A property of fermions at finite densities by a reduction formula of fermion determinant

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This talk is based on
KN, arXiv:1204.6480
KN, Nakamura, Motoki, PoS LATTICE2012 (2012) 094
KN, Motoki, Nakagawa, Nakamura, Saito, PTEP01A103('12)

JICFuS

N. E. C. O.

Introduction

- QCD at low-T and finite-μ is a challenge for lattice simulations: the puzzle about a value of μ at T=0 where n_q or <qbar q> start to change. [e.g.Gibbs('86), Glasgow group('91, '96)]

$$\mu_c = m_\pi/2 \neq m_N/3$$

- The early onset problem was discussed in
  - a propagator matrix [Gibbs('86), Glasgow group(e.g. '91, '96), etc]

$$am_\pi = -Nt^{-1} \ln \max_{|\lambda_n|<1} |\lambda_n|^2$$

  - $D(\gamma_0 D)$ in isospin μ [Davies, Klepfish '90, Cohen('04)]

$$\gamma_0(D+m)|\psi\rangle = \epsilon|\psi\rangle \qquad \epsilon_{\rm min} = m_\pi/2$$

- We need further investigation of early onset problem to study low-T regions in lattice simulations

Introduction

- We explain the early onset problem in terms of a Fermi distribution function obtained from quark number operator using a reduction formula.

Reduction formula for fermion determinant

Quark number operator
=
Fermi distribution for
single quark for each conf.

Lattice simulations for
reduced matrix

Early onset problem and quark's Fermi distribution

Reduction formula

- Reduction formula is a method to calculate the temporal part of the fermion determinant det Δ
  - for staggered and Wilson fermions

    [Gibbs ('86), Hasenfratz & Toussaint('92), Adams('03, '04), Borici('04), KN&AN('10), Alexandru &Wenger('10)]

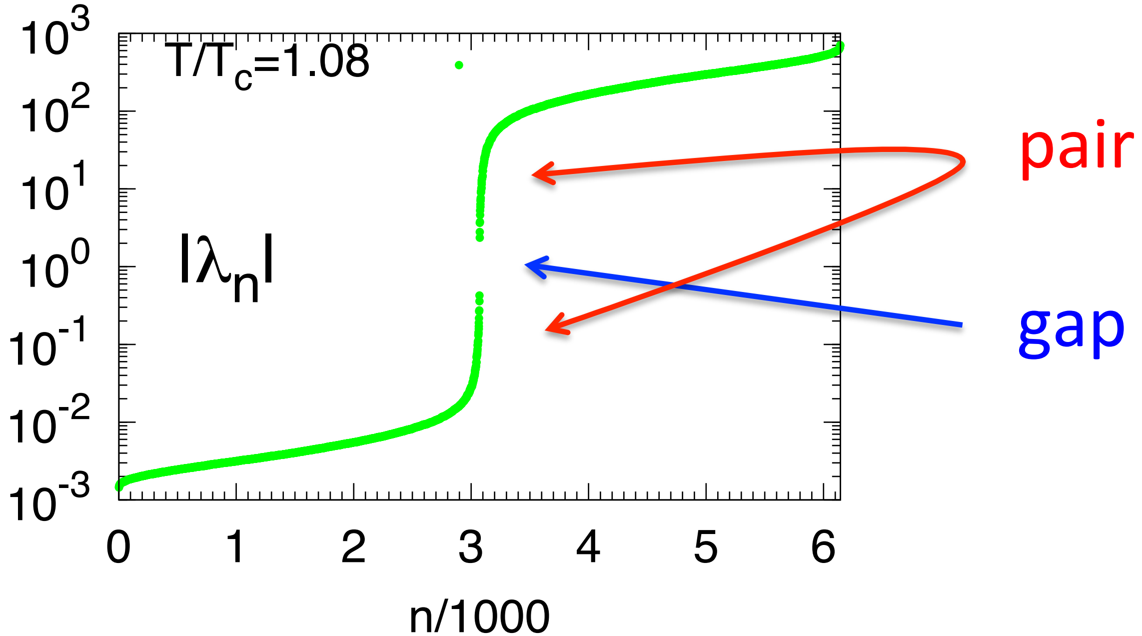
$$\det \Delta = \xi^{-N_{\mathrm{red}}/2} C_0 \det(\xi + \underline{Q})$$

$$\xi = e^{-\mu/T}$$

✓ Q and C_0 are functions of link variables
✓ $N_{\mathrm{red}}=12\, Ns^3$

Reduction formula

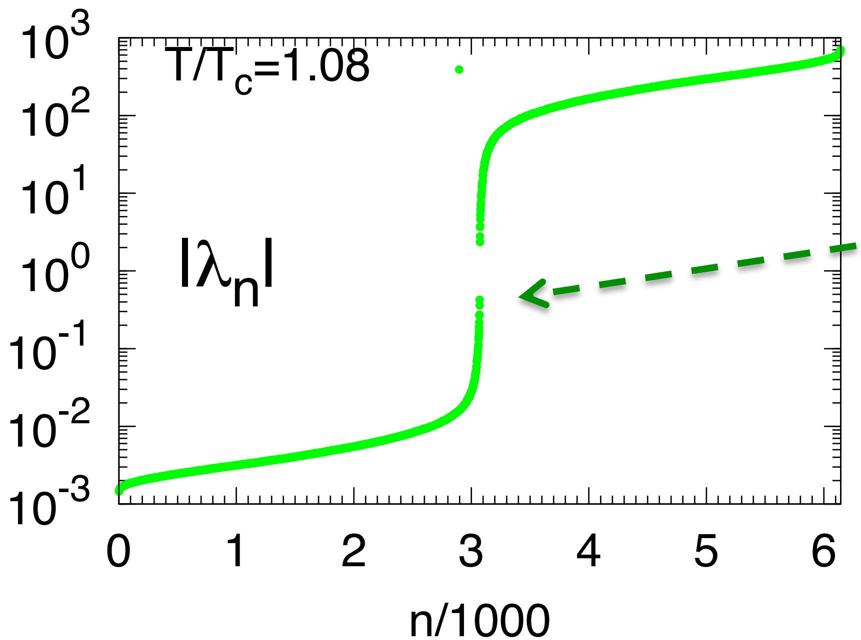
- Eigenvalues λ of Q
  - pair (γ_5-hermiticy) $\lambda_n \leftrightarrow 1/\lambda_n^*$
  - gap (for finite quark mass) $||\lambda| - 1| \neq 0$



Gap in spectrum of Q

- The gap size is related to the pion mass for large Nt.
- Onset of n_q is closely related to the gap for lager Nt.

  [Gibbs('86), Glasgow group('91, '97), etc]



$$am_\pi = -N_t^{-1} \ln \max_{|\lambda_n|<1} |\lambda_n|^2$$

$$am_\pi = \lim_{N_t \to \infty} \left(-\frac{1}{N_t} \ln \left\langle c \left| \sum_{n=1}^{3V} \lambda_n \right|^2 \right\rangle \right)$$

Gibbs('86) & Fodor, Szabo, Toth('07)

[We assume they are equivalent at T=0.]

Fermion determinant

- Fermion determinant can be written as

$$\det \Delta(\mu) = C_0 \xi^{-N_r/2} \prod_{n=1}^{N_r} (\lambda_n + \xi) \qquad \lambda_n \leftrightarrow 1/\lambda_n^*$$

$$= C_0 \prod_{n=1}^{N_r/2} (\lambda_n^*)^{-1} \prod_{n=1}^{N_r/2} (1 + \lambda_n \xi^{-1})(1 + \lambda_n^* \xi) \qquad |\lambda| < 1$$

  - analogous to Matsubara- frequency summation form [Adams '04]

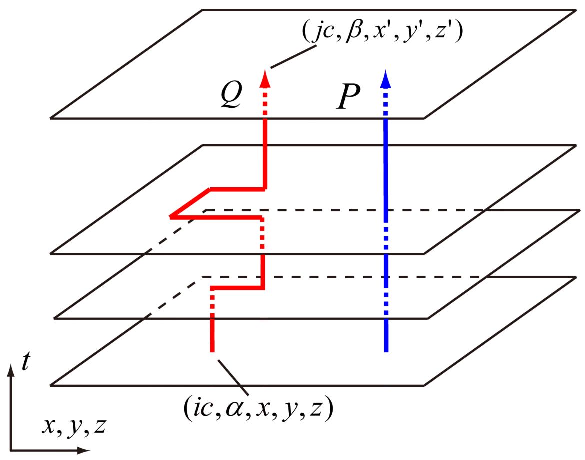
- Quark number operator is given by

$$\hat{n} = \frac{\partial}{\partial \mu / T} \ln \det \Delta(\mu) = \sum_{|\lambda|<1} \left(\frac{\lambda \xi^{-1}}{1 + \lambda \xi^{-1}} - \frac{\lambda^* \xi^{-1}}{1 + \lambda^* \xi^{-1}} \right)$$

Nt-dependence of spectrum of Q

- A reduced matrix Q describes the propagation of a quark between initial and final time slices
    - similar to the Polyakov loop

$$Q = A_1 A_2 \cdots A_{N_t}$$



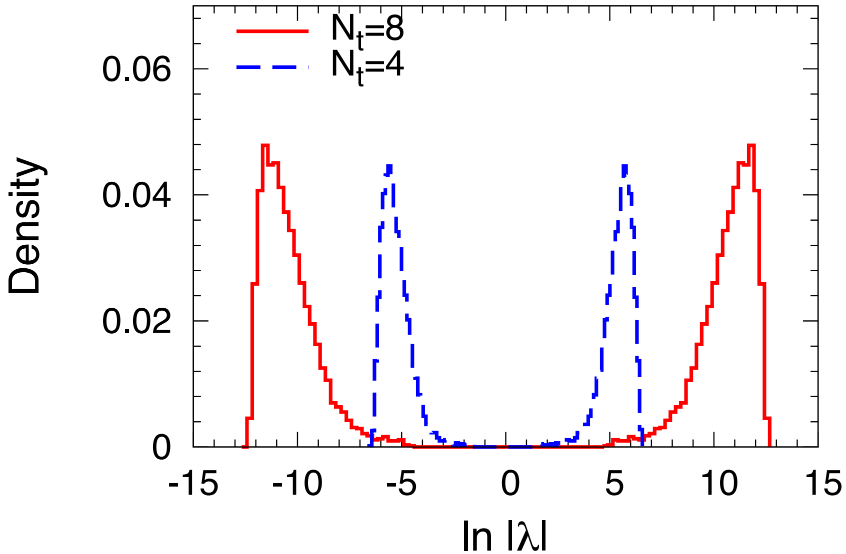
- Eigenvalues of Q are expected to scale as [KN, Nakamrua('10)]

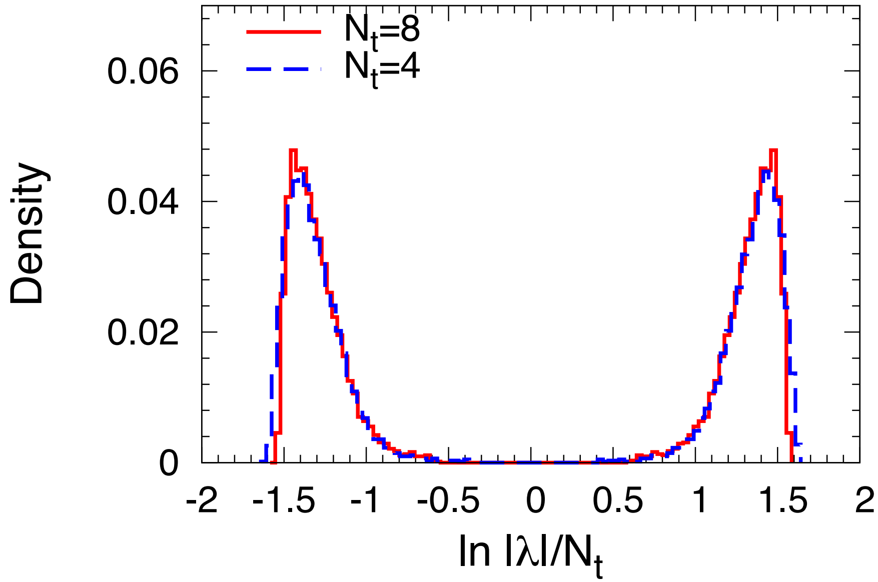
$$\lambda = l^{N_t} = e^{-\epsilon/T+i\theta}$$

- It should be confirmed in lattice simulations.

Eigenvalue density (original vs scaled)

$$\rho(\ln|\lambda_n|) \qquad \rho((\ln|\lambda_n|)/N_t)$$





[KN, Motoki, Nakagawa, Nakamura, Saito ('12)]

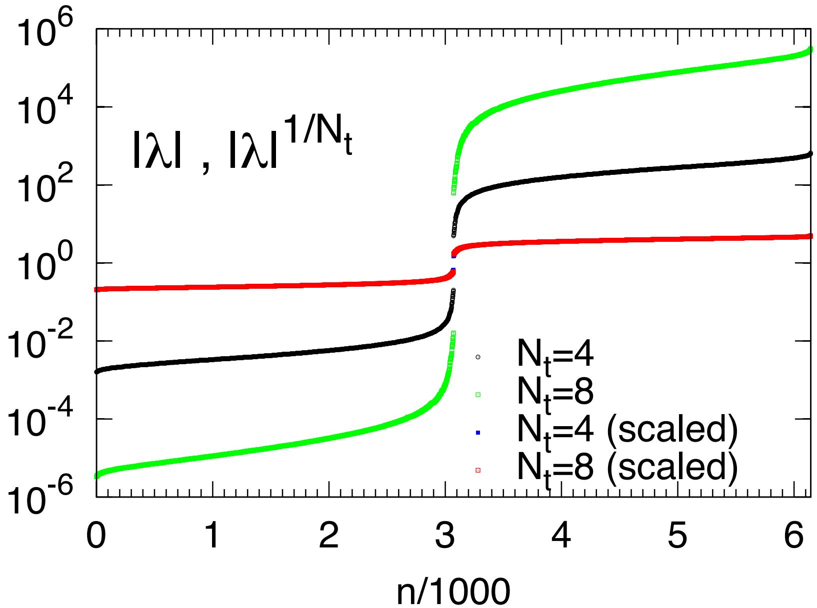
- Lattice results reproduce the scaling behavior

$$\lambda = e^{-\epsilon/T + i\theta}$$

- Lattice setup
  β=1.86(same value of a)
  8^3xNt, mps/mV=0.8,
  RG-improved gauge and clover-Wilson fermion with Nf=2

Eigenspectrum $|\lambda_n|$ (original vs scaled)

$$|\lambda_n|, |\lambda_n|^{1/N_t}$$



Nt=8 original

Nt=4 original

Nt=4, 8 scaled

- Lattice results reproduce the scaling behavior

$$\lambda = e^{-\epsilon/T+i\theta}$$

- Lattice setup
  β=1.86(same value of a)
  8^3xNt, mps/mV=0.8,
  RG-improved gauge and clover-Wilson fermion with Nf=2

Fermi distribution for each configuration

- Quark number operator is given by

$$\hat{n} = \sum_{n=1}^{N_r/2} \left(\frac{\lambda_n \xi^{-1}}{1+\lambda_n \xi^{-1}} - \frac{\lambda_n^* \xi}{1+\lambda_n^* \xi} \right) \qquad \lambda = e^{-\epsilon/T + i\theta}$$

$$= \sum_{n=1}^{N_r/2} \left(\frac{1}{1+e^{(\epsilon-\mu)/T - i\theta}} - \frac{1}{1+e^{(\epsilon+\mu)/T + i\theta}} \right)$$

- This is the Fermi distribution of single quark for each configuration. (winding number of λ is 1).
- Eigenvalues of the reduced matrix are identified as energy levels of single quark for each configuration.

Fermi distribution for each configuration

- Quark number operator is given by

$$\hat{n} = \sum_{n=1}^{N_r/2} \left(\frac{\lambda_n \xi^{-1}}{1+\lambda_n \xi^{-1}} - \frac{\lambda_n^* \xi}{1+\lambda_n^* \xi} \right) \qquad \lambda = e^{-\epsilon/T + i\theta}$$

$$= \sum_{n=1}^{N_r/2} \left(\frac{1}{1+e^{(\epsilon-\mu)/T - i\theta}} - \frac{1}{1+e^{(\epsilon+\mu)/T + i\theta}} \right)$$

- Onset of quark number at T=0

$$\hat{n} = 0 \;\; \text{for} \;\; \mu < \epsilon_{\min}$$

$$\neq 0 \;\; \text{for} \;\; \mu > \epsilon_{\min}$$

- Low-lying energy levels ~ eigenvalues near the gap

$$\epsilon_{\min} \sim m_\pi/2$$

Low-mode Approximation

- To obtain $\mu = m_N/3$, we need further investigations of eigenvalues near the gap (larger V, fluctuations, lower T...)
- However, the eigen-problem of Q becomes ill-conditioned at low-T, and requires large memory for large volume.
- High-lying modes(large and small λ) would be irrelevant of low-energy physics.

Low-mode Approximation

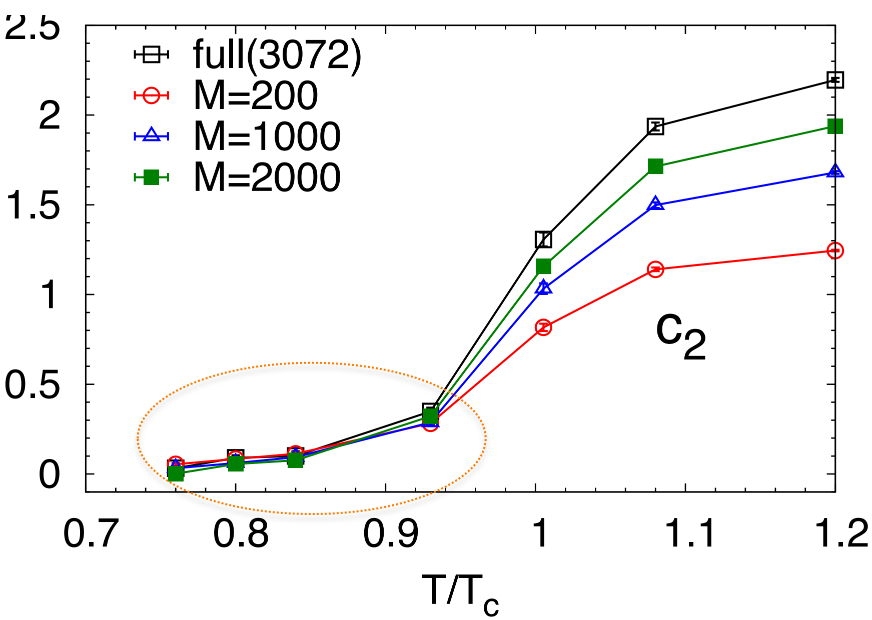
- We consider a low-mode expansion, e.g,

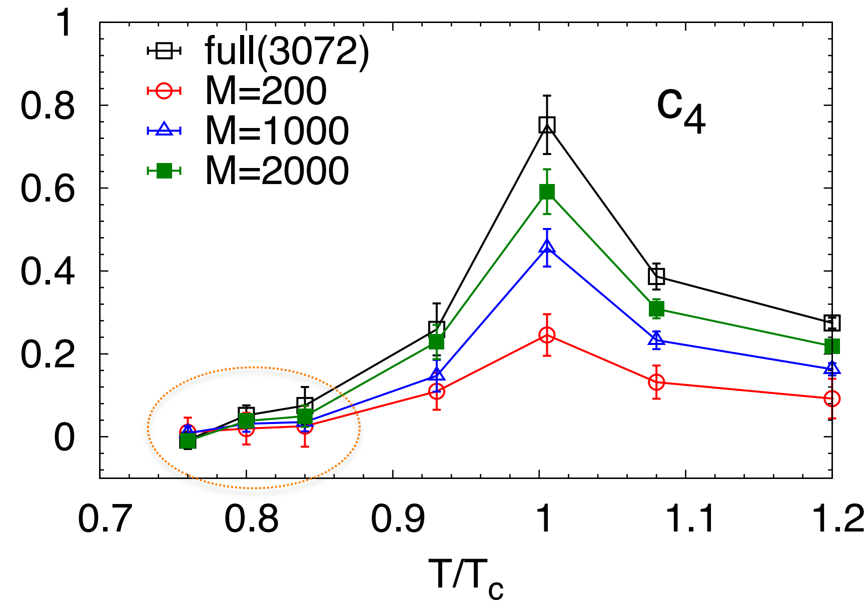
$$\hat{n} = \sum_{n=1}^{M} \left(\frac{\lambda_n \xi^{-1}}{1 + \lambda_n \xi^{-1}} - \frac{\lambda_n^* \xi}{1 + \lambda_n^* \xi} \right) + (n > M + 1)$$

  – We test the approximation for the Taylor coefficients of EoS (c_2 and c_4)

$$c_n = \frac{1}{n!} (N_t/N_s)^3 T^n \frac{\partial^n \ln Z}{\partial \mu^n}$$

Results





- Results with M=200, 1000, 2000, and 3072(all)
- Lattice setup

  RG-improved gauge and clover-Wilson fermion with Nf=2

  Mass & size : mps/mV=0.8 , 8^3x4

  Configs. : 10K trajectories at μ=0

  Measurement : 400 configs.

Summary

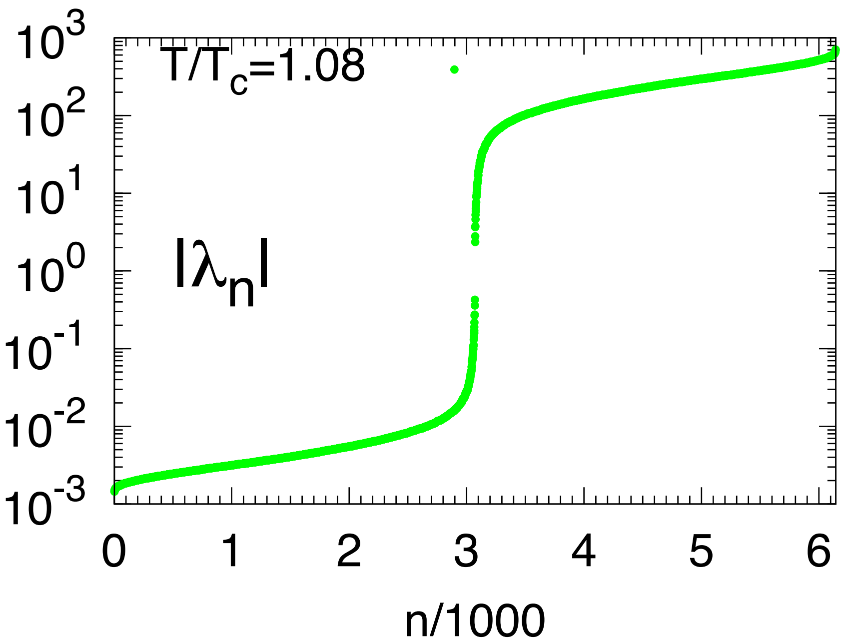
- Manifest representation of the early onset problem was shown using lattice QCD with the reduction formula.
  - The correspondence between the quark number operator and the Fermi distribution is clarified.
  - The eigenvalues of the reduced matrix are identified as energy levels of single quark for each configuration, which would be useful for applications.
  - We test the low-mode expansion, which is helpful for further studies of the early onset problem.

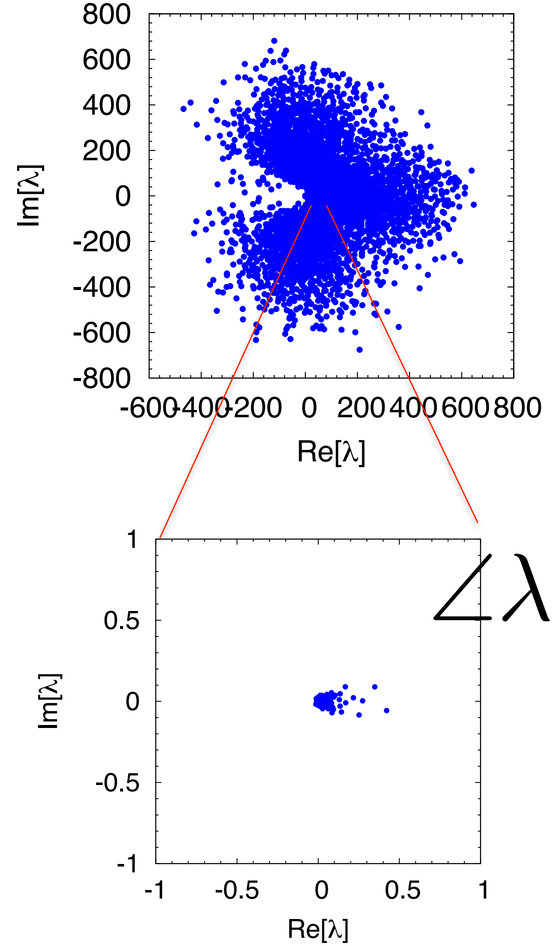
Buckup Slides

Reduction formula

- Eigenvalues λ of Q

  - pair $\lambda_n \leftrightarrow 1/\lambda_n^*$
  - gap $||\lambda| - 1| \neq 0$





$\angle\lambda$

Introduction

- Need better understanding spectrum of fermions in QCD
- We discuss the early onset problem using the reduced matrix Q in the reduction formula

  - lattice results of a *Nt-scaling* of eigenvalues of Q ('Lat12)
  - derivation of the Fermi distribution of single quark from fermion determinant
  - identification of eigenvalues of Q as energy levels of single quark for each configuration
  - application to the early onset problem and introduction of low-mode approximation

Reduction formula

- A fermion matrix in t-t matrix rep.

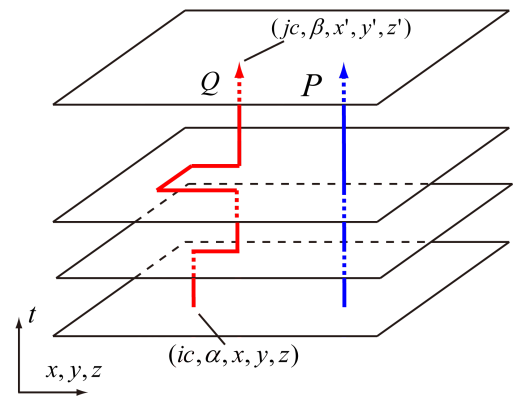
$$\Delta = {\color{red}B} - {\color{blue}e^{\mu a} V} - {\color{blue}e^{-\mu a} V^\dagger}$$

$$\Delta = \begin{pmatrix} \square & \triangle & & & \triangle \\ \triangle & \square & \triangle & & \\ & \triangle & \ddots & & \\ & & & \ddots & \triangle \\ \triangle & & & \triangle & \square \end{pmatrix}$$

[Gibbs ('86). Hasenfratz, Toussaint('92). Adams('03, '04), Borici('04). KN&AN('10), Alexandru &Wenger('10)]

$$\det \Delta = \xi^{-N_{\rm red}/2} C_0 \det(\xi + Q)$$

$$\xi = e^{-\mu/T}$$

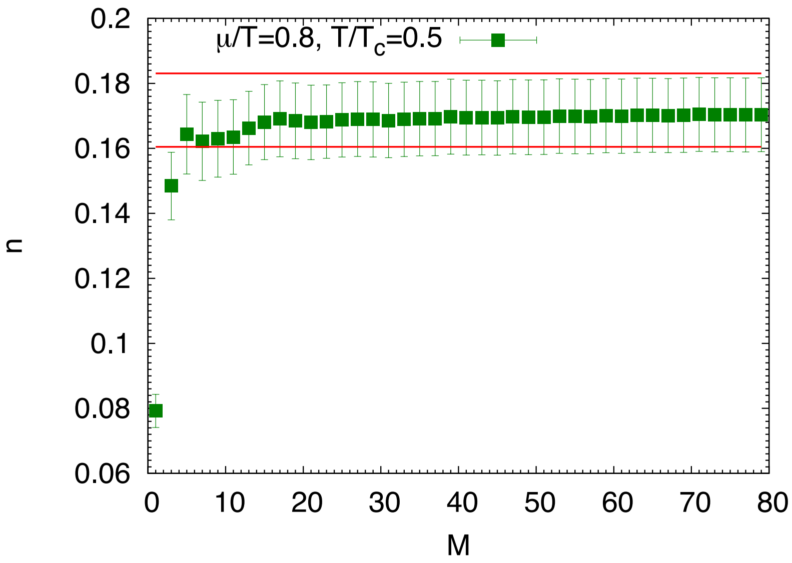


✓ Q and C_0 are functions link variables

✓ $N_{\rm red}$=12 Ns^3

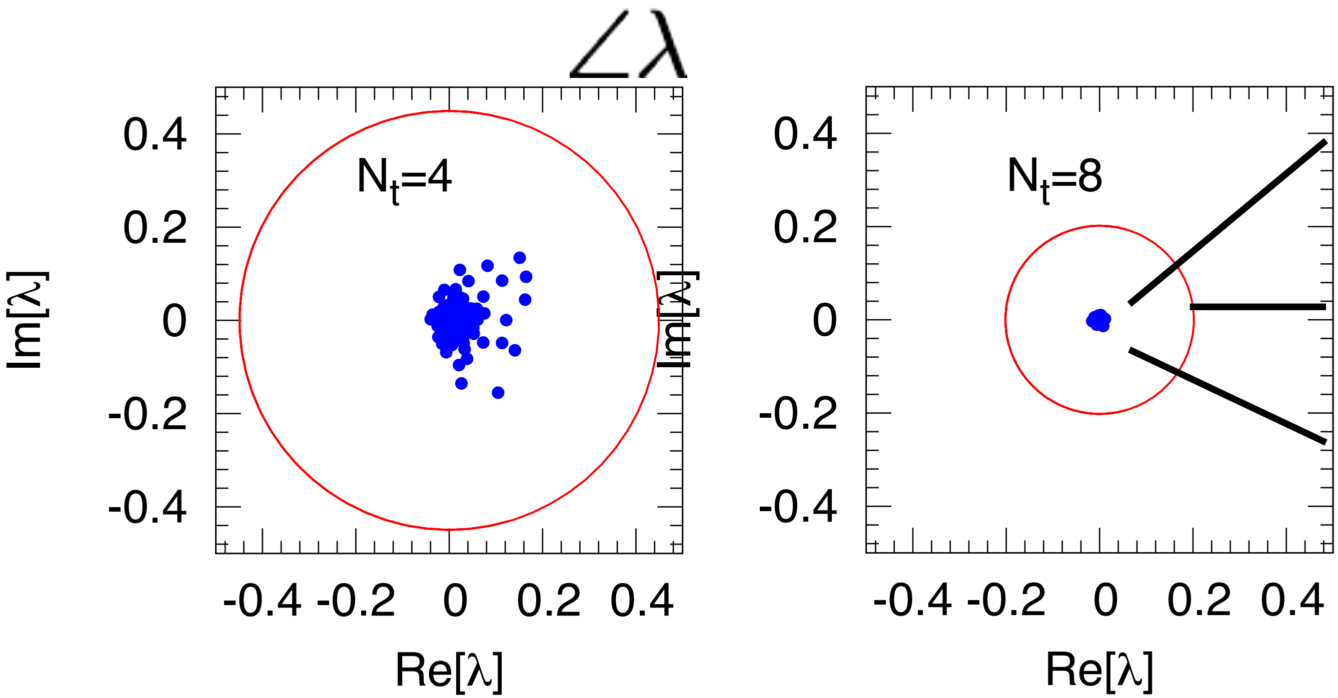
Approximaitons at low temperatures (T~0.5Tc)

- The average of the quark number density.



- It converges at small M~20.

Low Temperature Limit



$$|\lambda_n| = (l_n)^{N_t}$$

$$e^{-\mu/T} = e^{-0.2Nt}$$

$$\max_{|\lambda|<1} |\lambda| = e^{-m_\pi/(2T)}$$

Ev distribution is bounded by pion mass

$$\det \Delta = \xi^{-N_{\rm red}/2} C_0 \prod_{|\lambda|>1} (\cancel{\xi} + \lambda_n) \prod_{|\lambda|<1} (\xi + \cancel{\lambda_n})$$

$$= C_0 \prod_{|\lambda|>1} (\lambda_n)$$

$$\mu < m_\pi/2$$

Reduction formula

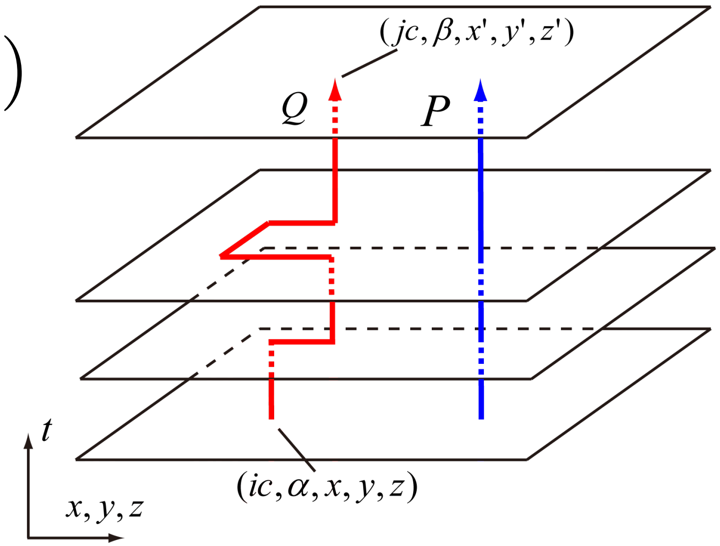
- Fermion determinant det Δ
  - calculating the temporal part of det Δ leads to

$$\det \Delta = \xi^{-N_{\rm red}/2} C_0 \det(\xi + Q)$$

$$\xi = e^{-\mu/T}$$

$$N_{\rm red} = 4 N_c N_x N_y N_z$$

$$Q = A_1 A_2 \cdots A_{N_t}$$



  - Q and C_0 are independent of μ
  - chemical potential and gauge fields are separated

Gibbs ('86). Hasenfratz &Toussaint('92). Adams('03, '04), Borici('04). KN&AN('10), Alexandru &Wenger('10)

Complex potential problem

A free energy or complex potential satisfies electromagnetic analogy (the same as Lee-Yang zero theorem)

$$h(\mu) = \frac{\ln \det \Delta(\mu)}{N_r}, (N_r = 4N_c V_s)$$

$$= \frac{\mu}{2} \ln \xi + N_r^{-1} \sum_{n=1}^{N_r} \ln(\xi + \lambda_n) + (\mu - \text{indep.})$$

$$(\partial_x^2 + \partial_y^2) \mathrm{Re}[h] = -\pi\delta(\xi) + 2\pi N_r^{-1} \sum_{n=1}^{N_r} \delta(\xi + \lambda_n)$$

$$= -\pi\delta(\xi) + 2\pi\rho(-\xi)$$

Complex potential problem

complex potential

$$h(\mu) = \frac{\ln\det\Delta(\mu)}{N_r}, (N_r = 4N_cV_s)$$

$$= \frac{\mu}{2}\ln\xi + N_r^{-1}\sum_{n=1}^{N_r}\ln(\xi+\lambda_n) + (\mu - \text{indep.})$$

h is analytic function of mu except for det =0,

$$\partial_x X = \partial_y Y$$

$$\partial_y X = -\partial_x Y$$

Cauchy-Riemann

$$\nabla_\xi^2 X = 0$$

$$\nabla_\xi^2 Y = 0$$

Laplace

Complex potential problem

Complex potential satisfies

$$(\partial_x^2 + \partial_y^2)\mathrm{Re}[h] = -\pi\delta(\xi) + 2\pi N_r^{-1}\sum_{n=1}^{N_r}\delta(\xi + \lambda_n)$$

$$= -\pi\delta(\xi) + 2\pi\rho(-\xi)$$

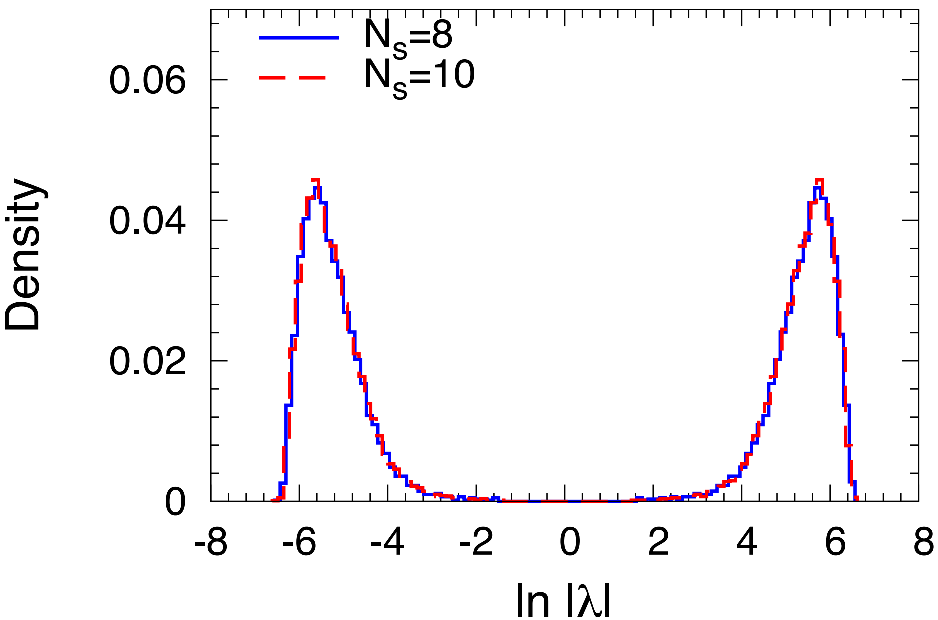
Gauss's law (2D electrostatic problem)

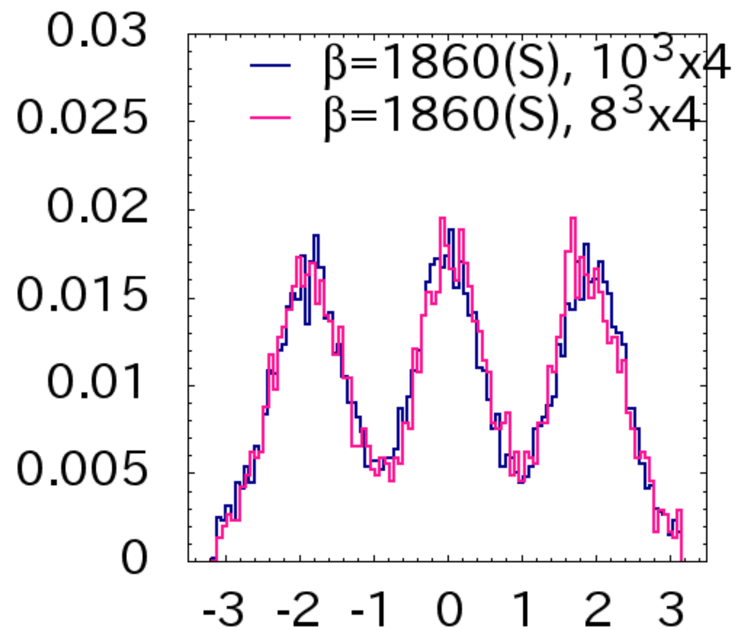
$$\vec{n} = \nabla\mathrm{Re}[h]$$

Electrostatic analogy (Lee-Yang('50))
quark number density ~ electric field
eigenvalues of reduced matrix ~ (opposite) location of charge

Spectral property - Volume

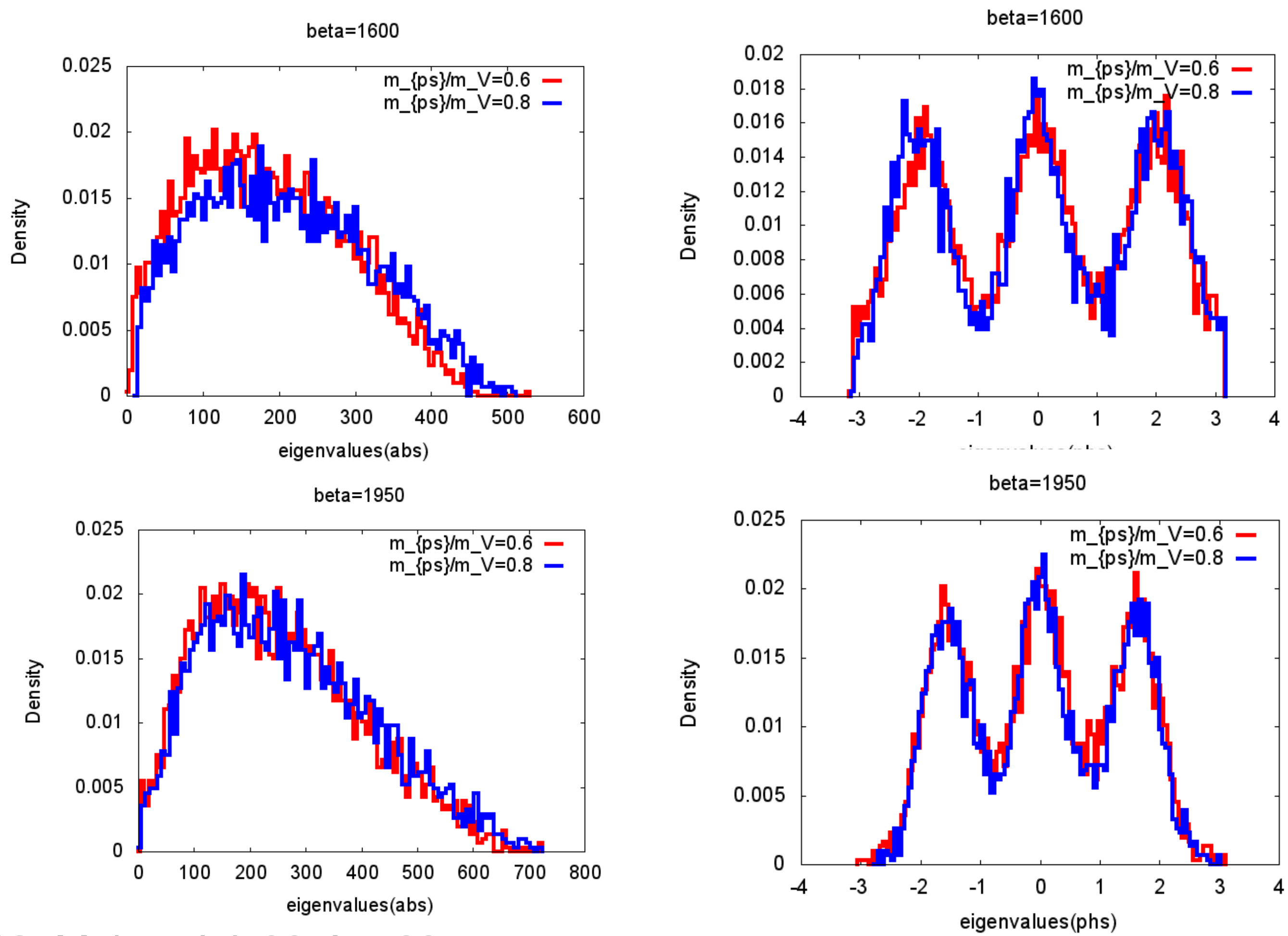
- Volume dependence





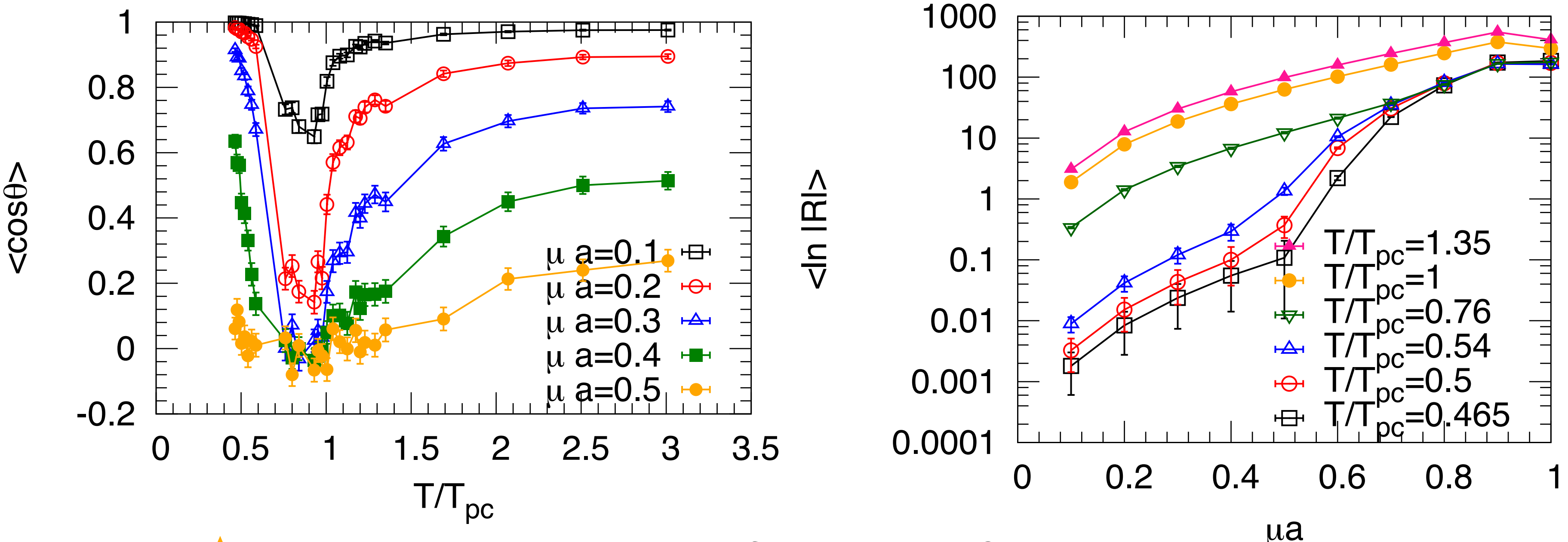
Quark mass dependence

mps / mv = 0.6 (red), 0.8 (blue)
Histograms : |ev| (Left), arg(ev) (Right)
confinement (top), deconfinement(bottom)



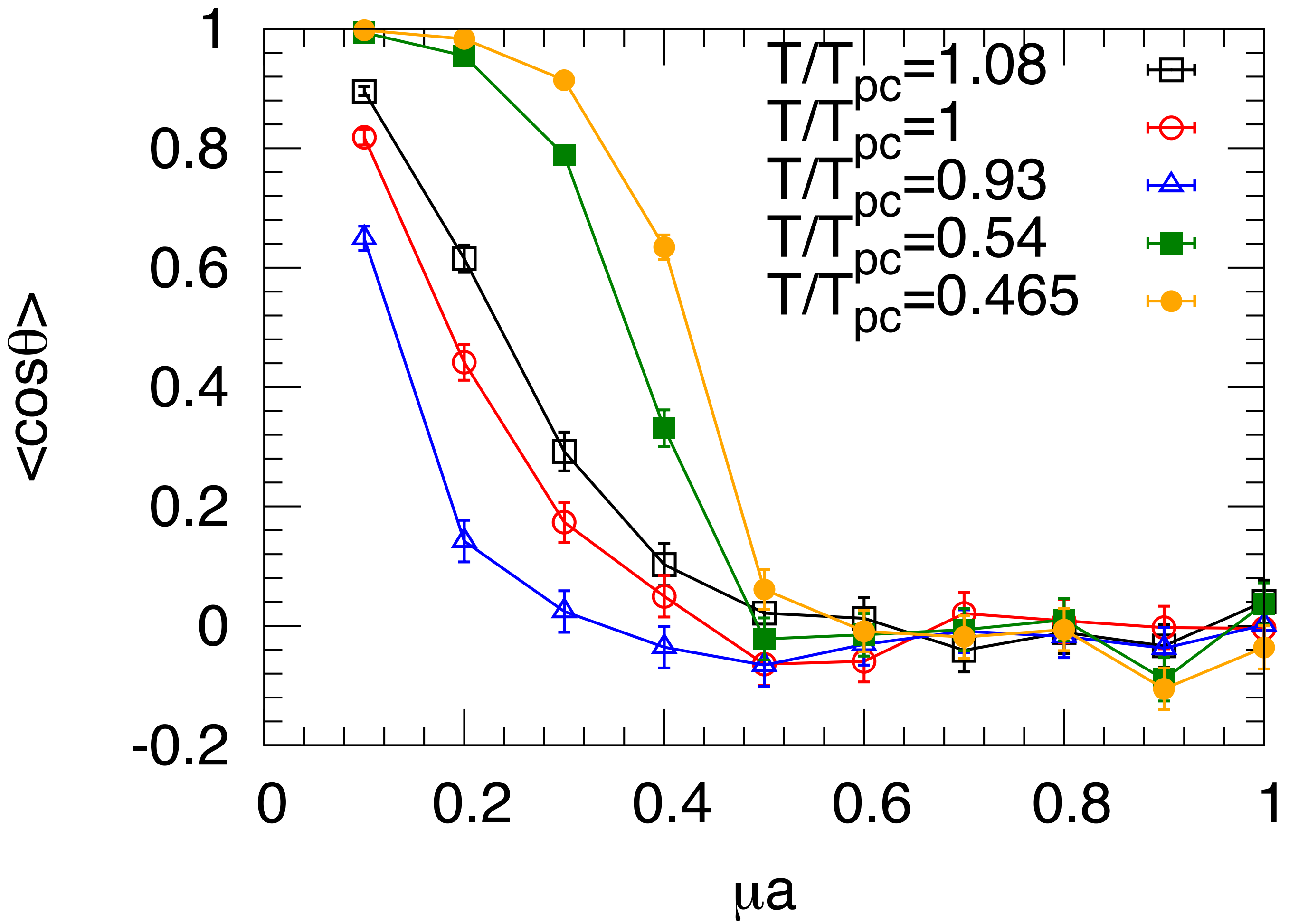
Chemical Potential Dependence at Low T

$$R = \left(\frac{\det \Delta(\mu)}{\det \Delta(0)} \right)^2 = |R| e^{i\theta}$$



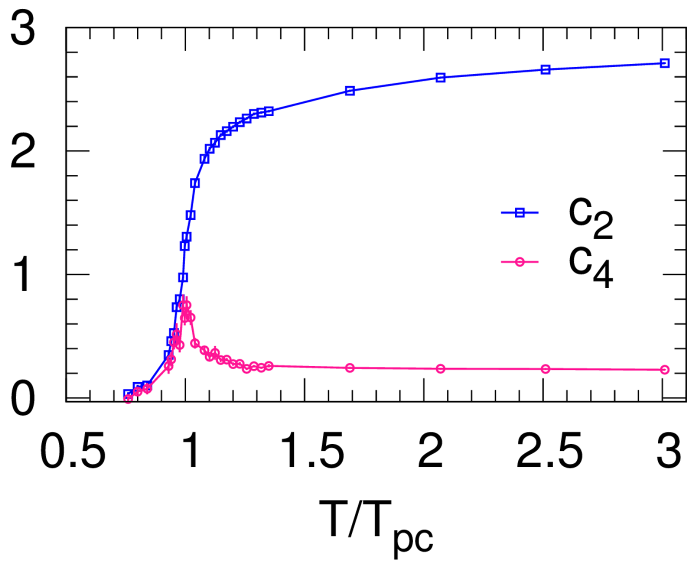
- det Δ is insensitive to μ for μa < 0.5.
- μ-dependence appears at μa = 0.5.
  - This value is close to $m_\pi/2$ in the present setup.

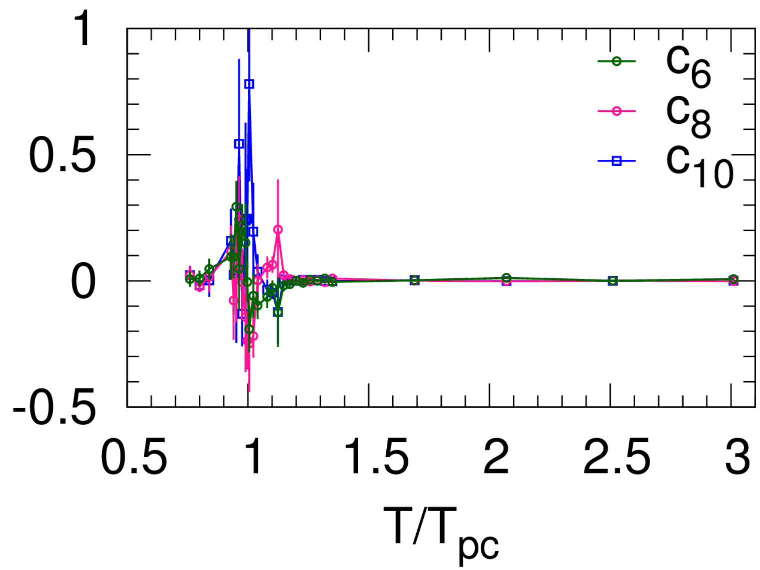
Average phase factor vs μ



Taylor coefficients of EoS

$$f(\mu) - f(0) = \sum_{n=1} c_{2n} (\mu/T)^{2n}$$





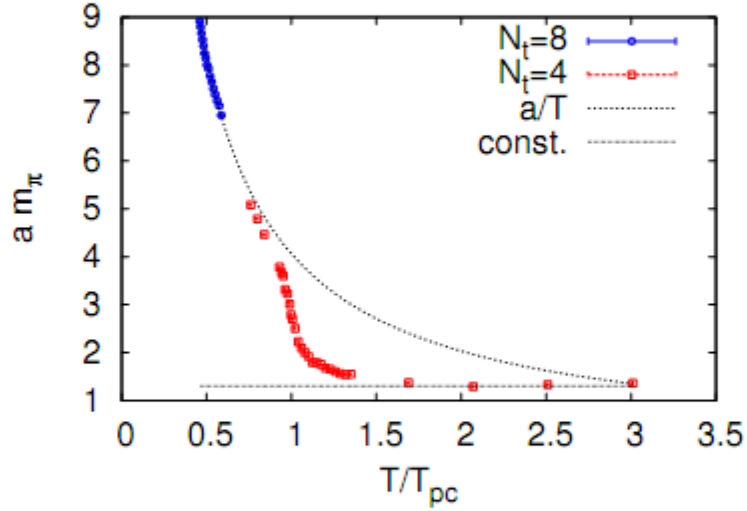
– slow convergence of the Taylor series of EoS
– small S/N ratio of chemical potential dependence

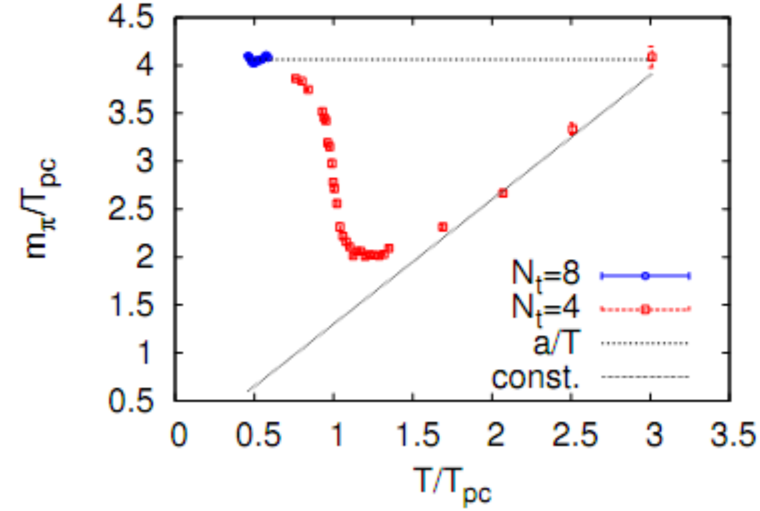
Gap is related to pion mass

$$am_\pi = -\frac{1}{Nt}\ln\max_{|\lambda_n|<1}|\lambda_n|^2$$

Gibbs('86). Eigenvalues and mpi

See also, Fodor, Szabo, Toth ('06). Eigenvalues and hadron spectrum





- *At low T, mpi/T is well fitted with a/T, a = 4 Tpc (mq heavy)*
- *At high T, mpi approaches to a constant*