# Crossing the Gribov horizon:

# an unconventional study of geometric properties

# of gauge-configuration space in Landau gauge

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# Abstract

We present a lower bound for the smallest nonzero eigenvalue of the Landau-gauge Faddeev-Popov matrix in terms of the smallest nonzero lattice momentum and of a parameter characterizing the geometry of the first Gribov region. This allows a simple and intuitive description of the infinite-volume limit in the ghost sector. In particular, we show how nonperturbative effects may be quantified by the rate at which typical thermalized and gauge-fixed configurations approach the Gribov horizon. Our analytic results are verified numerically through an informal, free and easy, approach.

# Confinement and Green's Functions

- How does linearly rising potential (seen in lattice QCD) come about?
- Green's functions carry all information of a QFT's physical and mathematical structure. Gluon propagator (two-point function) as the most basic quantity of QCD.
- Confinement given by behavior at large distances (small momenta) $\Rightarrow$ nonperturbative study of IR gluon propagator. Proposal by Mandelstam (1979) linking linear potential to infrared behavior of gluon propagator as $1/p^4$.

$$V(r) \sim \int \frac{d^3p}{p^4} e^{ip\cdot r} \sim r \ .$$

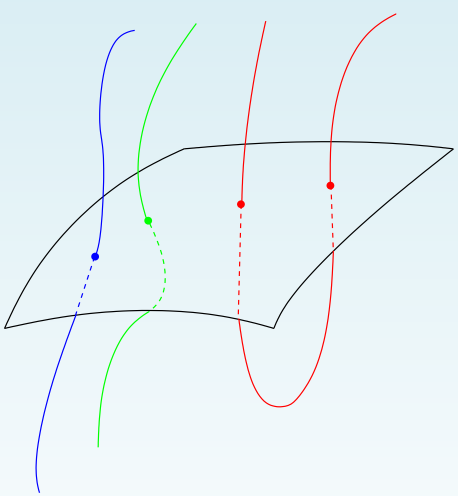
- Gribov-Zwanziger confinement scenario based on suppressed gluon propagator and enhanced ghost propagator in the infrared.

# Lattice Landau gauge

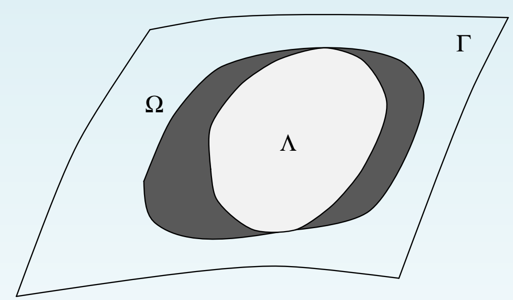
The (lattice) Landau gauge is imposed by minimizing the functional

$$S[U;\omega] \,=\, -Tr \sum_{x,\mu} \omega(x)\, U_\mu(x)\, \omega^\dagger(x + a\, e_\mu)\ ,$$

with respect to the lattice gauge transformation $\omega(x) \in SU(N)$ . This defines the first Gribov region $\Omega \equiv \{\, U \,:\, \partial \cdot A = 0,\, \mathcal{M} = -D \cdot \partial \,\geq\, 0 \,\}$ .



All gauge orbits intersect $\Omega$ (G.Dell'Antonio & D.Zwanziger, CMP 138, 1991) but the gauge fixing is not unique (Gribov copies). Absolute minima of $S[U;\omega]$ define the fundamental modular region $\Lambda$, free of Gribov copies on its interior. (Finding the absolute minimum is a spin-glass problem.)



# The Region $\Omega$: Properties

Three important properties have been proven (D.Zwanziger, NPB 209, 1982) for the Gribov region $\Omega$:

1. the trivial vacuum $A_\mu = 0$ belongs to $\Omega$;
2. the region $\Omega$ is convex;
3. the region $\Omega$ is bounded in every direction.

(The same properties can be proven also for the fundamental modular region $\Lambda$.)

The first property is trivial, since $A_\mu = 0$ implies that $\mathcal{M}(b, x; c, y)[0]$ is (minus) the Laplacian $-\partial^2$ (which is a semi-positive-definite operator).

# Convexity of $\Omega$

The gauge condition $\partial \cdot A = 0$ and the operators $D^{bc}(x,y)[A]$, $\mathcal{M}(b,x;c,y)[A] = -\partial^2 + \mathcal{K}[A]$ and $\mathcal{K}[A]$ are linear in the gauge field $A_\mu$:

$$\mathcal{M}[\gamma A_1 + (1-\gamma)A_2] \;=\; -\partial^2 + \mathcal{K}[\gamma A_1 + (1-\gamma)A_2]$$

$$=\; \gamma\left(-\partial^2 + \mathcal{K}[A_1]\right) \;+\; (1-\gamma)\left(-\partial^2 + \mathcal{K}[A_2]\right)$$

$$=\; \gamma\mathcal{M}[A_1] \;+\; (1-\gamma)\mathcal{M}[A_2]$$

and, for $\gamma \in [0,1]$, $\mathcal{M}[\gamma A_1 + (1-\gamma)A_2]$ is semi-positive definite if $\mathcal{M}[A_1]$ and $\mathcal{M}[A_2]$ are semi-positive definite. Also

$$\gamma\,\partial \cdot A_1 \;+\; (1-\gamma)\,\partial \cdot A_2 \;=\; 0$$

if $\partial \cdot A_1 = \partial \cdot A_2 = 0$. $\Longrightarrow$ The convex combination $\gamma A_1 + (1-\gamma)A_2$ belongs to $\Omega$, for any value of $\gamma \in [0,1]$, if $A_1, A_2 \in \Omega$.

# Boundary of $\Omega$

Using properties 1 and 2 and with $A_1 = 0$, $A_2 = A$, $1 - \gamma = \rho$ we have

$$\mathcal{M}[\rho A] \, = \, -\partial^2 + \mathcal{K}[\rho A] = \, (1-\rho)\,(-\partial^2) \, + \, \rho\,\mathcal{M}[A]$$

and, if $A \in \Omega$, then $\rho A \in \Omega$ for any value of $\rho \in [0, 1]$.

Since the color indices of $\mathcal{K}[A]$ are given by $\mathcal{K}^{bc}[A] \sim f^{bce} A^e_\mu$, we have that all the diagonal elements of $\mathcal{K}[A]$ are zero $\Longrightarrow$ the trace of the operator $\mathcal{K}[A]$ is zero.

The operator $\mathcal{K}^{bc}_{xy}[A]$ is real and symmetric (under simultaneous interchange of $x$ with $y$ and $b$ with $c$) and its eigenvalues are real $\Longrightarrow$ at least one of the eigenvalues of $\mathcal{K}[A]$ is (real and) negative. If $\phi_{neg}$ is the corresponding eigenvector, that for a sufficiently large (but finite) value of $\rho > 1$ the scalar product $(\phi_{neg}, \mathcal{M}[\rho A]\phi_{neg})$ must be negative $\Longrightarrow$ $\mathcal{M}[\rho A]$ is not semi-positive definite and $\rho A \notin \Omega$.

# The Infinite-Volume Limit (I)

In order to study the infra-red sector of the theory on the lattice we need to remove the infra-red cutoff $\Longrightarrow$ take the infinite-volume limit.

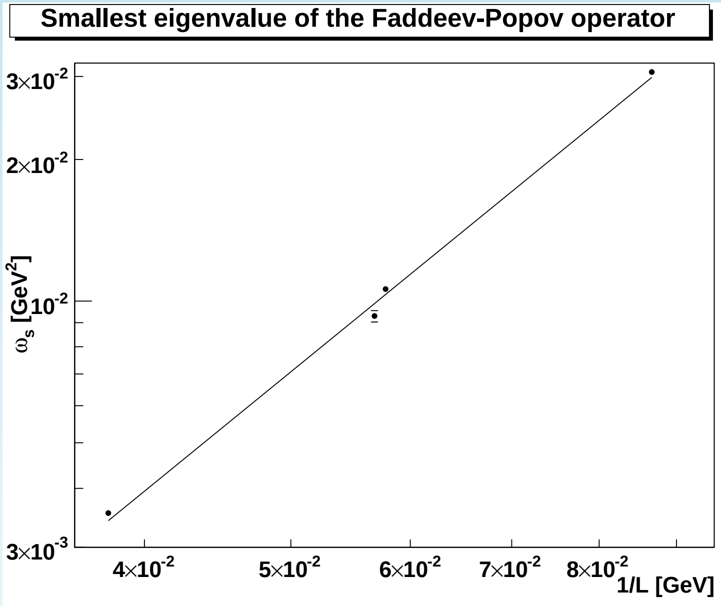
**The Main Axiom**

> *At very large volumes the functional integration gets concentrated on the boundary $\partial\Omega$ of the first Gribov region $\Omega$.*

For very large dimensionality and for large volumes, by considering the interplay among the volume of the configuration space, the Boltzmann weight and the step function used to constrain the functional integration to $\Omega$, one expects that entropy favors configurations near the boundary $\partial\Omega$.

# The Infinite-Volume Limit (II)

One can check if lattice data support $\lambda_1[A] \to 0$ in the infinite-volume limit $\Longrightarrow A \in \partial\Omega$.




Infinite-volume limit extrapolation $\lambda_1[A] \sim L^c$ for the $3d$ SU(2) case (A.C., A.Maas & T.Mendes, PRD 74, 2006). (Similar results in $2d$ and $4d$.)

# The Infinite-Volume Limit (III)

On the lattice, the ghost propagator is given by

$$G(p) \;=\; \frac{1}{N_c^2 - 1} \sum_{x\,,\,y\,,\,a} \frac{e^{-2\pi i\, k\cdot(x-y)}}{V} \, \mathcal{M}^{-1}(a,x;a,y)$$

$$=\; \frac{1}{N_c^2 - 1} \sum_{i,\lambda_i \neq 0} \frac{1}{\lambda_i} \sum_a |\widetilde{\psi}_i(a,p)|^2 \;,$$

where $\psi_i(a,x)$ and $\lambda_i$ are the eigenvectors and eigenvalues of the FP matrix. Then, one can prove (A.C. & T.Mendes, PRD 78, 2008) that

$$\frac{1}{N_c^2 - 1} \, \frac{1}{\lambda_1} \sum_a |\widetilde{\psi}_1(a,p)|^2 \;\leq\; G(p) \;\leq\; \frac{1}{\lambda_1} \;.$$

If $\lambda_1$ behaves as $L^{-2-\alpha}$ in the infinite-volume limit, $\alpha > 0$ is a necessary condition to obtain an IR-enhanced ghost propagator $G(p)$.

# The Infinite-Volume Limit (IV)

## New Axiom Formulation

*The key point seems to be the rate at which $\lambda_1$ goes to zero, which, in turn, should be related to the rate at which a thermalized and gauge-fixed configuration approaches $\partial\Omega$.*

These are only qualitative statements!

How do we relate $\lambda_1$

to the geometry of the Gribov region $\Omega$ ?

# Lower bound for $\lambda_1$ (I)

Consider a configuration $A'$ belonging to the boundary $\partial\Omega$ of $\Omega$ and write

$$\lambda_1 \left[ \mathcal{M}[\rho A'] \right] \,=\, \lambda_1 \left[ (1-\rho)\,(-\partial^2) \,+\, \rho\,\mathcal{M}[A'] \right] \;.$$

From the second property, $\rho A' \in \Omega$ for $\rho \in [0,1]$. Since

$$\lambda_1 \left[ (1-\rho)\,(-\partial^2) \,+\, \rho\,\mathcal{M}[A'] \right]$$

$$=\, \min_{\chi} \left( \chi \,, \left[\, (1-\rho)\,(-\partial^2) \,+\, \rho\,\mathcal{M}[A'] \,\right]\, \chi \right) \;,$$

with $(\chi\,,\chi) = 1$ and $\chi \neq$ constant, we can use the concavity of the minimum function

$$\boxed{\min_{\chi} \left(\chi, [M_1 + M_2]\,\chi\right) \,\geq\, \min_{\chi} \left(\chi, M_1\chi\right) + \min_{\chi} \left(\chi, M_2\chi\right) \,.}$$

# Lower bound for $\lambda_1$ (II)

We find

$$\lambda_1\left[\,\mathcal{M}[\rho A']\,\right] \;=\; \lambda_1\left[(1-\rho)\,(-\partial^2)\;+\;\rho\,\mathcal{M}[A']\right]$$

$$\geq (1-\rho)\min_{\chi}\left(\chi,(-\partial^2)\,\chi\right)\;+\;\rho\min_{\chi}\,\left(\chi\,,\mathcal{M}[A']\,\chi\right)$$

$$=\;(1-\rho)\,p^2_{min}\;.$$

Recall that $A' \in \partial\Omega \Longrightarrow$ the smallest non-trivial eigenvalue of the FP matrix $\mathcal{M}[A']$ is null, and that the smallest non-trivial eigenvalue of (minus) the Laplacian $-\partial^2$ is $p^2_{min}$.

In the Abelian case one has $\mathcal{M} = -\partial^2$ and $\lambda_1 = p^2_{min}$. $\Longrightarrow$ All non-Abelian effects are included in the $(1-\rho)$ factor (and in the inequality).

# Lower bound for $\lambda_1$ (III)

As the lattice side $L$ goes to infinity, $\lambda_1 \left[ \mathcal{M}[\rho A'] \right]$ cannot go to zero faster than $(1 - \rho)\, p_{min}^2$. Since $p_{min}^2 \sim 1/L^2$ at large $L \Longrightarrow$ $\lambda_1$ behaves as $L^{-2-\alpha}$ in the same limit, with $\alpha > 0$, only if $1 - \rho$ goes to zero at least as fast as $L^{-\alpha}$.

With $\rho A' = A$ the above inequality may also be written as

$$\boxed{\lambda_1 \left[ \mathcal{M}[A] \right] \; \geq \; [1 - \rho(A)]\, p_{min}^2 \; .}$$

Here $1 - \rho(A) \leq 1$ measures the distance of a configuration $A \in \Omega$ from the boundary $\partial\Omega$ (in such a way that $\rho^{-1} A \in \partial\Omega$). This result applies to any Gribov copy belonging to $\Omega$.

# Other Inequalities

As a consequence of the above result, we can find several new bounds with a simple geometrical interpretation. Using

$$G(A, p_{min}) = \frac{1}{p_{min}^2} \frac{1}{1 - \sigma(A, p_{min})} \leq \frac{1}{\lambda_1(A)}$$

we have

$$G(A, p) \leq \frac{1}{[1 - \rho(A)]\, p_{min}^2}$$

and

$$\sigma(A, p_{min}) \leq \rho(A) \; ,$$

which is a stronger version of the no-pole condition [$\sigma(A, p) < 1$ for $p^2 > 0$], used to impose the restriction of the physical configuration space to the region $\Omega$. Similarly, for the horizon function one can prove

$$\frac{H(A)}{dV(N_c^2 - 1)} \equiv h(A) \leq \rho(A) \; .$$

[Note: $\sigma(A, 0) = h(A)$ to all orders in the gauge coupling (M.A.L.Capri et al., PLB 2013).]

# Simulating the Math

We used 70 configurations, for the SU(2) case at $\beta = 2.2$, for $V = 16^4$, $24^4, 32^4, 40^4$ and 50 configurations for $V = 48^4, 56^4, 64^4, 72^4, 80^4$.

In order to verify the third property of the region $\Omega$ we applied scale transformations $\widehat{A}_\mu^{(i)}(x) = \tau_i A_\mu^{(i-1)}(x)$ to the gauge configuration $A$ with

- $\tau_0 = 1$,
- $\tau_i = \delta\,\tau_{i-1}$,
- $\delta = 1.001$ if $\lambda_1 \geq 5\,\times\,10^{-3}$,
- $\delta = 1.0005$ if $\lambda_1 \in [5\,\times\,10^{-4},\,5\,\times\,10^{-3})$
- and $\delta = 1.0001$ if $\lambda_1 < 5\,\times\,10^{-4}$,

where $\lambda_1$ is evaluated at the step $i-1$. After $n$ steps, the modified gauge field $\widehat{A}_\mu^{(n)}(x)$ does not belong to the region $\Omega$ anymore, i.e. the eigenvalue $\lambda_1$ of $\mathcal{M}[\widehat{A}^{(n)}]$ is negative (while $\lambda_2$ is still positive).

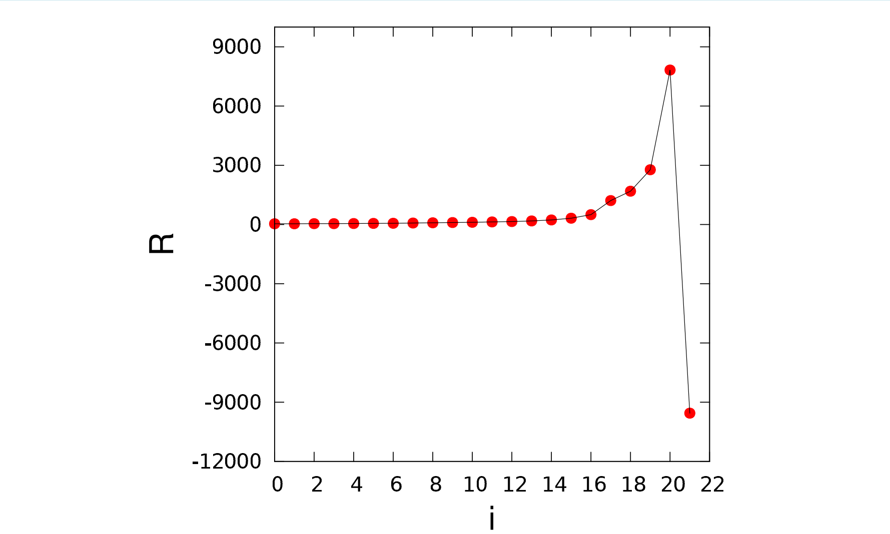
# Crossing the Horizon (I)

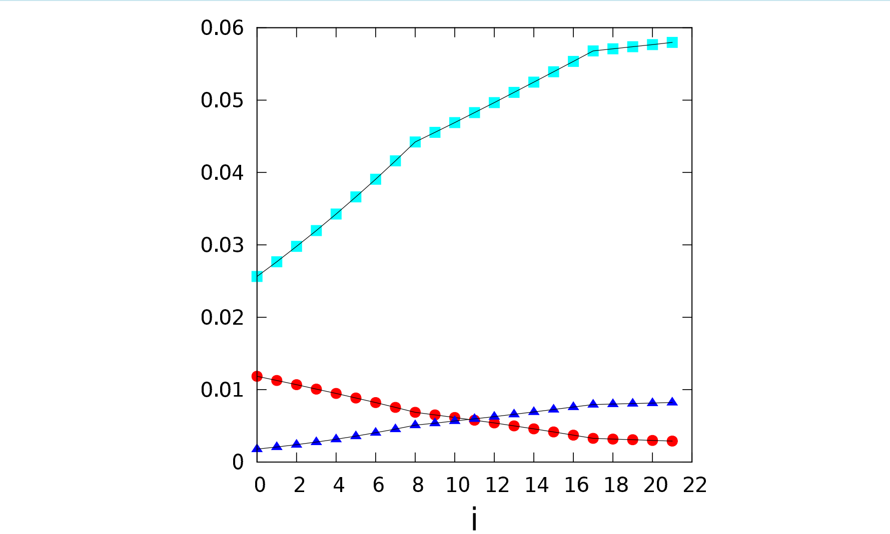
| $N$ | $\max(n)$ | $\min(n)$ | $\langle n \rangle$ | $R_{\rm before}/1000$ | $R_{\rm after}/1000$ |
|---|---|---|---|---|---|
| 16 | 30 | 6 | 17.2 | 15(3) | -30(12) |
| 24 | 27 | 4 | 15.1 | 20(7) | -26(6) |
| 32 | 19 | 5 | 11.7 | 26(9) | -51(20) |
| 40 | 18 | 4 | 9.4 | 155(143) | -21(6) |
| 48 | 13 | 2 | 7.8 | 21(5) | -21(5) |
| 56 | 12 | 3 | 7.6 | 16(4) | -21(7) |
| 64 | 11 | 2 | 6.8 | 20(7) | -42(18) |
| 72 | 11 | 2 | 6.1 | 129(96) | -42(13) |
| 80 | 12 | 3 | 6.1 | 15(4) | -24(4) |

The maximum, minimum and average number of steps $n$, necessary to "cross the Gribov horizon" along the direction $A_\mu^b(x)$, as a function of the lattice size $N$. We also show the ratio $R[A] = (S''')^2/(\lambda_1\, S'''')$ for the modified gauge fields $\tau_{n-1} A_\mu^b(x)$ and $\tau_n A_\mu^b(x)$, i.e. for the configurations immediately before and after crossing $\partial\Omega$.

# Crossing the Horizon (II)

The case of a typical configuration.



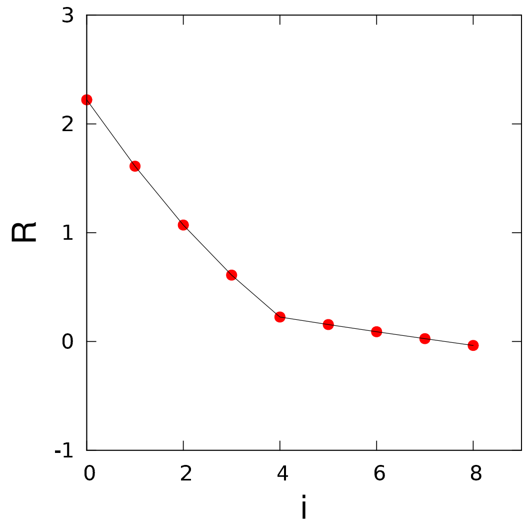



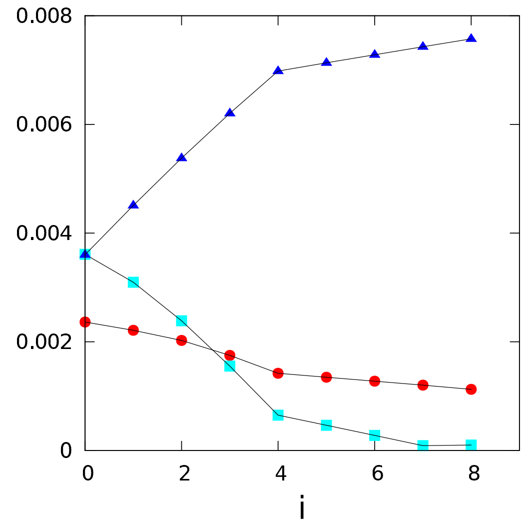

Plot of the ratio $R$, as a function of the iteration step $i$, for a configuration with lattice volume $16^4$.

Plot of $\lambda_2$ (full circes), $|\mathcal{E}'''|$ (full squares) and $\mathcal{E}''''$ (full triangles) as a function of the iteration step $i$, for the same configuration.

# Crossing the Horizon (III)

The case $R \approx 0$ (configuration on $\partial\Omega \cap \partial\Lambda$ ?).



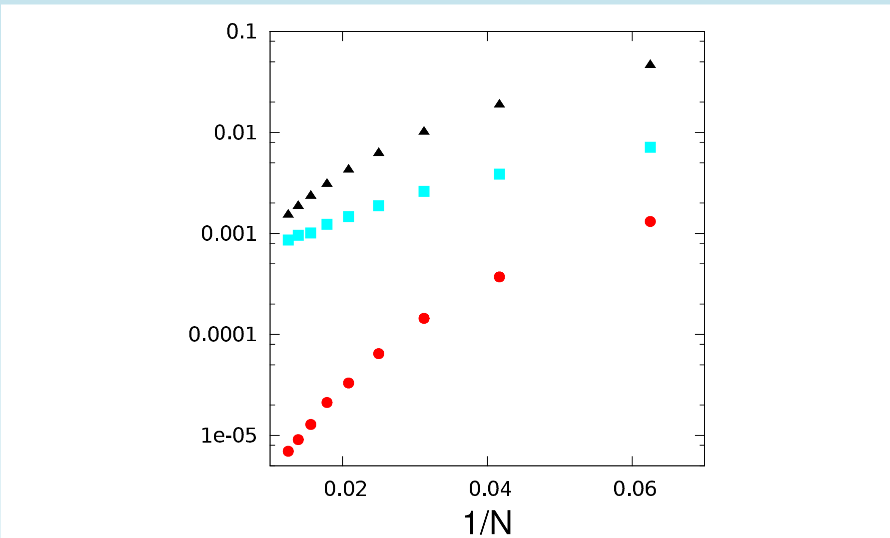



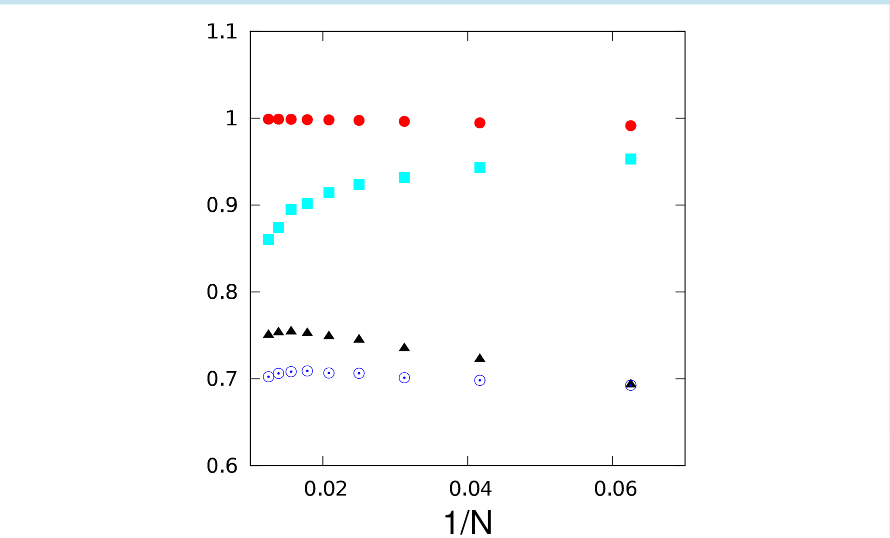

Plot of the ratio $R$, as a function of the iteration step $i$, for a configuration with lattice volume $48^4$.

Plot of $\lambda_2$ (full circes), $|\mathcal{E}'''|$ (full squares) and $\mathcal{E}''''$ (full triangles) as a function of the iteration step $i$, for the same configuration.

# New Inequalities

Using $A' = \widetilde{\tau}\, A \equiv A(\tau_{n-1} + \tau_n)/2 \in \partial\Omega$ and $\rho = 1/\widetilde{\tau} < 1$.



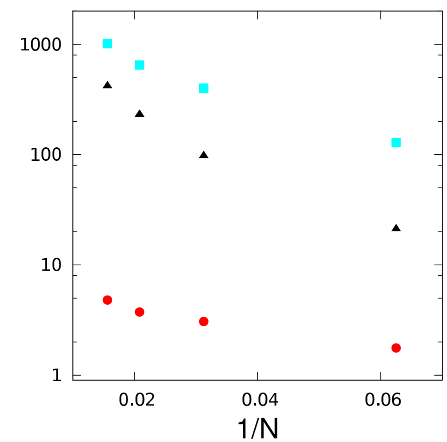

Plot of $1/G(p_{min})$ (full triangles), $\lambda_1$ (full squares) and of the quantity $(1-\rho)\, p_{min}^2$ (full circles) as a function of the inverse lattice size $1/N$.

Plot of $h$ (empty circles), $\sigma(p_{min})$ (full triangles), the quantity $1 - \lambda_1/p_{min}^2$ (full squares) and the upper bound $\rho$ (full circles) as a function of the inverse lattice size $1/N$. (Note $\lambda_1$ goes to zero as $p_{min}^2$.)

# New Inequalities

The new inequality $\lambda_1 \left[\, \mathcal{M}[A] \,\right] \geq [1 - \rho(A)]\, p_{min}^2$ becomes an equality only when the eigenvectors corresponding to the smallest nonzero eigenvalues of $\mathcal{M}[A]$ and $-\partial^2$ coincide. $\Longrightarrow$ The eigenvector $\psi_{min}$ is very different from the plane waves corresponding to $p_{min}$.

Plot of $G(p_{min})$ (full triangles), the lower bound (full circles) and the upper bound (full squares) as a function of the inverse lattice size $1/N$.




These results explain the non-enhancement of $G(p)$ in the IR.

# Conclusions

- Our new bounds suggest all non-perturbative features of a minimal-Landau-gauge configuration $A \in \Omega$ to be related to its normalized distance $\rho$ from the "origin" $A = 0$ or, equivalently, to its normalized distance $1 - \rho$ from the boundary $\partial\Omega$.
- Most lattice configurations are very close ($\rho[A] \approx 1$) to the first Gribov horizon $\partial\Omega$.
- Our data suggest that configurations producing an infrared-enhanced ghost propagator should almost saturate the new bound, i.e. their eigenvector $\psi_1$ should have a large projection on at least one of the plane waves corresponding to $p^2_{min}$.

This would imply that nonperturbative effects, such as color confinement, are driven by configurations whose FP matrix $\mathcal{M}$ is "dominated" by an eigenvector $\psi_1$ very similar to the corresponding eigenvector of $\mathcal{M} = -\partial_\mu \partial^\mu$ , i.e. to $\psi_1$ of the free case!