$B \rightarrow \pi$ semileptonic form factors from Lattice QCD

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Based on work done with J. A. Bailey, A. Bazavov, A. El-Khadra, S. Gottlieb, R. Jain, A. Kronfeld, Y. Liu, Y. Meurice, R. Van der Water, R. Zhou

Fermilab Lattice and MILC Collaborations

Lattice 2013, Aug 2nd, Mainz, Germany
Motivation

• Discrepancy between inclusive/exclusive determination of $|V_{ub}|$. Will it stay?

• Continuous experimental effort on $B \rightarrow \pi \ell \nu$ decays.
  
  *BaBar 1208.1253, Belle PRD83(071101)*

  Expect $\sim 4\%$ exp. error at Belle II. *Browder, ANL snowmass*

• Possible NP contribution in rare decays $B \rightarrow K(\pi)\ell^+\ell^-$ (more by Andreas Kronfeld and Chris Bouchard).
  
  First observation of $B^+ \rightarrow \pi^+ \mu^+\mu^-$ by LHCb. *1210.2645*
  
  But not seen in BaBar *1303.6010* and Belle *0804.3656v2*. Desy?

• **No** published update from LQCD since 2009. *FNAL/MILC 2008*

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<table>
<thead>
<tr>
<th>Quantity</th>
<th>CKM element</th>
<th>Present expt. error</th>
<th>2007 forecast lattice error</th>
<th>Present lattice error</th>
<th>2018 lattice error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B \rightarrow \pi \ell \nu$</td>
<td>$</td>
<td>V_{ub}</td>
<td>$</td>
<td>4.1%</td>
<td>–</td>
</tr>
</tbody>
</table>

2013 snowmass report (quark flavor)
Introduction and Notations

\[ \frac{d\Gamma}{dq^2}(B \rightarrow \pi \ell\nu) = \text{Phase space} \times |V_{ub}|^2 |f_+|^2 \]

\[ \langle \pi | V^\mu | B \rangle = f_+(q^2) \left( p_B^\mu + p_\pi^\mu - \frac{M_B^2 - M_\pi^2}{q^2} q^\mu \right) + f_0(q^2) \frac{M_B^2 - M_\pi^2}{q^2} q^\mu \]

\[ = \sqrt{2M_B} \left[ v^\mu f^\parallel(E_\pi) + p_\perp^\mu f^\perp(E_\pi) \right] \]

Convenient for lattice

\[ \frac{d\Gamma}{dq^2}(B \rightarrow \pi^+ \ell^-) = \text{Phase space} \times |V_{tb}V_{td}^*|^2 \left\{ |C_9^{eff} f_+ - C_7 \frac{2m_b f_T}{M_B + M_\pi}|^2 + |C_{10} f_+|^2 \right\} \]

Aliev & Savci 1999

\[ \langle \pi | \bar{d}\sigma^{\mu\nu} b | B \rangle q_{\nu} = -q^2(p + k)^\mu - (M_B^2 - M_\pi^2) q^\mu \frac{M_B}{M_B + M_\pi} f_T(q^2) \]
Overview

- What do we calculate?

\[
\langle \pi | \bar{q} i \gamma^\mu b | B \rangle \quad \rightarrow \quad \begin{cases} f_{||} \\ f_{\perp} \end{cases} \quad b \rightarrow u, V_{ub}, ...
\]

\[
\langle \pi | \bar{q} \sigma^{\mu\nu} b | B \rangle \quad \rightarrow \quad f_T \quad b \rightarrow d \ell^+ \ell^-
\]

- What is **new** (compared to FNAL/MILC 2008)?
  
  ✓ Major set of the new asqtad data (no \( u_0 \) tuning error)
  
  ✓ **12** ensembles vs 6
  
  ✓ **4** lattice spacings vs 2
  
  ✓ Mostly 3 times more configurations for the original ensembles still used
  
  ✓ Better understanding of the simulation parameters and the systematic errors (heavy quark mass, \( Z_v \), etc)
  
  ✓ New analysis methods (due to improved statistics)
MILC asqtad ensembles

Used in FNAL/MILC 2008
Used in this work

Lattice Spacing (fm)

$\frac{m_t}{m_s}$
Fitting correlation functions

- Method to extract the form factors $f_{||}, f_{\perp}, f_T$
  - Use the 3pt/$\sqrt{2}pt$s ratio to avoid using wave function overlaps. Iteratively average over two sink-source separations to suppress the contributions from oscillating states (opposite parity to the physical particles). Fermilab-MILC 2008

$$R(t) = \frac{\bar{C}_{3pt}(0, t, t_s; p_\pi)}{\sqrt{\bar{C}_{2pt}(t; p_\pi)C_{2pt}^B(t_s - t)}} \sqrt{2E_\pi e^{E_\pi,0t+m_B,0(t_s-t)}}$$

- Fit to plateau + exponential function reflecting the B meson excited state contribution.

$$R_{(||,\perp,T)}(t) = f_{(||,\perp,T)}^{LAT} \times \left(1 + A_{(||,\perp,T)}e^{-\frac{\Delta M_B}{2}(t_s-t)}\right)$$

- Combine the fit of ratio and B meson two-point function.
Fitting correlation functions

\[ R^\parallel(t) = \text{fit function} \]

\[ a = 0.12 fm, m_\ell/m_s = 0.1 \]
Chiral/continuum extrapolation

- The one-loop (NLO) HMs $\chi$PT expansion

$$f = f^{(0)}\left(c_0(1 + \log_{SU3}) + c_1 m_{q,val} + c_2 m_{q,sea} + c_3 E_\pi + c_4 E_\pi^2 + c_5 a^2\right)$$

- The non-analytic term $\log_{SU3}$ are replaced by their hard pion 1011.6531 and SU(2) limit, because it fails to describe $f_\parallel$ data (even with NNLO analytic terms!)

- Pions in the simulation are to energetic. Hard pion $\chi$PT integrates them out. So the $E_\pi$ dependence is essentially a phenomenological expansion.

- In hard pion limit, $f_\parallel, f_\perp, f_T$ share the same non-analytic term.

- The variables are normalized with the $\chi$PT breaking scale $\Lambda_\chi$.

- The $B-B^*-\pi$ coupling $g = 0.45(8)$. It has a small effect.
Chiral/continuum extrapolation

Chiral/continuum fit: chi2/[dof]=0.94[36], pvalue=0.56

$r_1 \approx 0.312 \text{ fm}$
Chiral/continuum extrapolation

\[ r_1 \approx 0.312 \text{ fm} \]

chiral/continuum fit: \( \frac{\chi^2}{[\text{dof}]} = 1.2 \), pvalue = 0.2
Chiral/continuum extrapolation

chiral/continuum fit: $\text{chi}^2/\text{dof} = 0.8$ [36], $p$-value = 0.8

$r_1 \approx 0.312$ fm

$\sqrt{f^{T}_1 r_1}$ vs. $E_\pi r_1$

- coarse 0.10ms (C005)
- coarse 0.14ms (C007)
- coarse 0.20ms (C010)
- fine 0.05ms (F00155)
- fine 0.10ms (F0031)
- fine 0.15ms (F00465)
- fine 0.20ms (F0062)
- superfine 0.10ms (S0018)
- superfine 0.14ms (S0025)
- superfine 0.20ms (S0036)
- superfine 0.40ms (S0072)
- ultrafine 0.20ms (U0028)

continuum physical()
$\chi^\text{PT}$ systematic errors

![Graph showing systematic error $f_{\perp}$ as a function of $E r_1$.](image)
**z-expansion**

- Measured partial rates are more reliable in low-$q^2$ region, while lattice has better confidence in large-$q^2$ region. To determine $|V_{ub}|$, need lattice results in full-$q^2$ range.
- The model-independent $z$-expansion is more natural than the $\chi$PT expansion ($|z| < 0.4$), plus important physical constraints (analyticity, unitarity, ...).
- A sequential fit ($\chi$PT/continuum $\rightarrow$ z expansion). Correcting an ill expansion with a well behaving expansion. **Fitting a function variable to another function form!**
- **Synthetic data** approach: sampling the $\chi$PT results at “certain” locations $\rightarrow$ a regular curve fitting problem.

**Issues**: how are points taken (number and locations)?

Is correlation correctly accounted? Systematic error?
z-expansion: a functional approach

- A generalization of data-point fitting to functional fitting.
- Covariance function (surface) $K_f(s, t)$ of a function $f(t)$ is a **Mercer kernel** (continuous, symmetry, positive semi-definite). By Mercer’s theorem

\[ K(s, t) = \sum_{i=1}^{L} \lambda_i \psi_i(s) \psi_i(t) \]

If $f(t)$ is a finite asymptotic expansion $\rightarrow L$ finite.

(Pseudo ) inverse of $K(s, t)$, $J(s, t)$

- Define an objective function (“$\chi^2$”)

\[
\mathcal{L} = \int_{z_1}^{z_2} \int_{z_1}^{z_2} ds dt [f(s) - g_f(s)] J(s, t) [f(t) - g_f(t)]
\]

\[
= \sum_{i=1}^{L} \frac{1}{\lambda_i} \left[ \int dt (f(t) - g_f(t)) \psi_i(t) \right]^2
\]

\[
\mathcal{O}_K \psi_i = \lambda_i \psi_i
\]

\[
\mathcal{O}_K \phi(s) \equiv \int K(s, t) \phi(t) dt
\]
- Data point samples: \( \{x_i, y_i + \delta y_i\} \)
- Target function: \( g(x; a_0, \ldots) \)
- Covariance matrix \( \text{Cov} \)
- Objective function: \( \chi^2 \)

\[
\chi^2 = \sum_{i,j} [y_i - g(x_i)] \text{Cov}^{-1}_{ij} [y_j - g(x_j)]
\]

- Summation over all data points \( \{x_i\} \)
- Principal decomposition:
  The eigen modes of \( \text{Cov} \)

\[
\text{cov } v_i = \lambda_i v_i
\]

- Degrees of freedom: \( N - n \)
  \( N \) - # of non-singular modes of \( \text{Cov} \)
  \( n \) - # of fit parameters

- Function samples: \( \{x, f(x) + \delta f(x)\} \)
- Target function: \( g_f(x; a_0, \ldots) \)
- Covariance function \( K(s, t) \)
- Objective function: \( \mathcal{L} \)

\[
\mathcal{L} = \iint dsdt [f(s) - g_f(s)] K^{-1}(s, t) [f(t) - g_f(t)]
\]

- Integral over the data range \([a, b]\)
- Principal decomposition:
  The eigen modes of \( K(s, t) \)

\[
\int dt K(s, t) \psi_i(t) = \lambda_i \psi_i(s)
\]

- Degrees of freedom: \( L - n \)
  \( L \) - # of non-zero eigenmodes of \( K(s, t) \)
  \( n \) - # of fit parameters
z-expansion: a functional approach

- Getting the covariance function $K(s, t)$.

  $$f(t) = \sum_i c_i y_i(t) \quad K_f(s, t) = \sum_{i,j} y_i(s) \text{Cov}_{ij} y_j(t)$$

- z-expansion

  $$z = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$$

  \[ t_+ = (M_B + M_\pi)^2 \]

  \[ t_0 \quad \text{choice to be optimized} \]

- BGL (Boyd-Grinstein-Lebed) expansion [hep-ph/0504209]

  $$g_f(z) = \frac{1}{P(z)\phi(z)} \sum_{n=0} a_n z^n$$

- BCL (Bourrely-Caprini-Lellouch) expansion: correcting large $q^2$ behavior [0807.2722v3]

  $$g_f(z) = \frac{1}{1 - q^2/M_B^2} \sum_{n=0}^{K-1} a_n \left( z^n - (-1)^{n-K} \frac{n}{K} z^K \right)$$
• BCL expansion
• Individual fits
• Expansion up to $z^3$
• One singular mode is cut from $K(s, t)$.

* The expansion for $f_0$ here is a polynomial expansion with no pole.
Adding Constraints

- Use Lagrange multiplier to add bounds on expansion coefficients.

\[ \sum_n |a_n|^2 \leq A \quad \text{BGL} \]
\[ \sum_{m,n} B_{mn} b_m b_n \leq A \quad \text{BCL} \]

For **Unitarity** bound \( A = 1 \), which is usually not saturated (bound is loose).

- **Analyticity** of form factors constrains the coefficients more tightly!

\[ A_+ \sim \frac{m_b^2}{3} \int_{t_+}^{\infty} \frac{[(t-t_+)(t-t_-)]^{3/2}}{t^5} |f_+|^2 \]
\[ \sim 0.03 \]

Replaced by leading order

A consistent check
Important for truncation error estimate.
**z-expansion systematic errors**

**Table I:**

<table>
<thead>
<tr>
<th>Integration Range</th>
<th>$[z_1, z_2]$</th>
<th>$[t_1, t_2]$</th>
<th>$\tilde{a}_0$</th>
<th>$\tilde{a}_1$</th>
<th>$\tilde{a}_2$</th>
<th>$f_+(q^2 = 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[-0.25, 0.07]</td>
<td>[17,26]</td>
<td>0.414(8)</td>
<td>-0.78(7)</td>
<td>-0.8(5)</td>
<td>0.133(57)</td>
<td></td>
</tr>
<tr>
<td>[-0.19, 0.05]</td>
<td>[18,25]</td>
<td>0.412(9)</td>
<td>-0.74(7)</td>
<td>-0.6(6)</td>
<td>0.146(65)</td>
<td></td>
</tr>
<tr>
<td>[-0.14,0.03]</td>
<td>[19,24]</td>
<td>0.413(9)</td>
<td>-0.74(7)</td>
<td>-0.6(5)</td>
<td>0.155(70)</td>
<td></td>
</tr>
<tr>
<td>[-0.09,0.00]</td>
<td>[20,23]</td>
<td>0.411(9)</td>
<td>-0.78(7)</td>
<td>-0.5(6)</td>
<td>0.160(74)</td>
<td></td>
</tr>
</tbody>
</table>

**BCL vs BGL**

**$t_0$ dependence**

**Truncation Error**
Error Estimate

In the region with lattice data:

- Statistics: 2~4%
- $\chi$PT systematic: 2~5% (the blowup is cured by $z$ expansion)
- Kappa tuning: 0.5%
- $z$-expansion: < 1%
- Heavy quark discretization: ?%
Summary and outlook

- We are working on an update to the lattice form factor calculations for semileptonic $B \rightarrow \pi$ decays. We also include the tensor form factor in the calculation. The improvement on the error is promising.
- While $\chi$PT systematic is a dominant source of systematic errors in low-$q^2$ region, it can be improved by $z$-expansion.
- A new functional $z$-expansion method is used to reparameterize the $\chi$PT results.
- Full error budget is in progress.
- $|V_{cb}|$ and $B \rightarrow \pi \ell^+ \ell^-$ prediction.