Confinement in Coulomb gauge
What does the lattice teach us?

Lattice 2013 - Mainz, July 29th 2013, Giuseppe Burgio
Together with:

- M. Quandt
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GZ-Confinement [Gribov NPB 1978; Zwanziger NPB 1997]

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- restrict gauge functional $F(A)$ to either:
  - $-\vec{D} \cdot \vec{\nabla} > 0$: Gribov Region \( \Omega \), local maxima
  - $-\vec{D} \cdot \vec{\nabla} - 1$ singular at \( \partial \Omega \) (\( \partial \Lambda \))!
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  - absolute maxima $F(A)$: Fundamental Modular Region $\Lambda$

- $(-\vec{D} \cdot \vec{\nabla})^{-1}$ singular at $\partial \Omega$ ($\partial \Lambda$)!
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- Natural test: $A_\mu, \psi$ dispersion relations; static potential
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  - fermion mass $M(\bar{p})$
GZ scenario in CG

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- What about $\omega_{\psi}$?
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Dual superconductor! [Reinhardt PRL 2008]

Extension to fermions: running mass $M(\vec{p})$
Everything static "by construction!"
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$\vec{p}$ proportional to vacuum dielectric function

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Goals of CG lattice investigation

From $d$, $V_C$, $\omega_A$, $M(\bar{\rho})$ (and possibly $\omega_\psi$) test:
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- “Built-in” differences to continuum. Issues to be addressed!
Lattice calculations

Correlators in CG

1. Static “by construction”

\[ G(\vec{p}) = |\vec{p}| - 2d(\vec{p}) = \delta_{ab} \langle \bar{c}_a(\vec{p}) c_b(-\vec{p}) \rangle = \langle -(\vec{D} \cdot \vec{\nabla}) - 1 \rangle V_C(\vec{p}) = g^2 \delta_{ab} \langle -(\vec{D} \cdot \vec{\nabla}) - 1 \rangle \]

Almost directly comparable to continuum

2. Time (i.e. energy!) dependent

\[ D(\vec{p}, p_0) = \delta_{ab} \delta_{ij} \langle A^a_i(\vec{p}, p_0) A^b_j(-\vec{p}, -p_0) \rangle \]

\[ S(\vec{p}, p_0) = \delta_{AB} \langle \bar{\psi}_A(\vec{p}, p_0) \psi_B(-\vec{p}, -p_0) \rangle \]

Equal-time component ⇔ integrate over \( p_0 \).

Sizable cut-off effects: scale with \( a^{-1} t \)...
Lattice calculations

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Equal-time component ⇔ integrate over \( p_0 \).
Sizable cut-off effects: scale with \( a_t^{-1} \).
Minimizing lattice artifacts I

Anisotropic action. Closer to the Hamiltonian limit $a_t \to 0$!

$$S = \beta \sum_x \left\{ \gamma \sum_{j>i=1}^{d} \left( 1 - \frac{1}{N_c} \Re [\text{Tr} (P_{ij}(x))] \right) + \frac{1}{\gamma} \sum_{i=1}^{d} \left( 1 - \frac{1}{N_c} \Re [\text{Tr} (P_{i,d+1}(x))] \right) \right\}$$

$\gamma$ bare anisotropy. Must fix $\xi = \frac{a_s}{a_t}$ non-perturbatively!

$T = 0$ simulations on a $L^3 \times (\xi L)$ lattice.
Need $\xi \gg 1$ for good scaling!
Strong effect, e.g. on $F(A)$ for fixed $a_s$

![Graph showing data points for $a_s$ values]

Large corrections, scale with $a_t^2$, $a_t^4$ (glueball spectrum!)
Minimizing lattice artifacts II

Scaling violations in static gluon \( \sum_{p_0} D(\vec{p}, p_0) \)
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Scaling violations in static gluon \( \sum_{p_0} D(\vec{p}, p_0) \)

- model explicit \( p_0 \) dependence to do \( \int d\rho_0 \) “analytically”

- Static \( D(\vec{p}) \) renormalizable, “agrees” with Gribov’s formula

\[
\omega_A(|\vec{p}|) \propto \sqrt{|\vec{p}|^2 + \frac{M_G^4}{|\vec{p}|^2}}
\]

\( M_G = 0.856(8) \text{GeV} \). See H. Vogt’s talk...
Minimizing lattice artifacts III

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- Download configurations from ILDG, e.g. Asqtad (MILC)
  - Parameters rarely test the “deep” IR
Ghost from factor \(d\) (see also H. Vogt’s talk)
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and/or static vertex might not be trivial...
Extracting Coulomb string tension

- Asymptotic + leading corrections:

\[ V_C(r) = \sigma_C r + \mu - \frac{\lambda}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \]
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Hard to get clean fit: see H. Vogt’s talk.
Quark

From the Dirac operator in CG we get

\[ S^{-1}(\vec{p}, p_0) = i\vec{p}A_s(\vec{p}) + ip_0A_t(\vec{p}) + B_m(\vec{p}) \]
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- Define \( S(\vec{p}) \), \( \omega_\psi \) from \( S^{-1}(\vec{p}, p_0) \)
$Z(\vec{p})$ renormalizable

28^3\times 96, a = 0.086 \text{ fm}, m = 27.1 \text{ MeV}

20^3\times 64, a = 0.121 \text{ fm}, m = 31.5 \text{ MeV}
\( \alpha(\vec{p}) \) scale invariant

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Plot of \( \alpha(\vec{p}) \) vs. \( |k| \) [GeV] for different lattice sizes and masses.}
\end{figure}

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- \( 20^3 \times 64, a = 0.121 \text{ fm}, m = 31.5 \text{ MeV}, \text{ IPG} \)
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- \( 20^3 \times 64, a = 0.119 \text{ fm}, m = 63.1 \text{ MeV}, \text{ IPG} \)
$M(\bar{p})$ scale invariant (for fixed quark mass!)

![Graph showing $M(\bar{p})$ scale invariant for different values of $|k|$ with various marked points and error bars, indicating data points for different lattice resolutions and quark masses.](image-url)
Chiral limit for $M(\vec{p})$

\[ M(|\vec{k}|, m_b) = \frac{m_\chi(m_b)}{1 + b \frac{|\vec{k}|^2}{\Lambda^2} \log \left( e + \frac{|\vec{k}|^2}{\Lambda^2} \right)^{-\gamma}} + \frac{m_r(m_b)}{\log \left( e + \frac{|\vec{k}|^2}{\Lambda^2} \right)^{\gamma}} \]

\[ b = 2.9(1), \gamma = 0.84(2), \Lambda = 1.22(6) \text{ GeV}, m_\chi(0) = 0.31(1) \text{ GeV}, \frac{\chi^2}{\text{d.o.f.}} = 1.06 \]
Define $S(\vec{p})$, $\omega_\psi(|\vec{p}|) I$

Analogy with free fermion: Hamiltonian

$$S^H(\vec{p}) = \int d\rho_0 \ S(\vec{p}, \rho_0) \propto H$$
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$$S^H(\vec{p}) = \frac{Z(\vec{p})}{\alpha(\vec{p})} \frac{\sqrt{\vec{p}^2 + M^2(|\vec{p}|)}}{i \vec{p} + M(\vec{p})} = \frac{Z(\vec{p})}{\alpha(\vec{p})} \frac{-i \vec{p} + M(\vec{p})}{\sqrt{\vec{p}^2 + M^2(|\vec{p}|)}}$$

No divergences 😊
Define $S(\vec{p})$, $\omega_\psi(|\vec{p}|)$

Analogy with free fermion: Hamiltonian

$$S^H(\vec{p}) = \int dp_0 \, S(\vec{p}, p_0) \propto H$$

$$S^H(\vec{p}) = \frac{Z(\vec{p})}{\alpha(\vec{p})} \sqrt{\vec{p}^2 + M^2(|\vec{p}|)} = \frac{Z(\vec{p})}{\alpha(\vec{p})} \frac{-i\vec{p} + M(\vec{p})}{\sqrt{\vec{p}^2 + M^2(|\vec{p}|)}}$$

No divergences 😊

Coefficient$^{-1}$ of $-i\vec{p} + M(\vec{p})$ eigenvalue of $H$: quark effective energy!

$$\omega^H_\psi(|\vec{p}|) = \frac{\alpha(|\vec{p}|)}{Z(|\vec{p}|)} \sqrt{\vec{p}^2 + M^2(|\vec{p}|)}$$
Define \( S(\vec{p}) \), \( \omega_\psi(|\vec{p}|) \) II

Analogy with free fermion: Euclidean

Consistency between \( \int dp_0 \, S(\vec{p}, p_0) \) and \( \int dp_0 \, S^{-1}(\vec{p}, p_0) \)
Define $S(\vec{p})$, $\omega_\psi(|\vec{p}|)$ II

Analogy with free fermion: Euclidean

Consistency between $\int dp_0 S(\vec{p}, p_0)$ and $\int dp_0 S^{-1}(\vec{p}, p_0)$

$S^E(\vec{p}) = \Lambda \frac{Z(\vec{p})}{i\vec{p} + M(\vec{p})}$

$\Lambda \propto a_t^{-1} \rightarrow \infty \smile$
Define $S(\vec{p})$, $\omega_\psi(|\vec{p}|)$ II

Analogy with free fermion: Euclidean

Consistency between $\int dp_0 S(\vec{p}, p_0)$ and $\int dp_0 S^{-1}(\vec{p}, p_0)$

- $S^E(\vec{p}) = \Lambda \frac{Z(\vec{p})}{i\vec{p} + M(\vec{p})}$

- $\Lambda \propto a_t^{-1} \rightarrow \infty \ominus$

- Quark effective energy

$$\omega_\psi^E(|\vec{p}|) = \int dp_0 S^2(\vec{p}, p_0) = \frac{\alpha(|\vec{p}|)}{Z^2(|\vec{p}|)} \sqrt{\vec{p}^2 + M^2(|\vec{p}|)}$$
\[ \omega_{\psi}^{H,E}(|\vec{p}|) = \frac{\alpha(|\vec{p}|)}{Z(2)(|\vec{p}|)} \sqrt{p^2 + M^2(|\vec{p}|)} \]

\[ M(|\vec{p}|) \to m_\chi. \text{ Only } \frac{\alpha}{Z(2)} \text{ relevant for IR...} \]

Both IR enhanced! What happens at lower momenta?
Summary

- Static propagators in CG renormalizable
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- \( M(\vec{p}) \) well describes \( \chi \)-symmetry breaking
Outlook
Outlook

See H. Vogt’s talk!
Thanks!