The Schrödinger Functional in Numerical Stochastic Perturbation Theory

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31st International Symposium on Lattice Field Theories
July-August 2013, Mainz, Germany

In collaboration with
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Overview

• Stochastic quantization and NSPT
• NSPT in the Schrödinger functional
• The running SF coupling in NSPT
Preliminaries

We work in the Schrödinger Functional as defined in [Lüscher, Narayanan, Weisz, and Wolff 1992].

I.e. the SU(3) lattice gauge fields $U_\mu(x)$ obey the **boundary conditions**

\[
U_\mu(x + \hat{k}L) = U_\mu(x)
\]

\[
U_k(x)|_{x_0=0} = e^{C(x)}
\]

\[
U_k(x)|_{x_0=T} = e^{C'(x)}
\]

and may be decomposed (in the perturbative regime) as

\[
U_\mu(x) = e^{a_0 q_\mu(x)} V_\mu(x)
\]

(sometimes I’ll also write $U_{x\mu}$)
**Stochastic Quantization**

Introduce an extra d.o.f., the *stochastic time*. The field evolution in stochastic time is given by the Langevin Equation.

\[
\partial_t U_{x\mu}(t; \eta) = - \left\{ \nabla_{x\mu} S_G[U(t; \eta)] + \eta_{x\mu}(t) \right\} U_{x\mu}(t; \eta)
\]

with Gaussian noise \( \eta = \eta^a T^a \)

\[
\langle \eta^a_{x\mu}(t) \rangle_\eta = 0, \quad \langle \eta^a_{x\mu}(t) \eta^b_{y\nu}(u) \rangle_\eta = 2\delta^{ab} \delta_{xy} \delta_{\mu\nu} \delta(t - u)
\]

[Parisi & Wu, 1981]
Stochastic Quantization

The expectation value of an observable w.r.t. the stochastic noise may be written as

$$\langle O \rangle_\eta = \frac{1}{Z_\eta} \int \mathcal{D}[\eta] O[U(t; \eta)] e^{-\frac{1}{4} \int \mathrm{d}x \mathrm{d}t \eta^2(x,t)}$$

$$= \int \mathcal{D}U O[U] P[U, t]$$

Where the time evolution of the weight is given by the Fokker-Planck equation

$$\dot{P}[U, t] = \int \mathrm{d}x \mathrm{d}\mu \frac{\delta}{\delta U_{x\mu}} \left( \frac{\delta S[U]}{\delta U_{x\mu}} + \frac{\delta}{\delta U_{x\mu}} \right) P[U, t]$$
Stochastic Quantization

One can prove that in a well-defined sense the weight converges to the one encountered in the path integral formalism, order by order...

\[ P^{(0)}[U(\eta; t)] \xrightarrow{t \to \infty} P_{\text{PI}}^{(0)}[U] = \frac{1}{Z^{(0)}} e^{-S_{G}^{(0)}[U]}, \]

\[ P^{(1)}[U(\eta; t)] \xrightarrow{t \to \infty} P_{\text{PI}}^{(1)}[U], \ldots \]

Hence one may obtain perturbative expansion of any observable in the large-time-limit

\[ \lim_{t \to \infty} \left\langle \mathcal{O} \left[ \sum_{k} g_{0}^{k} U^{(k)}(t; \eta) \right] \right\rangle_{\eta} = \sum_{k} g_{0}^{k} \mathcal{O}^{(k)} \]
Numerical Stochastic Perturbation Theory

Solve the Langevin equation on a lattice numerically e.g. using an Euler Scheme

\[ U_{x\mu}(t + \epsilon; \eta) = e^{-F_{x\mu}[U]}U_{x\mu}(t; \eta), \quad F_{x\mu}[U] = \epsilon \nabla_{x\mu} S_G[U] - \sqrt{\epsilon} \eta_{x\mu} \]

... the perturbative expansion of which may be consistently truncated ...

\[ U^{(0)} \rightarrow U^{(0)}, \quad U^{(1)} \rightarrow U^{(1)} - F^{(1)} \]

\[ U^{(2)} \rightarrow U^{(2)} - F^{(2)} + \frac{1}{2} \left( F^{(1)} \right)^2 - F^{(1)}U^{(1)}, \ldots \]

[Di Renzo, Marchesini, et al. 1994]
Numerical Stochastic Perturbation Theory

Set up a computer simulation, storing all gauge fields as a series

\[ U = \sum_{k=0}^{N} g_0^k U^{(k)} \]

and perform all computations order by order, e.g.

\[ (U \cdot U')^{(k)} = \sum_{l=0}^{N} U^{(l)} U'^{(k-l)} \]

• Obtain expectation value through single long history
• Perform the limit \( \epsilon \to 0 \)
Sample History

\( \dot{\epsilon}_m, \quad L/a = 6, \quad \epsilon = 0.0125 \)
Gauge Suppression

Historically: Stochastic gauge fixing.

\[ F_{x\mu}[U] = \varepsilon \nabla_{x\mu} S_G[U] - \sqrt{\varepsilon} \eta_{x\mu} \]

Drives *transversal* gauge modes to classical solution.

Random walk.

Perform properly chosen gauge transformation to suppress longitudinal modes and avoid growing variance.

[Zwanziger 1981]
Numerical Tests

Without gauge suppression

With gauge suppression
Numerical Tests

\[ \dot{\xi}_s^{(2)} \]
Running Coupling
In the SF, we can define a running coupling, using the external scale $L$, employing the free energy

$$e^{-\Gamma} = \int \mathcal{D}[U] e^{-S_G[U]}$$

with which we define

$$\bar{g}_{SF}^2(L) = \frac{k}{\partial_{\eta} \Gamma \mid_{\eta=0}} = g_0^2 + m_1(L/a) g_0^4 + m_2(L/a) g_0^6 + \ldots$$

[Lüscher, Weisz, Wolff 1991]
Known values from [ALPHA Collaboration 1998].
Two Loops

Known values from [ALPHA Collaboration 1998].

Statistical errors only.
**Comparison with known results**

Defining \( \delta m_i^a = \frac{m_i^{a,\text{NSPT}}}{m_i^a} - 1 \), we obtain

<table>
<thead>
<tr>
<th>( L/a )</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>12</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta m_1^a )</td>
<td>0.0017(21)</td>
<td>-0.0016(31)</td>
<td>0.0041(56)</td>
<td>-0.001(10)</td>
<td>-0.019(16)</td>
</tr>
<tr>
<td>( \delta m_2^a )</td>
<td>-0.014(12)</td>
<td>-0.007(15)</td>
<td>-0.003(26)</td>
<td>0.001(58)</td>
<td>-0.045(91)</td>
</tr>
</tbody>
</table>

Statistical errors only.

<table>
<thead>
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<th>6</th>
<th>8</th>
<th>12</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta m_1^a )</td>
<td>0.002(11)</td>
<td>-0.0016(79)</td>
<td>0.0041(60)</td>
<td>-0.001(11)</td>
<td>-0.019(17)</td>
</tr>
<tr>
<td>( \delta m_2^a )</td>
<td>-0.01(19)</td>
<td>-0.01(17)</td>
<td>-0.003(69)</td>
<td>0.001(72)</td>
<td>-0.05(20)</td>
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</table>

Statistical and systematic errors.
Numerical Effort

Final result for $L/a = 12$

$$m_2^a = 0.0684(49)$$

Corresponding to a precision of

$$\frac{\delta m_2^a}{m_2^a} \approx 7.2\%$$

With wall-clock time of 210h on a single node of TCD’s Lonsdale cluster (AMD Opteron, 8 cores / node)

<table>
<thead>
<tr>
<th>$L$</th>
<th>$\epsilon$</th>
<th>$\tau_{\text{int}}$</th>
<th>$N_{\text{eff}}$</th>
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<tr>
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<td>0.05</td>
<td>1.032(22)</td>
<td>55882</td>
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<tr>
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<td>7.78(31)</td>
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<td>5.58(19)</td>
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<td>8</td>
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<td>8.19(40)</td>
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<td>5.20(21)</td>
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<tr>
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<td>12.19(97)</td>
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<td>9.18(65)</td>
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<td>16</td>
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<td></td>
<td>0.05</td>
<td>5.57(53)</td>
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</tr>
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</table>
Conclusion & Outlook

Conclusion

☑️ NSPT is a useful tool to extract perturbative expansions with decent precision.

☑️ Valuable for gradient flow observables (next talk!).

Shopping list

☑️ MPI parallelization
☐ Fermions
☐ Higher order integrator(s)
Appendix
Gauge Suppression

To construct a gauge transformation that suppresses the unwanted modes, one minimizes

\[ W[U] = -a^2 \sum_{(x,\mu) \in \Lambda} \cosh\{a g_0 q_\mu(x)\} . \]

This is archived by applying a gauge transformation \( \Omega(x) = e^{-i \epsilon \omega(x)} \)

\[ \omega(x) = \begin{cases} 
0 & x_0 = T , \\
-\alpha \left( \frac{a}{L} \right)^3 \sum_{x,x_0=0} \sinh \{a g_0 q_0(x)\} j_j \big|_{\text{traceless}} & x_0 = 0 , \\
-\alpha \sum_\mu D^*_\mu \sinh \{a g_0 q_\mu(x)\} \big|_{\text{traceless}} & \text{else} . 
\end{cases} \]