Study of Anomalous Mass Generation in $N_f = 1$ QCD

Luchang Jin

Dept. of Physics, Columbia University

Friday August 2 @ Lattice 2013, Mainz, Germany

Introduction

- $U(1)$ axial symmetry is anomalously broken.
- The quark mass is not protected by symmetry, and could receive arbitrary quantum corrections.
- Should there be an additive renormalization to the quark mass?
  - Convention field theory shows that there isn't, because of the lack of enough small instantons.
  - The quark mass is ambiguous. [Creutz, 2004]

Conclusion of this talk:

- For $N_f = 1$ QCD, there can be additive renormalization of the fermion mass generated by lattice scale instantons for a class of lattice actions.

Outline

- Anomalous mass generation by lattice scale instantons and a numerical study.
- A theoretical estimation of the density of lattice scale instantons and continuum limit.
- Explore the design space of lattice action.

Anomalous mass generation by lattice scale instantons

A mass term would be generated by instanton-like gauge field configurations.['t Hooft, 1976]

$$m_{\text{anom}} \sim \frac{a^2}{m}\rho_a \tag{1}$$

- a is lattice spacing
- m is the input quark mass
- ρ_a is the density of the lattice scale instantons

't Hooft Effective Lagrangian ['t Hooft, 1976]

The fermion zero mode $u_0(x)$ of an instanton of radius R at origin would contribute to the Green's function outside the instanton as:

$$\begin{aligned} \langle q(x)\,\overline{q}(y)\rangle &= u_0(x)\,\frac{1}{m}\,\overline{u}_0(y) \\ &= \frac{R\gamma^\mu x^\mu}{x^4}\,\frac{1}{m}\,\frac{R\gamma^\nu y^\nu}{y^4} \end{aligned} \tag{2}$$

In momentum space, the contribute to the propagator by fermion zero modes is:

$$\begin{aligned} S(p) &= \int \mathrm{d}^4 x e^{-ip\cdot x}\,\langle q(x)\,\overline{q}(0)\rangle \\ &= \frac{1}{\not p}\,\frac{R^2}{m}\,\rho(R)\,\mathrm{d}R\,\frac{1}{\not p} \qquad \left(p \ll \frac{1}{R}\right) \end{aligned} \tag{3}$$

$\rho(R)\,\mathrm{d}R$ is the density of instanton of radius R

't Hooft Effective Lagrangian ['t Hooft, 1976]

A mass term will be generated by instantons upto the scale of instanton size.

$$\frac{R^2}{m}\rho(R)\,\mathrm{d}R \tag{4}$$

We are interested in the "hard" fermion mass which act like a normal fermion mass term in all scale. This mass term cannot be generated by intantons of physical size, it can only come from lattice scale instantons, so

$$m_{\text{anom}} \sim \frac{a^2}{m}\rho_a \tag{5}$$

Anomalous mass generation by lattice scale instantons

The Landau-gauge-fixed fermion propagator using volume source and volume sink takes the following form

$$S(p) = \frac{1/Z_q\,(p)}{i\bar{\not p} + m_R\,(p)} \qquad (6)$$
$$\bar{p}_\mu = \sin p_\mu$$

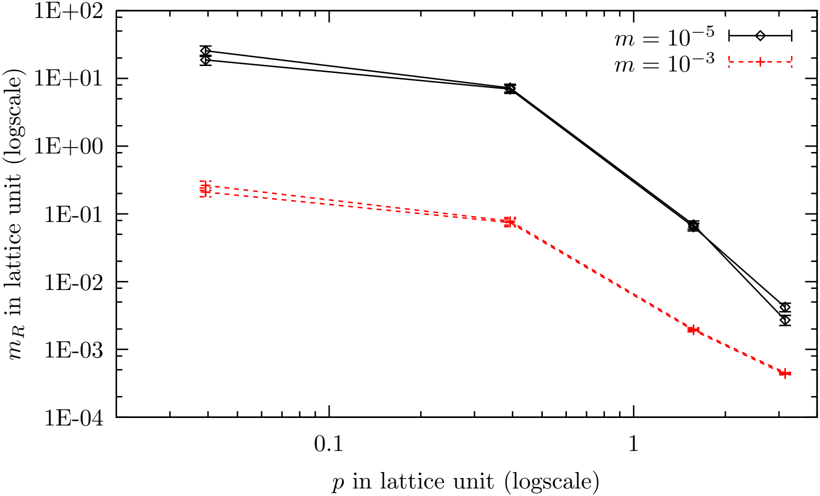
The renormalization factor Z_q and the renormalized mass m_R could be extracted by (RI/MOM [Sturm et al., 2009])

$$m_R\,(p) = 1/Z_q\,(p)\,\mathrm{Tr}\left[S^{-1}(p)\right]$$
$$1/Z_q\,(p) = \frac{ip^2}{\mathrm{Tr}\left[\not p S^{-1}(p)\right]} \qquad (7)$$

$$\mathcal{A} = \frac{\beta}{3}\left(\sum_{x;\mu<\nu}(1-8c_1)\,P^{1\times1}_{\mu\nu} + c_1\sum_{x;\mu\neq\nu}P^{1\times2}_{\mu\nu}\right) \qquad (8)$$

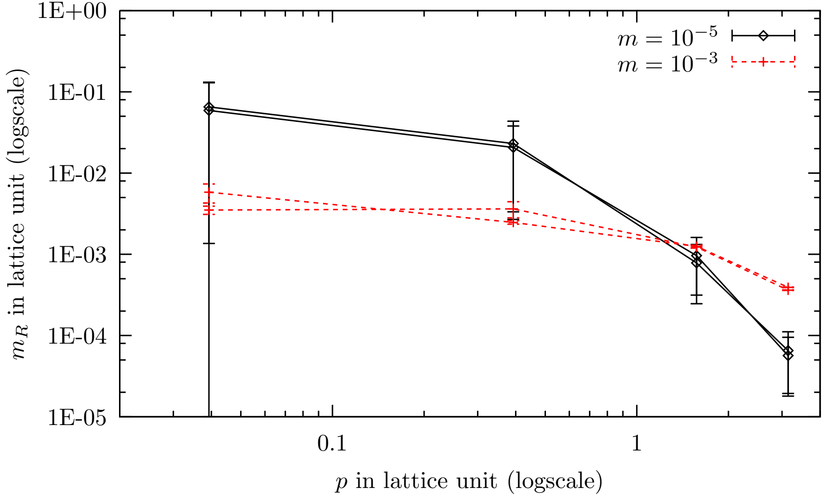
Anomalous mass generation by lattice scale instantons

Figure : m_R for $\beta = 8.2$ $c_1 = 0.05$, quenched lattice 16^4, with $M = 1.8$, $L_s = 64$



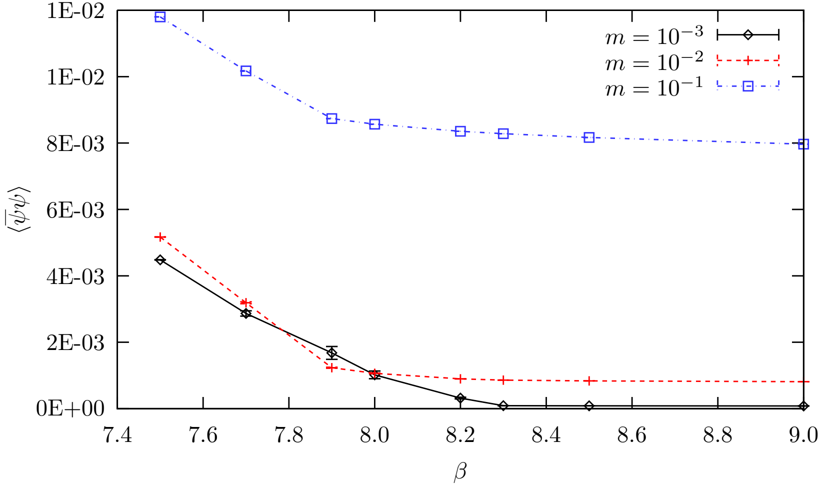
Anomalous mass generation by lattice scale instantons

Figure : m_R for $\beta = 8.3$ $c_1 = 0.05$, quenched lattice 16^4, with $M = 1.8$, $L_s = 64$



Quark condensate, T_c and momentum scale

Figure : Determine the momentum scale relative to T_c



$\beta = 8.3, 8.2$ at or a little below T_c

Anomalous mass generation by lattice scale instantons

- We do see a $1/m$ enhanced mass term suggesting an anomalous mass.
- This behavior survives up to $p \sim 1/a$ implying a hard mass up to the lattice scale.
- However, it would be very hard to study the continuum limit.
- Instead, try a theoretical analysis.

Density of lattice scale instantons and continuum limit

For $N_f = 1$, the density of instantons $\rho(R)$ of radius R is approximately['t Hooft, 1976]

$$\rho(R)\,\mathrm{d}R \sim mR\frac{\mathrm{d}R}{R^5}\exp\left(-\frac{8\pi^2}{g(R)^2}\right) \tag{9}$$

Above formula should be most accurate when R is small. One might expect for lattice scale instantons

$$\rho_a \sim ma\frac{1}{a^4}\exp\left(-\frac{8\pi^2}{g_a^2}\right) \tag{10}$$

Here, g_a is the coupling constant at lattice scale. Unfortunately, the ρ_a and m_{anom} would vanish in the continuum limit ($a, g_a \to 0$ follows renormalization group equation).

Density of lattice scale instantons and continuum limit

Assume the minimum action of a lattice-scale instanton is

$$\mathcal{A}_{\text{inst}} = \alpha \frac{8\pi^2}{g_a^2} \tag{11}$$

The density of lattice scale instanton should be

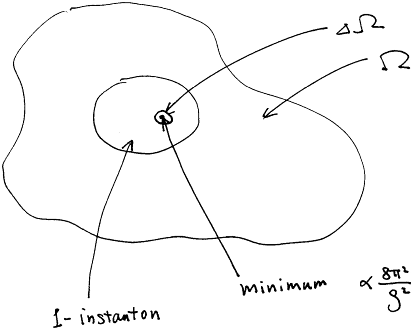
$$\rho_a \sim ma \frac{1}{a^4} \exp\left(-\alpha \frac{8\pi^2}{g_a^2}\right) \tag{12}$$

Assumptions

- For an instanton-like gauge configuration, the fermion determinent would contribute only a factor of ma, since other modes are not affected much by the instanton.
- If we divide the infinite lattice into size-fixed sub-blocks(e.g. 16^4), the probabilities of having an instanton in each block are independent.

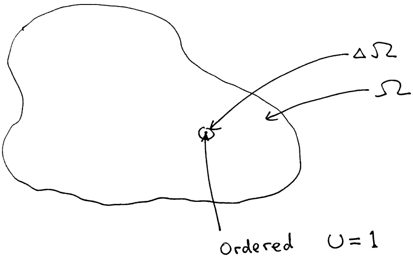
The lower bound of the probability of having an instanton in a pure gauge, size-fixed lattice, e.g. 16^4

$$p^{16^4}_{\text{pure gauge}} = \frac{\int_{\text{instanton}} [\mathcal{D}U] \exp\left(-\mathcal{A}[U]\right)}{\int [\mathcal{D}U] \exp\left(-\mathcal{A}[U]\right)} > \frac{\Delta\Omega \exp\left(-\alpha \frac{8\pi^2}{g_a^2} - \frac{\epsilon'}{g^2}\right)}{\Omega} > \exp\left(-(\alpha+\epsilon)\frac{8\pi^2}{g_a^2}\right) \tag{13}$$



The upper bound of the probability of having an instanton in a pure gauge, size-fixed lattice, e.g. 16^4

$$
\begin{aligned}
p^{16^4}_{\text{pure gauge}} &= \frac{\int_{\text{instanton}} [\mathcal{D}U] \exp\left(-\mathcal{A}[U]\right)}{\int [\mathcal{D}U] \exp\left(-\mathcal{A}[U]\right)} \\
&< \frac{\Omega \exp\left(-\alpha \frac{8\pi^2}{g_a^2}\right)}{\Delta\Omega \exp\left(-\frac{\epsilon'}{g_a^2}\right)} \\
&< \exp\left(-(\alpha - \epsilon)\frac{8\pi^2}{g_a^2}\right)
\end{aligned}
\tag{14}
$$



Density of lattice scale instantons and continuum limit

$$m_{\text{anom}} \sim \frac{a^2}{m}\rho_a \tag{15}$$

$$\rho_a \sim ma\frac{1}{a^4}\exp\left(-\alpha\frac{8\pi^2}{g_a^2}\right) \tag{16}$$

with the renormalization equation

$$\frac{8\pi^2}{g_a^2} \approx \left(11 - \frac{2}{3}N_f\right)\ln\frac{1}{a} \tag{17}$$

We got our final expression for the anomalous quark mass term.

$$m_{\text{anom}} \sim a^{\frac{31}{3}\alpha - 1} \tag{18}$$

Density of lattice scale instantons and continuum limit

Given the minimum action of a lattice-scale instanton

$$\mathcal{A}_{\text{inst}} = \alpha \frac{8\pi^2}{g_a^2} \tag{19}$$

The generated anomalous mass would scale like

$$m_{\text{anom}} \sim a^{\frac{31}{3}\alpha - 1} \tag{20}$$

So, if the generated anomalous mass term does not vanish in the continuum limit, we should have

$$\alpha \leq \frac{3}{31} \approx 0.097 \tag{21}$$

Above criteria is the necessary and sufficient condition.

Explore the design space of lattice action

Wilson Action

$$\mathcal{A} = \frac{\beta}{3} \sum_{x;\mu<\nu} P_{\mu\nu}^{1\times1} \tag{22}$$

$$\alpha < 0.83 \tag{23}$$

Rectangular Action

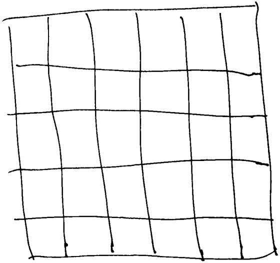
$$\mathcal{A} = \frac{\beta}{24} \sum_{x;\mu\neq\nu} P_{\mu\nu}^{1\times2} \tag{24}$$

$$\alpha < 0.69 \tag{25}$$

Recall the definition of α,

$$\mathcal{A}_{\text{inst}} = \alpha \frac{8\pi^2}{g_a^2} \tag{26}$$

Explore the design space of lattice action



$16 \times 16 \times 16 \times 16.$

Sample Instanton

U

$U^{16\times16\times16\times16}_{\text{sample inst}}$

$$
\begin{aligned}
\mathcal{A}[U] &= \mathcal{A}_{\text{wilson}}[U] - N_{\text{sample inst}}[U]\Delta\mathcal{A} \\
\Delta\mathcal{A} &= \mathcal{A}_{\text{wilson}}[U^{16\times16\times16\times16}_{\text{sample inst}}] - \alpha\frac{8\pi^2}{g_a^2}
\end{aligned}
\tag{27}
$$

Conclusion

For $N_f = 1$ QCD, there can be additive renormalization of the fermion mass generated by lattice scale instantons for a class of lattice actions.

Thank you

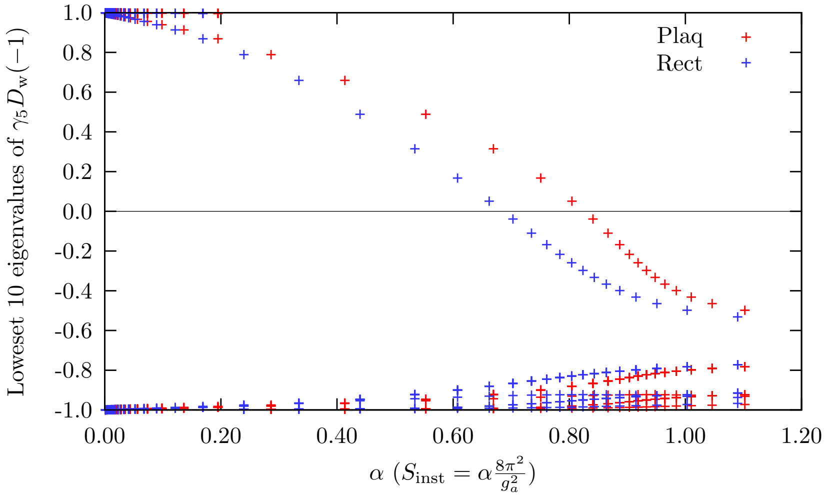
Michael Creutz. Ambiguities in the up-quark mass. *Phys.Rev.Lett.*, 92:162003, 2004. doi: 10.1103/PhysRevLett.92.162003.

C. Sturm, Y. Aoki, N.H. Christ, T. Izubuchi, C.T.C. Sachrajda, et al. Renormalization of quark bilinear operators in a momentum-subtraction scheme with a nonexceptional subtraction point. *Phys.Rev.*, D80:014501, 2009. doi: 10.1103/PhysRevD.80.014501.

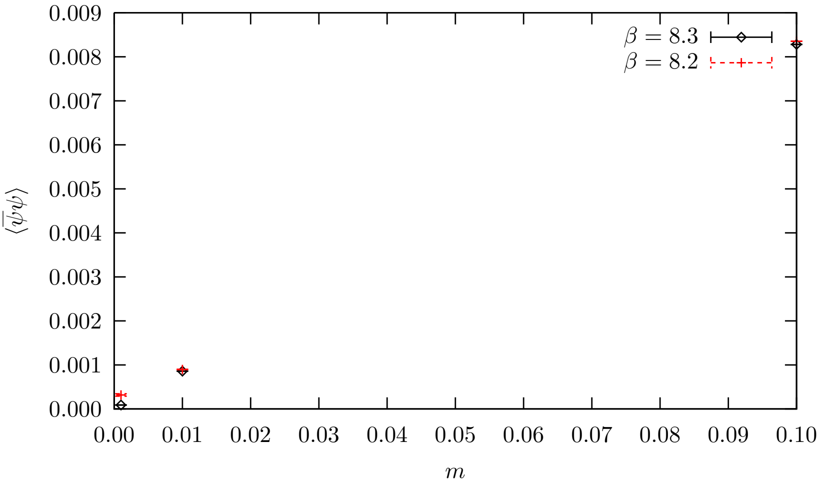
Gerard 't Hooft. Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle. *Phys.Rev.*, D14:3432–3450, 1976. doi: 10.1103/PhysRevD.18.2199.3,10.1103/PhysRevD.14.3432.

Calculate minimum instanton action α

Figure : Cooling configuration with one instanton



Quark condensate and T_c



Quark condensate and T_c