A property of fermions at finite densities by a reduction formula of fermion determinant

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This talk is based on
KN, arXiv:1204.6480
KN, Motoki, Nakagawa, Nakamura, Saito, PTEP01A103('12)
Introduction

- QCD at low-T and finite-$\mu$ is a challenge for lattice simulations: the puzzle about a value of $\mu$ at $T=0$ where $n_q$ or $\langle q\bar{q} \rangle$ start to change. [e.g. Gibbs('86), Glasgow group('91, '96)]

$$\mu_c = m_\pi / 2 \neq m_N / 3$$

- The early onset problem was discussed in
  - a propagator matrix [Gibbs('86), Glasgow group(e.g. ‘91, ’96), etc ]

$$am_\pi = -Nt^{-1} \ln \max_{|\lambda_n|<1} |\lambda_n|^2$$

- $D(\gamma_0 D)$ in isospin $\mu$ [Davies, Klepfish ’90, Cohen(‘04)]

$$\gamma_0 (D + m) |\psi\rangle = \epsilon |\psi\rangle \quad \epsilon_{\min} = m_\pi / 2$$

- We need further investigation of early onset problem to study low-T regions in lattice simulations
Introduction

• We explain the early onset problem in terms of a Fermi distribution function obtained from quark number operator using a reduction formula.

Reduction formula for fermion determinant

Quark number operator = Fermi distribution for single quark for each conf.

Lattice simulations for reduced matrix

Early onset problem and quark’s Fermi distribution
Reduction formula

- Reduction formula is a method to calculate the temporal part of the fermion determinant $\det \Delta$
  - for staggered and Wilson fermions

$$\det \Delta = \xi^{-N_{\text{red}}/2} C_0 \det(\xi + Q)$$

$$\xi = e^{-\mu/T}$$

✓ $Q$ and $C_0$ are functions of link variables
✓ $N_{\text{red}} = 12 N_s^3$
Reduction formula

- Eigenvalues $\lambda$ of $Q$
  - pair ($\gamma_5$-hermiticity)
    \[ \lambda_n \leftrightarrow 1/\lambda_n^* \]
  - gap (for finite quark mass)
    \[ ||\lambda| - 1| \neq 0 \]
Gap in spectrum of $Q$

- The gap size is related to the pion mass for large $N_t$.
- Onset of $n_q$ is closely related to the gap for larger $N_t$.

$[\text{Gibbs('86), Glasgow group('91, '97), etc}]$

\[ am_\pi = -N_t^{-1} \ln \max_{|\lambda_n|<1} |\lambda_n|^2 \]

\[ am_\pi = \lim_{N_t \to \infty} \left( -\frac{1}{N_t} \ln \left| c \sum_{n=1}^{3V} \lambda_n \right| \right) \]

Gibbs('86) & Fodor, Szabo, Toth('07)

[We assume they are equivalent at $T=0$.]
Fermion determinant

- Fermion determinant can be written as

\[
\det \Delta(\mu) = C_0 \xi^{-N_r/2} \prod_{n=1}^{N_r} (\lambda_n + \xi) \quad \lambda_n \leftrightarrow 1/\lambda_n^*
\]

\[
= C_0 \prod_{n=1}^{N_r/2} (\lambda_n^*)^{-1} \prod_{n=1}^{N_r/2} (1 + \lambda_n \xi^{-1})(1 + \lambda_n^* \xi) \quad |\lambda| < 1
\]

- analogous to Matsubara- frequency summation form [Adams ’04]

- Quark number operator is given by

\[
\hat{n} = \frac{\partial}{\partial \mu/T} \ln \det \Delta(\mu) = \sum_{|\lambda|<1} \left( \frac{\lambda \xi^{-1}}{1 + \lambda \xi^{-1}} - \frac{\lambda^* \xi^{-1}}{1 + \lambda^* \xi^{-1}} \right)
\]
Nt-dependence of spectrum of $Q$

- A reduced matrix $Q$ describes the propagation of a quark between initial and final time slices
  - similar to the Polyakov loop

  $$Q = A_1 A_2 \cdots A_{N_t}$$

- Eigenvalues of $Q$ are expected to scale as [KN, Nakamura('10)]

  $$\lambda = l^{N_t} = e^{-\epsilon/T + i\theta}$$

- It should be confirmed in lattice simulations.
Eigenvalue density (original vs scaled)

\[ \rho(\ln |\lambda_n|) \quad \rho((\ln |\lambda_n|)/N_t) \]

- Lattice results reproduce the scaling behavior

\[ \lambda = e^{-\epsilon/T} + i\theta \]

[KN, Motoki, Nakagawa, Nakamura, Saito (‘12)]

- Lattice setup
  \[ \beta=1.86 (\text{same value of } a) \]
  \[ 8^3 \times N_t, \text{mps/mV}=0.8, \]
  \[ \text{RG-improved gauge and clover-Wilson fermion with } N_f=2 \]
Eigenspectrum $|\lambda_n|$ (original vs scaled)

$|\lambda_n|, |\lambda_n|^{1/N_t}$

• Lattice results reproduce the scaling behavior

$$\lambda = e^{-\epsilon/T} + i \theta$$

• Lattice setup
  $\beta=1.86$ (same value of $a$)
  $8^3 \times N_t$, $\text{mps/mV}=0.8$, RG-improved gauge and clover-Wilson fermion with $N_f=2$
Fermi distribution for each configuration

• Quark number operator is given by

\[
\hat{n} = \sum_{n=1}^{N_r/2} \left( \frac{\lambda_n \xi^{-1}}{1 + \lambda_n \xi^{-1}} - \frac{\lambda_n^* \xi}{1 + \lambda_n^* \xi} \right) \quad \lambda = e^{-\epsilon/T+i\theta}
\]

\[
= \sum_{n=1}^{N_r/2} \left( \frac{1}{1 + e^{(\epsilon-\mu)/T-i\theta}} - \frac{1}{1 + e^{(\epsilon+\mu)/T+i\theta}} \right)
\]

• This is the Fermi distribution of single quark for each configuration. (winding number of $\lambda$ is 1).

• Eigenvalues of the reduced matrix are identified as energy levels of single quark for each configuration.
Fermi distribution for each configuration

• Quark number operator is given by

\[ \hat{n} = \sum_{n=1}^{N_r/2} \left( \frac{\lambda_n \xi^{-1}}{1 + \lambda_n \xi^{-1}} - \frac{\lambda_n^* \xi}{1 + \lambda_n^* \xi} \right) \]

\[ \lambda = e^{-\epsilon/T+i\theta} \]

\[ \hat{n} = \sum_{n=1}^{N_r/2} \left( \frac{1}{1 + e^{(\epsilon-\mu)/T-i\theta}} - \frac{1}{1 + e^{(\epsilon+\mu)/T+i\theta}} \right) \]

• Onset of quark number at T=0

\[ \hat{n} = 0 \quad \text{for} \quad \mu < \epsilon_{\text{min}} \]

\[ \neq 0 \quad \text{for} \quad \mu > \epsilon_{\text{min}} \]

• Low-lying energy levels \( \sim \) eigenvalues near the gap

\[ \epsilon_{\text{min}} \sim m_\pi / 2 \]
Low-mode Approximation

• To obtain $\mu = m_N/3$, we need further investigations of eigenvalues near the gap (larger $V$, fluctuations, lower $T$...)

• However, the eigen-problem of $Q$ becomes ill-conditioned at low-$T$, and requires large memory for large volume.

• High-lying modes (large and small $\lambda$) would be irrelevant of low-energy physics.
Low-mode Approximation

• We consider a low-mode expansion, e.g,

$$\hat{n} = \sum_{n=1}^{M} \left( \frac{\lambda_n \xi^{-1}}{1 + \lambda_n \xi^{-1}} - \frac{\lambda_n^* \xi}{1 + \lambda_n^* \xi} \right) + (n > M + 1)$$

– We test the approximation for the Taylor coefficients of EoS ($c_2$ and $c_4$)

$$c_n = \frac{1}{n!} \left( \frac{N_t}{N_s} \right)^3 T^n \frac{\partial^n \ln Z}{\partial \mu^n}$$
Results

- Results with M=200, 1000, 2000, and 3072(all)
- Lattice setup
  RG-improved gauge and clover-Wilson fermion with Nf=2
  Mass & size : mps/mV=0.8, 8^3x4
 Configs. : 10K trajectories at \( \mu = 0 \)
  Measurement : 400 configs.
Summary

• Manifest representation of the early onset problem was shown using lattice QCD with the reduction formula.

  – The correspondence between the quark number operator and the Fermi distribution is clarified.

  – The eigenvalues of the reduced matrix are identified as energy levels of single quark for each configuration, which would be useful for applications.

  – We test the low-mode expansion, which is helpful for further studies of the early onset problem.
Reduction formula

- Eigenvalues $\lambda$ of $Q$
  - pair $\lambda_n \leftrightarrow 1/\lambda^*_n$
  - gap $||\lambda| - 1| \neq 0$
Introduction

• Need better understanding spectrum of fermions in QCD
• We discuss the early onset problem using the reduced matrix $Q$ in the reduction formula

- lattice results of a $Nt$-scaling of eigenvalues of $Q$ (‘Lat12)
- derivation of the Fermi distribution of single quark from fermion determinant
- identification of eigenvalues of $Q$ as energy levels of single quark for each configuration
- application to the early onset problem and introduction of low-mode approximation
Reduction formula

- A fermion matrix in t-t matrix rep.

\[ \Delta = B - e^{\mu a} V - e^{-\mu a} V^\dagger \]

\[ \Delta = \begin{pmatrix} \Box & \bigtriangleup & \bigtriangleup & \bigtriangleup \\ \bigtriangleup & \Box & \bigtriangleup & \bigtriangleup \\ \bigtriangleup & \bigtriangleup & \ddots & \bigtriangleup \\ \bigtriangleup & \bigtriangleup & \bigtriangleup & \Box \end{pmatrix} \]

\[ \text{det} \Delta = \xi^{-N_{\text{red}}/2} C_0 \text{det}(\xi + Q) \]

\[ \xi = e^{-\mu/T} \]

- \( Q \) and \( C_0 \) are functions link variables
- \( N_{\text{red}} = 12 \ Ns^3 \)

Approximations at low temperatures ($T \sim 0.5 T_c$)

- The average of the quark number density.

- It converges at small $M \sim 20$. 

![Graph showing the average quark number density versus $M$ with a horizontal line indicating convergence at small $M$]
Low Temperature Limit

\[ \det \Delta = \xi^{-N_{\text{red}}/2} C_0 \prod_{|\lambda|>1} (\xi + \lambda_n) \prod_{|\lambda|<1} (\xi + \lambda_n) \]

\[ = C_0 \prod_{|\lambda|>1} (\lambda_n) \]

\[ \mu < m_\pi / 2 \]

\[ e^{-\mu/T} = e^{-0.2Nt} \]

\[ \max |\lambda| = e^{-m_\pi/(2T)} \]

Ev distribution is bounded by pion mass
Reduction formula

- Fermion determinant $\det \Delta$
  - calculating the temporal part of $\det \Delta$ leads to

\[
\det \Delta = \xi^{-N_{\text{red}}/2} C_0 \det(\xi + Q)
\]

\[
\xi = e^{-\mu/T}
\]

\[
N_{\text{red}} = 4N_c N_x N_y N_z
\]

\[
Q = A_1 A_2 \cdots A_{N_t}
\]

- $Q$ and $C_0$ are independent of $\mu$
- chemical potential and gauge fields are separated

Complex potential problem

A free energy or complex potential satisfies electromagnetic analogy (the same as Lee-Yang zero theorem)

\[ h(\mu) = \frac{\ln \det \Delta(\mu)}{N_r}, \quad (N_r = 4N_c V_s) \]

\[ = \frac{\mu}{2} \ln \xi + N_r^{-1} \sum_{n=1}^{N_r} \ln(\xi + \lambda_n) + (\mu - \text{indep.}) \]

\[ (\partial_x^2 + \partial_y^2) \text{Re}[h] = -\pi \delta(\xi) + 2\pi N_r^{-1} \sum_{n=1}^{N_r} \delta(\xi + \lambda_n) \]

\[ = -\pi \delta(\xi) + 2\pi \rho(-\xi) \]
Complex potential problem

complex potential

\[ h(\mu) = \frac{\ln \det \Delta(\mu)}{N_r}, \quad (N_r = 4N_cV_s) \]

\[ = \frac{\mu}{2} \ln \xi + N_r^{-1} \sum_{n=1}^{N_r} \ln(\xi + \lambda_n) + (\mu - \text{indep.}) \]

h is analytic function of mu except for det = 0,

\[ \partial_x X = \partial_y Y \]
\[ \partial_y X = -\partial_x Y \]
\[ \nabla_\xi^2 X = 0 \]
\[ \nabla_\xi^2 Y = 0 \]

Cauchy-Riemann

Laplace
Complex potential problem

Complex potential satisfies

\[(\partial_x^2 + \partial_y^2)\text{Re}[h] = -\pi \delta(\xi) + 2\pi N_r^{-1} \sum_{n=1}^{N_r} \delta(\xi + \lambda_n)\]

\[= -\pi \delta(\xi) + 2\pi \rho(-\xi)\]

Gauss’s law (2D electrostatic problem)

\[\vec{n} = \nabla \text{Re}[h]\]

Electrostatic analogy (Lee-Yang (‘50))

quark number density \(\sim\) electric field

eigenvalues of reduced matrix \(\sim\) (opposite) location of charge
Spectral property - Volume

- Volume dependence

![Graph showing spectral property volume dependence with density on the y-axis and ln |λ| on the x-axis. Two plots are shown: one for N_s = 8 and another for N_s = 10. Two more plots compare densities for different β values with 10^3 x 4 and 8^3 x 4 configurations.]
Quark mass dependence

\[ m_{ps}/m_{V} = 0.6 \text{ (red), 0.8 (blue)} \]

Histograms: $|\text{ev}|$ (Left), arg(ev) (Right), confinement (top), deconfinement (bottom)
Chemical Potential Dependence at Low $T$

$$R = \left( \frac{\text{det } \Delta(\mu)}{\text{det } \Delta(0)} \right)^2 = |R|e^{i\theta}$$

- $\text{det } \Delta$ is insensitive to $\mu$ for $\mu a < 0.5$.
- $\mu$-dependence appears at $\mu a = 0.5$.
  - This value is close to $m_\pi/2$ in the present setup.

**Diagram:**
- Graph showing $\langle \cos \theta \rangle$ vs. $T/T_{pc}$ for different $\mu a$ values.
- Graph showing $\ln |R|$ vs. $\mu a$ for different $T/T_{pc}$ values.

Lattice 2013, Mainz, July28-Aug03
Average phase factor vs $\mu$
Taylor coefficients of EoS

\[ f(\mu) - f(0) = \sum_{n=1}^{\infty} c_{2n} (\mu/T)^{2n} \]

– slow convergence of the Taylor series of EoS
– small S/N ratio of chemical potential dependence
Gap is related to pion mass

\[ a m_\pi = - \frac{1}{Nt} \ln \max_{|\lambda_n| < 1} |\lambda_n|^2 \]

Gibbs('86). Eigenvalues and mpi

See also, Fodor, Szabo, Toth ('06). Eigenvalues and hadron spectrum

- At low \( T \), \( mpi/T \) is well fitted with \( a/T \), \( a = 4 \ Tpc \) (\( mq \) heavy)
- At high \( T \), \( mpi \) approaches to a constant