Comparing Tensor Renormalization Group and MC for Spin and Gauge Models (arXiv1307.6543)

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Work done in part with Alan Denbleyker, Yuzhi "Louis" Liu, Judah Unmuth-Yockey, Tao Xiang, Zhiyuan Xie, Ji-Feng Yu and Haiyuan Zou

Lattice 2013, July 29, Mainz

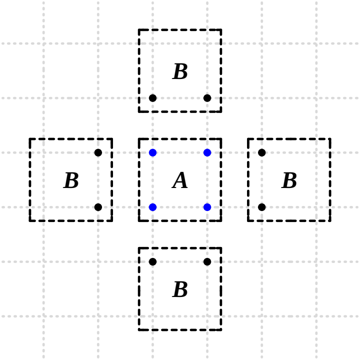
Content of the talk

- Blocking in configuration space: it's hard! (Ising 2 example)
- Tensor Renormalization Group (TRG): blocking is simple and exact!
- TRG for spin models ($O(N)$ and principal chiral models)
- TRG for gauge models (Ising, $U(1)$ and $SU(2)$)
- Numerical applications: beating the sign problem at complex β and chemical potential for spin models
- Conclusions

For details see: PRB 87 064422 (2013) and arXiv1307.6543. There are two preprints in progress with numerical applications.

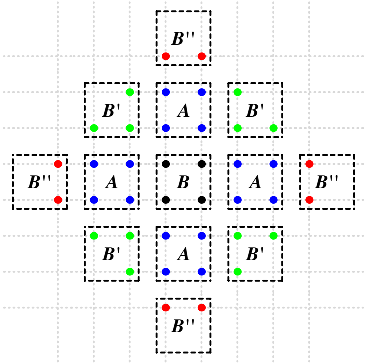
Block Spining in Configuration Space: Step 1

Square blocks in $A-B$ checkerboard; B blocks are fixed backgrounds.
We can block spin in the A blocks. Example:
$Pr(\phi_A = 4|\phi_i^{background}) = \exp(\beta(4 + \sum_{i=1}^{8} \phi_i^{background}))$



Block Spining in Configuration Space: Step 2 etc.

The next step is to try to block spin in the B blocks. We can use step 1 and block spin in a given B block but there are 20 background spins. This does not seem to stop and finding the effective energy function is nontrivial.



TRG blocking: it's simple and exact!

For each link:

$$\exp(\beta\sigma_1\sigma_2) = \cosh(\beta)(1 + \sqrt{\tanh(\beta)}\sigma_1\sqrt{\tanh(\beta)}\sigma_2) =$$
$$cosh(\beta)\sum_{n_{12}=0,1}(\sqrt{\tanh(\beta)}\sigma_1\sqrt{\tanh(\beta)}\sigma_2)^{n_{12}}.$$

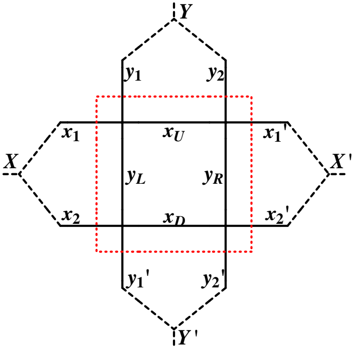
Regroup the four terms involving a given spin σ_i and sum over its two values ±1. The results can be expressed in terms of a tensor: $T^{(i)}_{xx'yy'}$ which can be visualized as a cross attached to the site i with the four legs covering half of the four links attached to i. The horizontal indices $x,\ x'$ and vertical indices $y,\ y'$ take the values 0 and 1 as the index n_{12}.

$$T^{(i)}_{xx'yy'} = f_x f_{x'} f_y f_{y'} \delta\left(\mathrm{mod}[x + x' + y + y', 2]\right) \ ,$$

where $f_0 = 1$ and $f_1 = \sqrt{\tanh(\beta)}$. The delta symbol is 1 if $x + x' + y + y'$ is zero modulo 2 and zero otherwise.

TRG blocking (graphically)

Exact form of the partition function: $Z = (\cosh(\beta))^{2V}\,\mathrm{Tr}\prod_i T^{(i)}_{xx'yy'}$.
Tr mean contractions (sums over 0 and 1) over the links.
Reproduces the closed paths of the HT expansion.
TRG blocking separates the degrees of freedom inside the block which are integrated over, from those kept to communicate with the neighboring blocks. Graphically :



TRG Blocking (formulas)

Blocking defines a new rank-4 tensor $T'_{XX'YY'}$ where each index now takes four values.

$$T'_{X(x_1,x_2)X'(x'_1,x'_2)Y(y_1,y_2)Y'(y'_1,y'_2)} = \sum_{x_U,x_D,x_R,x_L} T_{x_1 x_U y_1 y_L} T_{x_U x'_1 y_2 y_R} T_{x_D x'_2 y_R y'_2} T_{x_2 x_D y_L y'_1} \ ,$$

where $X(x_2, x_2)$ is a notation for the product states e. g. , $X(0,0) = 1,\ X(1,1) = 2,\ X(1,0) = 3,\ X(0,1) = 4$. The partition function can be written as

$$Z = \mathrm{Tr} \prod_{2i} T'^{(2i)}_{XX'YY'} \ ,$$

where $2i$ denotes the sites of the coarser lattice with twice the lattice spacing of the original lattice.

Accurate exponents from approximate tensor renormalizations (YM, Phys. Rev. B 87, 064422 2013)

- For the Ising model on square and cubic lattices, truncation method (HOSVD) sharply singles out a surprisingly small subspace of dimension two.
- In the two states limit, the transformation can be handled analytically yielding a value 0.964 for the critical exponent ν much closer to the exact value 1 than 1.338 obtained in Migdal-Kadanoff approximations. Alternative blocking procedures that preserve the isotropy can improve the accuracy to $\nu = 0.987$ and 0.993 respectively.
- Applications to other classical lattice models are possible, including models with fermions. TRG could become a competitor for the Monte Carlo method suitable to calculate accurately critical exponents, take continuum limits and study near-conformal systems in large volumes.

$O(2)$ model

$$Z = \int \prod_i \frac{d\theta_i}{2\pi} \mathrm{e}^{\beta \sum_{<ij>} \cos(\theta_i - \theta_j)} .$$

$$\mathrm{e}^{\beta \cos(\theta_i - \theta_j)} = \sum_{n_{ij}=-\infty}^{+\infty} \mathrm{e}^{i n_{ij}(\theta_i - \theta_j)} I_{n_{ij}}(\beta) ,$$

where the I_n are the modified Bessel functions of the first kind. In two dimensions, we obtain the factorizable expression:

$$\begin{aligned} T^i_{n_{ix}, n_{ix'}, n_{iy}, n_{iy'}} &= \sqrt{I_{n_{ix}}(\beta)} \sqrt{I_{n_{iy}}(\beta)} \sqrt{I_{n_{ix'}}(\beta)} \sqrt{I_{n_{iy'}}(\beta)} \\ & \quad \delta_{n_{ix}+n_{iy}, n_{ix'}+n_{iy'}} . \end{aligned}$$

The partition function and the blocking of the tensor are similar to the Ising model. The only difference is that the sums run over the integers. As the $I_n(\beta)$ decay rapidly for large n and fixed β (namely like $1/n!$) there is no convergence issue.
The generalization to higher dimensions is straightforward.

$O(3)$ model

$H = -\sum_{<ij>} \cos\gamma_{ij}$ with
$\cos\gamma_{ij} = \cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j \cos(\phi_i - \phi_j)$.

$$e^{\beta\cos\gamma_{ij}} = \sum_{l=0}^{\infty} A_l(\beta) \sum_{m=-l}^{l} Y^*_{lm}(\theta_j, \phi_j) Y_{lm}(\theta_i, \phi_i),$$

$$A_l(\beta) = 4\pi \sum_{n=l}^{\infty} I_n(\beta) \int_{-1}^{1} T_n(x) P_l(x) dx.$$

$$Y_{l_1 m_1}(\theta, \phi) Y_{l_3 m_3}(\theta, \phi) = \sum_{L=l_{min}}^{l_{max}} G_L^{(m_1, m_3, l_1, l_3)} Y_L^{m_1+m_3}(\theta, \phi) \ .$$

$$T_{(l_1,m_1),(l_2,m_2),(l_3,m_3),(l_4,m_4)} = \delta_{m_1+m_3, m_2+m_4}$$
$$\sum_L G_L^{(m_1,m_3,l_1,l_3)} G_L^{*(m_2,m_4,l_2,l_4)} \sqrt{A_{l_1} A_{l_2} A_{l_3} A_{l_4}}.$$

TRG for $SU(2)$ Principal Chiral Models

$$Z = \prod_{n} \int dU(n) \prod_{ni} \exp\left\{ \frac{\beta}{2} \mathrm{Re}[\mathrm{tr}\left[U(n) U^{\dagger}(n+i) \right]] \right\}.$$

$$\begin{aligned} & T_{(r_1,m_1,n_1)(r_2,m_2,n_2)(r_3,m_3,n_3)(r_4,m_4,n_4)} = \\ & (F_{r_1}(\beta) F_{r_2}(\beta) F_{r_3}(\beta) F_{r_4}(\beta))^{\frac{1}{2}} \\ & \times \sum_{r',m',n'} d_{r'}^{-1} (-1)^{m'-n'} \\ & C^{r_1\, r_2\, r'}_{m_1\, m_2\, m'} C^{r_1\, r_2\, r'}_{n_1\, n_2\, n'} C^{r_3\, r_4\, r'}_{m_3\, m_4\, -m'} C^{r_3\, r_4\, r'}_{n_3\, n_4\, -n'}. \end{aligned}$$

TRG Formulation of 3D Z_2 Gauge Theory

$$Z = \sum_{\{\sigma\}} \exp\left(\beta \sum_P \sigma_{12}\sigma_{23}\sigma_{34}\sigma_{41}\right),$$

For each plaquette the weight is

$$\sum_{n=0,1} (\sqrt[4]{\tanh(\beta)}\sigma_{12}\sqrt[4]{\tanh(\beta)}\sigma_{23}\sqrt[4]{\tanh(\beta)}\sigma_{34}\sqrt[4]{\tanh(\beta)}\sigma_{41})^n.$$

Regrouping the factors with a given σ_l and summing over ± 1 we obtain a tensor attached to this link

$$A^{(l)}_{n_1 n_2 n_3 n_4} = \left(\sqrt[4]{\tanh\beta}\right)^{n_1+n_2+n_3+n_4} \times \delta\left(\mathrm{mod}[n_1+n_2+n_3+n_4, 2]\right).$$

A and B tensors

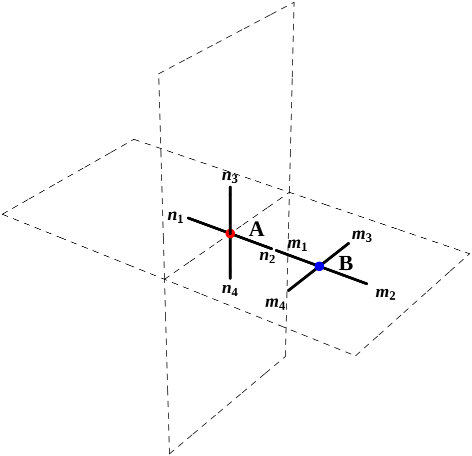
The four links attached to a given plaquette p must carry the same index 0 or 1. For this purpose we introduce a new tensor

$$\begin{aligned} B^{(p)}_{m_1 m_2 m_3 m_4} &= \delta(m_1, m_2, m_3, m_4) \\ &= \begin{cases} 1, & \text{all } n_i \text{ are the same} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The partition function can now be written as

$$Z = (2\cosh\beta)^{3V}\, \mathrm{Tr} \prod_{l} A^{(l)}_{n_1 n_2 n_3 n_4} \prod_{p} B^{(p)}_{m_1 m_2 m_3 m_4},$$

A and B tensors graphically



Asymmetric Formulation Graphically

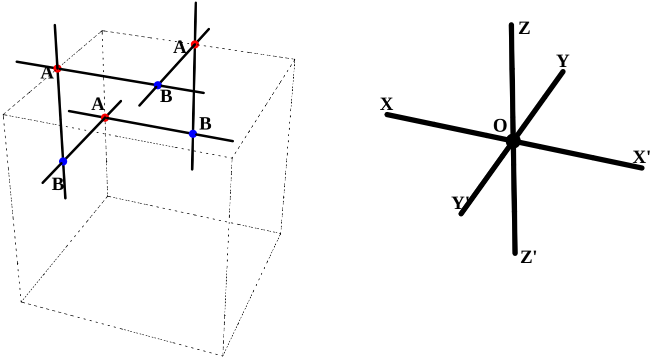


Figure: A new basic cell in an original cube. The equivalent T_6 tensor, its center is (1/4, 3/4, 3/4) in the original cube. Each leg is a double index. Left-Right $A - B$ asymmetric in each direction.

Assymetric formulation: the partition function

By using 3 A tensors and 3 B tensors, a basic cell can be constructed. There are twelve external legs. We can recombine the indices attached to the legs pointing in the same directions using product states (labeled by capital letters). For instance $X = x_1 \otimes x_2$ and similarly with the other directions. Proceeding this way, we obtain a new tensor $T_{6\,XX'YY'ZZ'}$ which can be treated as in the case of a 3D spin model. However, in the positive (X, Y, Z) and negative (X', Y', Z') directions, the opposite legs are associated with different tensors. For instance X is associated with A and X' with B.
The partition function can be rewritten as the tensor-network state of the new T_6 tensor at each cube c,

$$Z = (2\cosh\beta)^{3V} \mathrm{Tr} \prod_c T_{6\,XX'YY'ZZ'}^{(c)} \, .$$

Assymetric formulation: blocking

To blockspin, we can use anisotropic steps by contracting the lattice alternatively in the x axis, y axis, and z axis directions. In each step, the lattice size is reduced by a factor of 2 in the appropriate direction and a new T'_6 tensor is generated as,

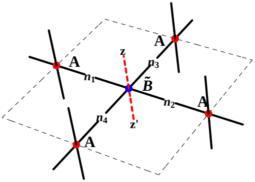
$$
\begin{aligned}
&T'_{6\,XX''\tilde{Y}(Y_1,Y_2)\tilde{Y}'(Y'_1,Y'_2)\tilde{Z}(Z_1,Z_2)\tilde{Z}'(Z'_1,Z'_2)} \\
&= \sum_{X'} T_{6\,XX'Y_1Y'_1Z_1Z'_1} T_{6\,X'X''Y_2Y'_2Z_2Z'_2},
\end{aligned}
$$

where $\tilde{Y}(Y_1, Y_2)$ is the notation for the product states $\tilde{Y} = Y_1 \otimes Y_2$ and similarly with the other directions. The partition function can then be rewritten as the trace of product of T'_6 tensors as before blocking. It is also possible to find tensors associated with the partition function in the temporal gauge. The *A* tensor on the temporal links disappear while those on the space links have a space-time asymmetry. This will be important for numerical applications.

Symmetric Formulation: the $\tilde{B}$ tensor

The A and B tensors do not suffer from this asymmetry. However they do not close under blocking. We can try to combine the B tensors of two adjacent plaquettes in the same plane into a new one. This does not work because the A tensor on the common link induces two new legs orthogonal to the plane and pointing in opposite directions (eliminated in Migdal-Kadanoff by bond-sliding). For an exact formula we modify the B tensor to form a $\tilde{B}$ tensor with 6 indices with initial value

$$\tilde{B}_{n_1 n_2 n_3 n_4 z z'} = B_{n_1 n_2 n_3 n_4} \delta_{z z'} \ ,$$



Symmetric Formulation: the C tensor

The new legs piercing the plaquettes can be traced by introducing a new tensor $C_{xx'yy'zz'}$ at the center of the cubes with initial value

$$C_{xx'yy'zz'} = \delta_{xx'}\delta_{yy'}\delta_{zz'},$$

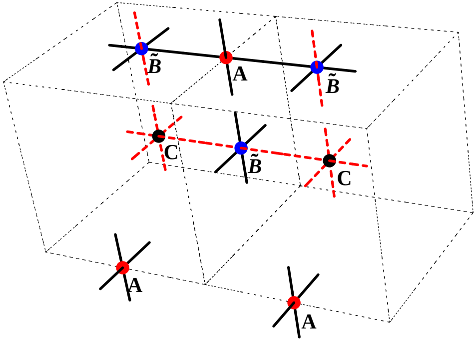
We can now rewrite the partition function as

$$Z = K(2\cosh\beta)^{3V}\,\mathrm{Tr}\prod_{l} A^{(l)}\prod_{p}\tilde{B}^{(p)}\prod_{c} C^{(c)}\ ,$$

where the indices are implicit to keep the formula short. The Kronecker delta in the initial values can be summed along open or closed lines (depending on the boundary conditions) and give rise to a power of 2 that can be eliminated by adjusting the constant K. The other traces are as in the assymetric expression of the partition function.

Symmetric formulation: Blocking (graphically)

A blocking procedure can be constructed by sequentially combining two cubes into one in each of the directions.



Symmetric formulation: blocking formulas

On the link of the new lattice formed by two cubes, two parallel A tensors form the new A' tensor with product states (capital letters). Each tensor element is

$$
\begin{aligned}
&A'_{X(x_1,x_2)X'(x'_1,x'_2)Y(y_1,y_2)Y'(y'_1,y'_2)} \\
&= A_{x_1x'_1y_1y'_1} \times A_{x_2x'_2y_2y'_2}.
\end{aligned}
$$

On the new face, two $\tilde{B}$ tensors and one A tensor form a new $\tilde{B}'$ tensor,

$$
\begin{aligned}
&\tilde{B}'_{xx'Y(y_1,y_2)Y'(y'_1,y'_2)Z(z_1,z_2,z_3)Z'(z'_1,z'_2,z'_3)} \\
&= \sum_{x_3,x'_3} \tilde{B}_{xx_3y_1y'_1z_1z'_1} A_{x_3x'_3z_3z'_3} \tilde{B}_{x'_3x'y_2y'_2z_2z'_2}.
\end{aligned}
$$

At the center, two C tensors and one $\tilde{B}$ tensor form a new C' tensor,

$$
\begin{aligned}
&C'_{xx'Y(y_1,y_2,y_3)Y'(y'_1,y'_2,y'_3)Z(z_1,z_2,z_3)Z'(z'_1,z'_2,z'_3)} \\
&= \sum_{x_2,x'_2} C_{xx_2y_1y'_1z_1z'_1} \tilde{B}_{x_2x'_2y_2y'_2z_2z'_2} C_{x'_2x'y_3y'_3z_3z'_3}.
\end{aligned}
$$

$U(1)$ Gauge Models

$$Z = \prod_{\langle ij \rangle} \int_{-\pi}^{\pi} \frac{d\theta_{ij}}{2\pi} \exp\left(\beta \sum_P \cos(\theta_{12} + \theta_{23} - \theta_{43} - \theta_{14})\right),$$

where the product is running through all the links of the lattice and the sum is over all the plaquettes. Using the Fourier expansion with the Bessel functions and collecting the factors for each link, we obtain the A tensor

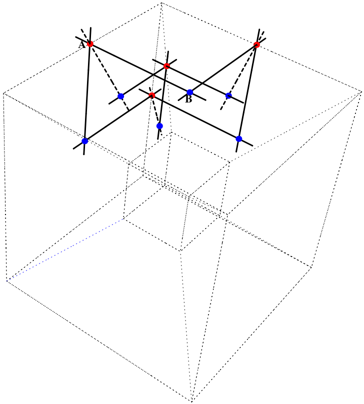
$$A_{n_1 \dots n_{2(D-1)}} = \prod_{i=1}^{2(D-1)} \sqrt[4]{I_{n_i}(\beta)} \delta\left(\sum_{i=1}^{2(D-1)} (-1)^{i+1} n_i\right),$$

where the I_ns are the modified Bessel functions. For any D, we can use a B tensor that ensures that the four indices attached to a plaquette are identical just like for the Z_2 case. The partition function can be written as

$$Z = \mathrm{Tr} \prod_l A^{(l)}_{n_1 \dots n_{2(D-1)}} \prod_p B^{(p)}_{m_1 m_2 m_3 m_4}.$$

$D = 4$

The basic cell of tensors in $D = 4$ has A tensors with six legs and six B tensors in one basic cell of the hyper-cube. There are 3 legs pointing in each of the directions. Following the asymmetric procedure, we can combine each of these three legs into a single index, build a rank 8 tensor. It seems possible to follow the symmetric procedure.



TRG Formulation of 3D $SU(2)$ Gauge

$$Z = \prod_{ni} \int dU(ni) \prod_{nij} \exp\left\{ \frac{\beta}{4} \mathrm{Re}[\mathrm{tr}\,[U(nij)]] \right\},$$

We can re-write the action as a character expansion

$$\mathrm{e}^{-\beta S_p} = \sum_r F_r(\beta) \chi^r(U(nij)).$$

Using

$$\chi^r(U_1 U_2 U_3 U_4) = D^r_{ij}(U_1) D^r_{jk}(U_2) D^r_{kl}(U_3) D^r_{li}(U_4),$$

we can perform the product over plaquettes of the lattice, and gather together the four D-functions which all share the same link variable. Explicit expressions can be obtained from the orthogonality of the Wigner D-functions. This situation is similar to the 2D Principal Chiral model.

3D $SU(2)$ Gauge: A and B tensors

Initial values of the A and B tensors:

$$
\begin{aligned}
& A_{(r_1,m_1,n_1)(r_2,m_2,n_2)(r_3,m_3,n_3)(r_4,m_4,n_4)} = \\
& (F_{r_1}(\beta)F_{r_2}(\beta)F_{r_3}(\beta)F_{r_4}(\beta))^{\frac{1}{4}} \\
& \times \sum_{r',m',n'} d_{r'}^{-1}(-1)^{m'-n'} \\
& C^{r_1\ r_2\ r'}_{m_1\ m_2\ m'} C^{r_1\ r_2\ r'}_{n_1\ n_2\ n'} C^{r_3\ r_4\ r'}_{m_3\ m_4\ -m'} C^{r_3\ r_4\ r'}_{n_3\ n_4\ -n'} .
\end{aligned}
$$

$$
\begin{aligned}
& \tilde{B}_{(r,i,i')(r',j,j')(r'',k,k')(r''',l,l')} \\
& = B_{rr'r''r'''}\delta_{i,j}\delta_{j',k}\delta_{k',l}\delta_{l',i'} \ .
\end{aligned}
$$

We can now proceed as in the 3D Abelian case to write the partition function and perform blockings using A and B tensors. The only difference is that the single indices of the Abelian formulas need to be replaced by three indices.

Comparison with exact and MC results (in progress)

In practice, the TRG blocking needs (optimal) truncations (see PRB 87 064422 and refs. therein).

Successful results have been obtained by Haiyuan Zou, Ji-Feng Yu and Alan Denbleyker for

- 2D Ising at complex β (sign problem) and finite volume compared to exact Onsager-Kaufman; except near Fisher's zeros, TRG can reach much larger values of Imβ than MC
- 2D $O(2)$ at finite volume: Fisher's zero (compared with MC)
- Critical properties of the 2D $O(2)$ at large volume (compared with MC)
- 2D $O(2)$ with chemical potential (sign problem resolved)

Comparison of TRG and exact results for 2D Ising

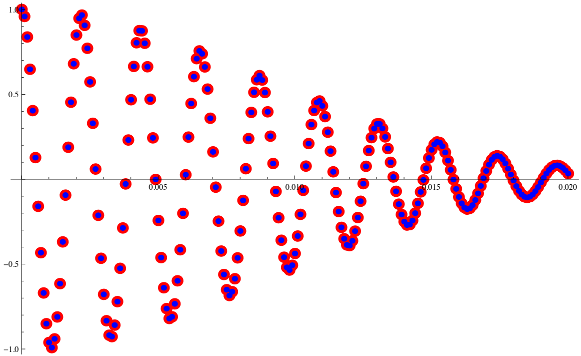


Figure: The real part of the partition function for $\beta = 0.3 + ix$ vs. x; red: result from HOTRG with 20 states; blue: exact solution (Onsager-Kaufmann).

Comparison of TRG and exact results for 2D Ising: Fisher's zeros

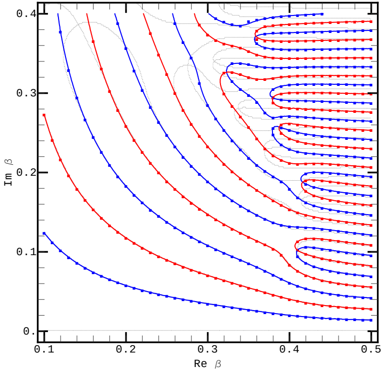


Figure: Zeros of the real and imaginary parts of the partition function from HOTRG with 40 states for 8x8 Ising. Blue: ReZ=0. Red: ImZ=0. Squares: TRG. Thick line: exact solution. Gray line: MC.

Fisher's zeros of the 2D $O(2)$ model at small volume

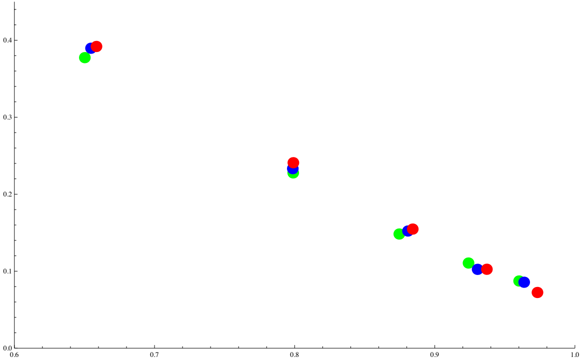


Figure: Fisher's zeros of XY model with $L = 4, 8, 16, 32, 64$ and number of states 16 (green), 20 (blue), 30 (red).

Fisher's zeros of the 2D $O(2)$ model: TRG and MC

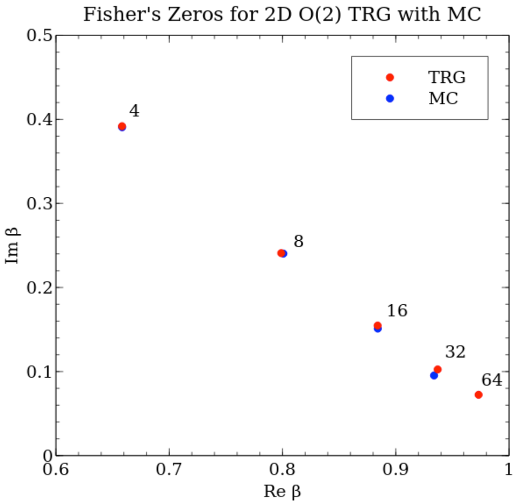


Figure: Fisher's zeros of XY model with $L = 4, 8, 16, 32, 64$ for 30 states compared to MC

Other possible applications

- Modulated phases in clock and $O(2)$ models with chemical potential.
  See: The sign problem and Abelian lattice duality, Michael Ogilvie and Peter Meisinger, Mon, 16:30, Seminar Room B
- Comparison between classical and quantum tensor formulations of the 2D Schwinger model.
  See: Matrix Product States for Lattice Field Theories, Mari Carmen Banuls, Krzysztof Cichy, Karl Jansen, and Ignacio Cirac Tue, 17:00, Seminar Room E
- Proof of confinement or the absence thereof. Refinement of the Migdal-Kadanoff bounds (this could help distinguishing between $U(1)$ and $SU(2)$ in Tomboulis's approach)

Conclusions

- The TRG method allows us to achieve the Wilsonian program (block spinning) in an exact way
- It applies to most classical lattice models
- Successful numerical calculations for models with sign problems
- Many new applications possible
- Thanks!