WTI check configuration by configuration
- Size of additional non-transverse term in VP

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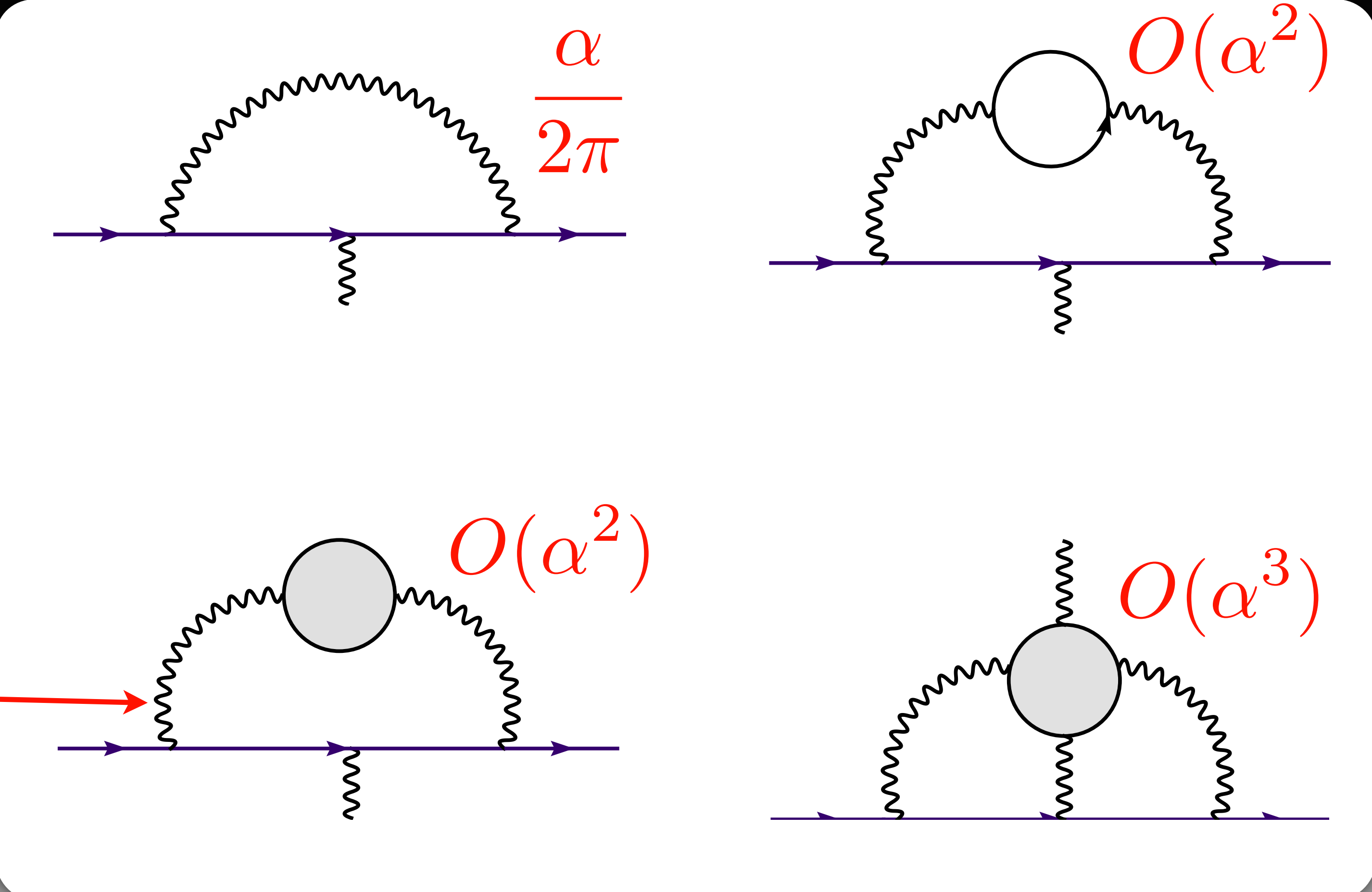
Muon (g-2)/2



Leading QCD contribution

$$a_\mu^{\mathrm{HLO}} = 4\alpha^2 \int_0^\infty dp^2\, f(p^2)\left(\Pi^{\mathrm{em}}(0) - \Pi^{\mathrm{em}}(p^2)\right) ,$$

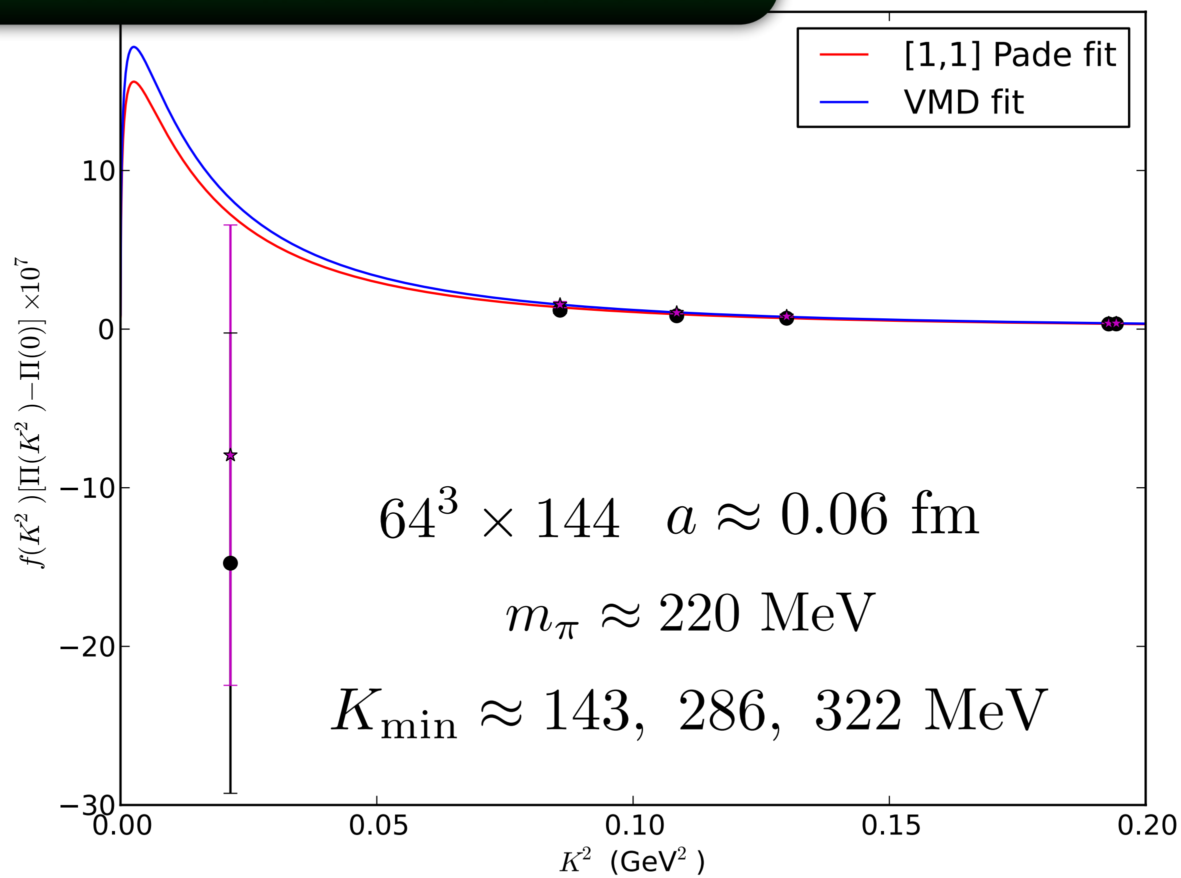
$$f(p^2) = m_\mu^2 p^2 Z^3(p^2)\,\frac{1 - p^2 Z(p^2)}{1 + m_\mu^2 p^2 Z^2(p^2)} ,$$

$$Z(p^2) = \left(\sqrt{(p^2)^2 + 4m_\mu^2 p^2} - p^2\right)/(2m_\mu^2 p^2) ,$$

Many groups working on g-2

C. Aubin and T. Blum, PRD 75, 114502, etc. [Staggered, 2+1]
X. Feng, et al, PRL 107, 081802, etc. [tmWilson, 2; 2+1+1]
M. Della Morte et al, JHEP 1203, 055, etc. [w/ twisted BC's, 2]
Boyle et al, PRD 85 074504 (2012) [DWF, 2+1]
Talks this session...

Problem with finite volume



Peak is at roughly half the muon mass squared

Largest errors on smallest accessible momentum

Small difference in fits correspond to ~12% difference in g-2 [ABGP-PRD 86 (2012) 054509]

Need to access smaller momentum
– Twisted boundary conditions

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Twisted Boundary Conditions

Della Morte et al, JHEP 1204, 055 (2012)
P. F. Bedaque, PLB 593, 82 (2004)
G. M. de Divitiis, R. Petronzio and N. Tantalo, PLB 595, 408 (2004)
C. T. Sachrajda and G. Villadoro, PLB 609, 73 (2005)

Define quarks with twisted boundary conditions:

$$\begin{aligned} q_t(x) &= e^{-i\theta_\mu}\, q_t(x+L_\mu) \\ \overline{q}_t(x) &= \overline{q}_t(x+L_\mu)\, e^{i\theta_\mu} \end{aligned}$$

Momenta are no longer restricted to integer multiples of $2\pi/L$:

$$p_\mu = \frac{2\pi n_\mu + \theta_\mu}{L_\mu}\ , \qquad n_\mu \in \{0, 1, \ldots, L_\mu - 1\}$$

We define two currents (for naïve quarks; for staggered, just replace $\gamma_\mu \to \eta_\mu(x)$).

$$\begin{aligned} j^+_\mu(x) &= \frac{1}{2}\left(\overline{q}(x)\gamma_\mu U_\mu(x) q_t(x+\mu) + \overline{q}(x+\mu)\gamma_\mu U^\dagger_\mu(x) q_t(x)\right) \\ j^-_\mu(x) &= \frac{1}{2}\left(\overline{q}_t(x)\gamma_\mu U_\mu(x) q(x+\mu) + \overline{q}_t(x+\mu)\gamma_\mu U^\dagger_\mu(x) q(x)\right) \end{aligned}$$

Ward-Takahashi Identity

We have a mixed-action theory, with periodic sea quarks and twisted valence quarks (which are thus quenched, and we can formally introduce ghosts to cancel the quark det).

The action is invariant under the isospin-like symmetry:

$$\begin{aligned} \delta q(x) &= i\alpha^+(x) e^{-i\theta x/L} q_t(x) \;, & \delta \overline{q}(x) &= -i\alpha^-(x) e^{i\theta x/L} \overline{q}_t(x) \;, \\ \delta q_t(x) &= i\alpha^-(x) e^{i\theta x/L} q(x) \;, & \delta \overline{q}_t(x) &= -i\alpha^+(x) e^{-i\theta x/L} \overline{q}(x) \;, \end{aligned}$$

$$\theta x/L = \sum_\mu \theta_\mu x_\mu / L_\mu$$

Following the standard procedure, we find that under this symmetry:

$$\begin{aligned} &\sum_\mu \partial_\mu^- \left\langle j_\mu^+(x) j_\nu^-(y) \right\rangle + \frac{1}{2}\delta(x-y) \left\langle \overline{q}_t(y+\nu)\gamma_\nu U_\nu^\dagger(y) q_t(y) - \overline{q}(y)\gamma_\nu U_\nu(y) q(y+\nu) \right\rangle \\ &\quad -\frac{1}{2}\delta(x-\nu-y) \left\langle \overline{q}(y+\nu)\gamma_\nu U_\nu^\dagger(y) q(y) - \overline{q}_t(y)\gamma_\nu U_\nu(y) q_t(y+\nu) \right\rangle = 0 \;, \end{aligned}$$

Second two terms can be written as a total derivative in the zero twist case.

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Ward-Takahashi Identity

A natural definition of the vacuum polarization tensor is

$$\begin{aligned}\Pi^{+-}_{\mu\nu}(x-y) &= \langle j^{+}_{\mu}(x) j^{-}_{\nu}(y)\rangle \\ &\quad -\frac{1}{4}\delta_{\mu\nu}\delta(x-y)\Big(\langle \overline{q}(y)\gamma_\nu U_\nu(y) q(y+\nu) - \overline{q}(y+\nu)\gamma_\nu U^\dagger_\nu(y) q(y)\rangle \\ &\qquad + \langle \overline{q}_t(y)\gamma_\nu U_\nu(y) q_t(y+\nu) - \overline{q}_t(y+\nu)\gamma_\nu U^\dagger_\nu(y) q_t(y)\rangle\Big)\end{aligned}$$

Its divergence is non-zero for non-zero twist, but we have:

$$\sum_\mu \partial^-_\mu \Pi^{+-}_{\mu\nu}(x-y) + \frac{1}{4}\left(\delta(x-y) + \delta(x-\nu-y)\right)\langle j^t_\nu(y) - j_\nu(y)\rangle = 0$$

with

$$\begin{aligned} j_\mu(x) &= \frac{1}{2}\left(\overline{q}(x)\gamma_\mu U_\mu(x) q(x+\mu) + \overline{q}(x+\mu)\gamma_\mu U^\dagger_\mu(x) q(x)\right) \;, \\ j^t_\mu(x) &= \frac{1}{2}\left(\overline{q}_t(x)\gamma_\mu U_\mu(x) q_t(x+\mu) + \overline{q}_t(x+\mu)\gamma_\mu U^\dagger_\mu(x) q_t(x)\right) \;. \end{aligned}$$

There are other definitions of $\Pi_{\mu\nu}$ possible, but the second term in some form will always exist (for this particular choice, it vanishes for zero twist).

Subtraction of contact term

We can decompose this tensor as

$$\Pi^{+-}_{\mu\nu}(\hat{p}) = \left(\hat{p}^2\delta_{\mu\nu} - \hat{p}_\mu\hat{p}_\nu\right)\Pi^{+-}(\hat{p}^2) + \frac{\delta_{\mu\nu}}{a^2}X_\nu(\hat{p})$$

$$\hat{p}_\mu = \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)$$

$$i\sum_\mu \hat{p}_\mu \Pi^{+-}_{\mu\nu}(\hat{p}) = -\cos(ap_\nu/2)\langle j^t_\nu(0)\rangle = i\frac{\hat{p}_\nu}{a^2}X_\nu(\hat{p})$$

so we extract X from the twisted current:

$$X_\nu(\hat{p}) = \frac{i}{2}\cot\left(ap_\nu/2\right)a^3\langle j^t_\nu(0)\rangle$$

Pole in X only when

$$\pi n_\nu + \theta_\nu/2 = k\pi L_\nu/a$$

We can see from dimensional analysis and axis reversal symmetry:

$$\langle j^t_\nu(y)\rangle = -i\frac{c}{a^2}\hat{\theta}_\nu\left(1 + O(\hat{\theta}^2)\right)$$

$$\hat{\theta}_\mu = \theta_\mu/L_\mu$$

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Perturbation Theory (one-loop)

$$\Pi^{+-}_{\mu\nu}(p) \;=\; -\frac{N_c}{V}\sum_k \mathrm{tr}\left[\gamma_\mu \frac{\cos\left(k_\mu + p_\mu/2\right)}{i\sum_\kappa \gamma_\kappa \sin(k_\kappa + p_\kappa) + m}\,\gamma_\nu \frac{\cos\left(k_\nu + p_\nu/2\right)}{i\sum_\lambda \gamma_\lambda \sin k_\kappa + m}\right]$$

$$+\frac{i}{2}\,\delta_{\mu\nu}\,\frac{N_c}{V}\sum_k \mathrm{tr}\left[\gamma_\nu\left(\frac{\sin k_\nu}{i\sum_\kappa \gamma_\kappa \sin k_\kappa + m} + \frac{\sin\left(k_\nu + \hat{\theta}_\nu\right)}{i\sum_\kappa \gamma_\kappa \sin\left(k_\kappa + \hat{\theta}_\kappa\right) + m}\right)\right]$$

$$i\sum_\mu \hat{p}_\mu \Pi^{+-}_{\mu\nu}(p) \;=$$

$$-2i\cos\left(p_\nu/2\right)\frac{N_c}{V}\sum_k\left(\frac{\sin(2k_\nu)}{\sum_\kappa \sin^2 k_\kappa + m^2} - \frac{\sin(2(k_\nu + \hat{\theta}_\nu))}{\sum_\kappa \sin^2\left(k_\kappa + \hat{\theta}_k\right) + m^2}\right)$$

$$=\; 2i\cos\left(p_\nu/2\right)\hat{\theta}\left[\frac{N_c}{V}\sum_k\left(\frac{2\cos(2k_\nu)}{\sum_k \sin^2 k_\kappa + m^2} - \frac{\sin^2(2k_\nu)}{(\sum_k \sin^2 k_\kappa + m^2)^2}\right)\right] + O(\hat{\theta}^3)$$

For realistic lattices the effect can be small;
the free theory result gives:

$$V = 48^3 \times 144,\;\; am = 0.0036,\;\; \theta_i = 0.28\pi$$

$$\langle j^t_\nu(0)\rangle \approx 7.30 \times 10^{-5} i$$

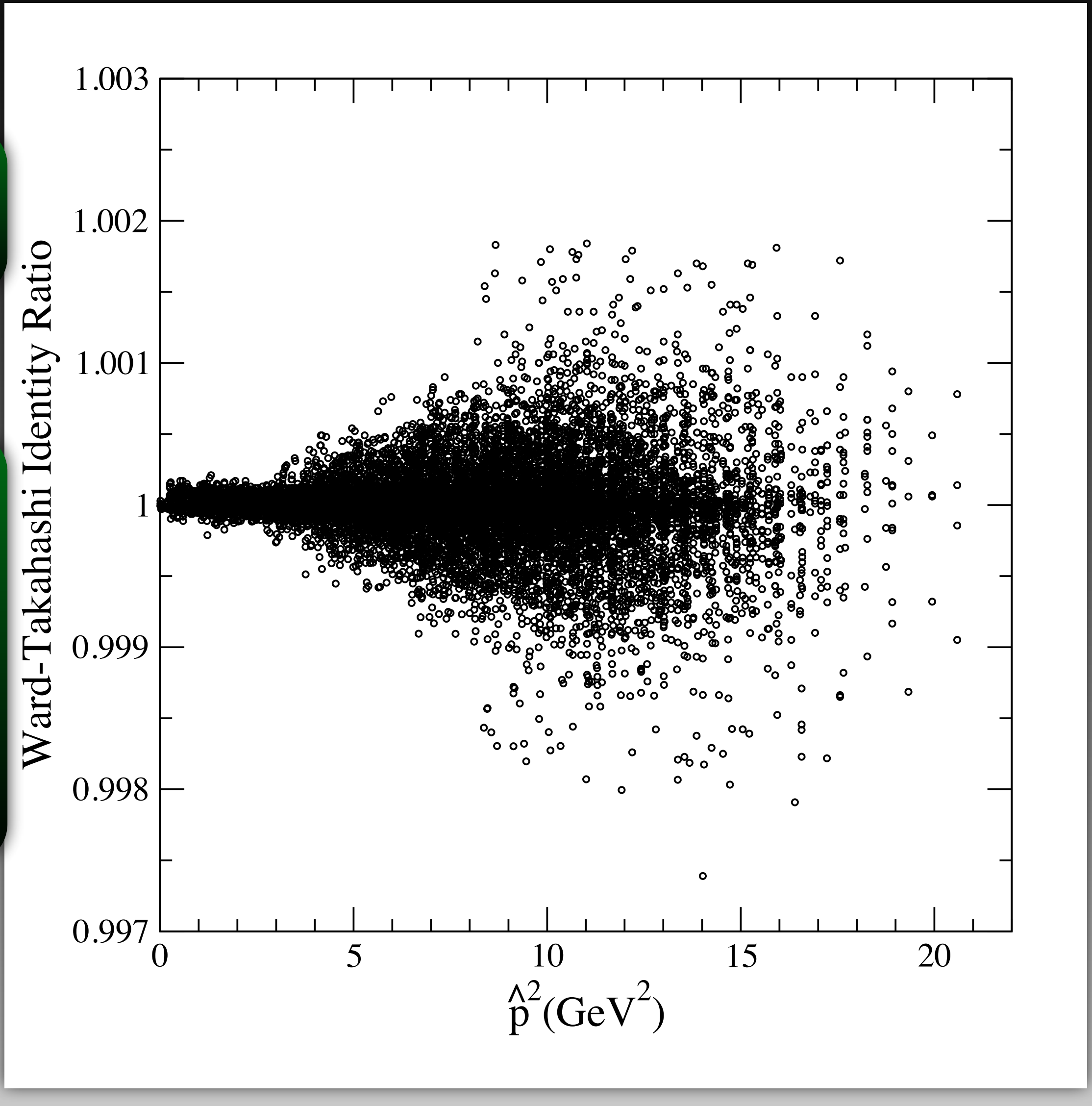
Numerical Checks

A single gauge configuration, with the vacuum polarization evaluated exactly (CG res = 10^{-8})

While our full calculation uses all-mode averaging to reduce statistical errors [Blum, Izubuchi, Shintani, arXiv:1208.4349], this is just the "exact" part.

Plot is the RHS/LHS of

$$i \sum_{\mu} \hat{p}_{\mu} \Pi^{+-}_{\mu\nu}(\hat{p}) \;=\; -\cos\left(ap_{\nu}/2\right)\langle j^{t}_{\nu}(0)\rangle$$



$$V = 48^3 \times 144,\ \ am = 0.0036,\ \ \theta_i = 0.28\pi$$

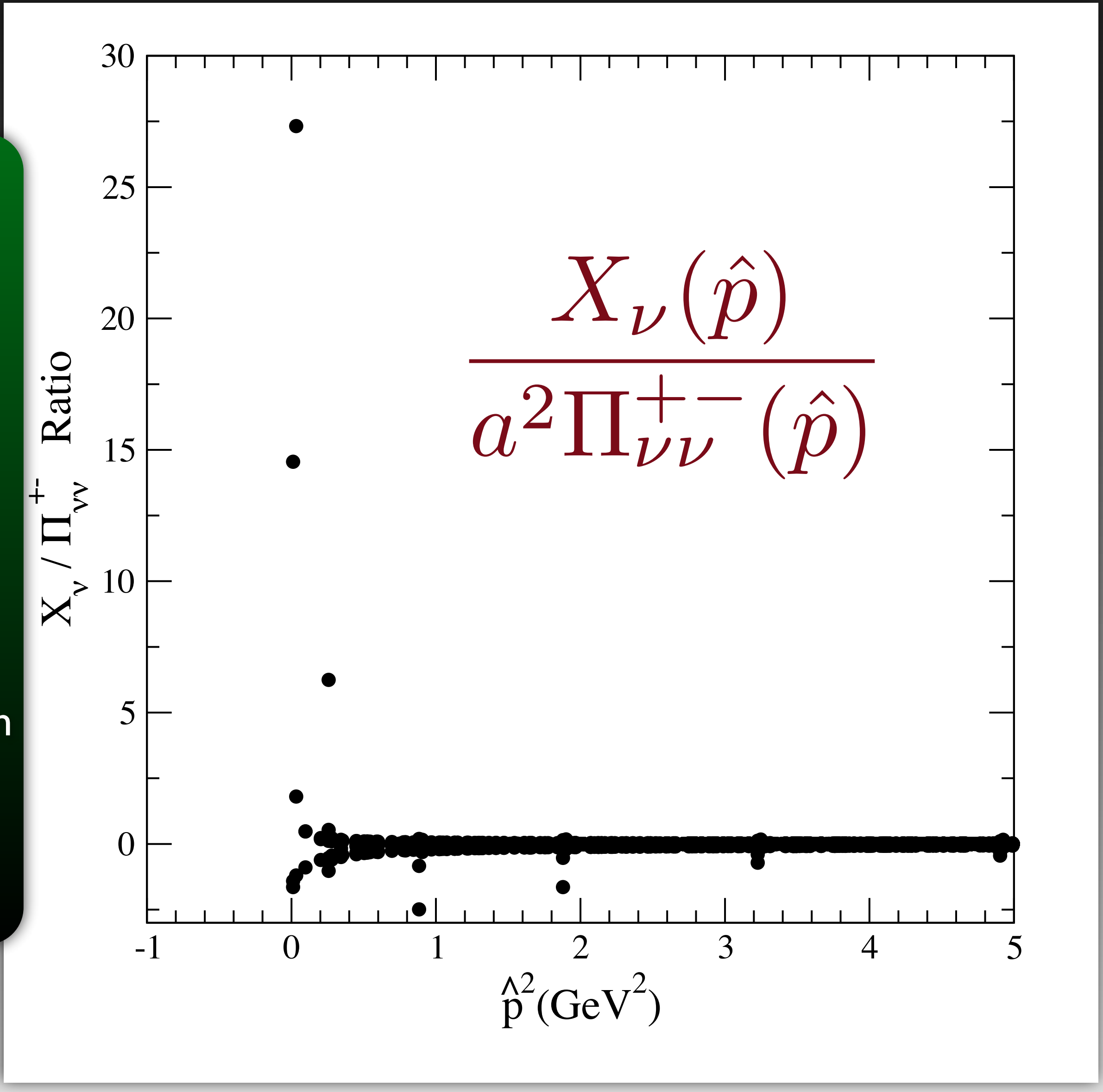
Numerical Checks

Another check – for small momentum X(p) can be very large on a single configuration.

Same configuration as before (and still just the "exact" term).

Averaging over configurations, we find large cancellations so the overall effect is extremely small.

Recall: Even small effects at low momentum can make a significant impact on the extracted value of *g*-2.



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Conclusions

When using twisted boundary conditions, a new term arises in the expression for the vacuum polarization.

While the term may be negligible after averaging over gaugefield configurations, it can be extremely large on a single configuration – recall it is quadratically divergent!

To ensure minimizing uncertainties, which is crucial in extracting the muon *g*-2, it is straightforward to subtract this contamination.