Singularities around the QCD critical point in the complex chemical potential plane

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1. Introduction

QCD phase diagram

Critical (end) point?

notorious sign problem for $\mu \neq 0$

- Taylor series
- imaginary chemical potential
- reweighing
- strong coupling expansion
- etc

Complex chemical potential plane

- Partition function zeroes: complementary view of critical phenomena
  Lee-Yang (1952)
- scaling in the complex plane
  Itzykson-Pearson-Zuber (1983)
- critical point and singularities in QCD
  Stephanov (2006)
- in connection with experimental data
  Nakamura-Nagata (2013)
In this talk, we have a look at complex effective potential in terms of complex order parameter for complex $\mu$.

By using an effective theory based on a mean field theory, we study

- singularities in the complex $\mu$ plane
- extrema of real part of the effective potential
- the Stokes lines
Plan

1. Introduction (√)
2. a QCD effective theory
3. Singularities in the complex $\mu$ plane
4. Stokes lines
5. Summary
2. a QCD effective theory: mean field approach

\[ N_f = 2 \]

\[ \text{tricritical point} \]

\[ m = 0 \quad \text{tricritical point: upper critical dimension} \]

\[ d = 3 \]

mean field description is expected to be valid (up to log corrections), because the system is effectively in three dimensions at finite \( T \).

\[ m \neq 0 \quad \text{critical point = liquid-gas phase transition} \]

\[ \sigma \quad \text{massless} \]

\[ \pi \quad \text{massive} \]

\[ M = \langle \bar{q}_L q_R \rangle = \sigma + i \vec{\tau} \cdot \vec{\pi} \]
a QCD effective theory based on mean field approach

\[ \Omega = -m\sigma + \frac{a}{2}\sigma^2 + \frac{b}{4}\sigma^4 + \frac{c}{6}\sigma^6, \]

expands around the tricritical point (TCP) \( a = 0, b = m = 0 \)

\[
\begin{align*}
a(T,\mu) &= C_a (T - T_3) + D_a (\mu - \mu_3) \\
b(T,\mu) &= C_b (T - T_3) + D_b (\mu - \mu_3),
\end{align*}
\]

\( C_b D_a - C_a D_b > 0. \)

Hatta - Ikeda (2003)

\[ m = 0 \]
By switching on $m$, the condition for the critical end point

$$\frac{\partial \Omega(T_E, \mu_E, \sigma_0)}{\partial \sigma} = \frac{\partial^2 \Omega(T_E, \mu_E, \sigma_0)}{\partial \sigma^2} = \frac{\partial^3 \Omega(T_E, \mu_E, \sigma_0)}{\partial \sigma^3} = 0$$

$$\Rightarrow a(T_E, \mu_E) = \frac{9b(T_E, \mu_E)^2}{20c}, \quad -b(T_E, \mu_E) = \frac{5}{54^{1/5}}c^{3/5}m^{2/5}, \quad \sigma_0 = \sqrt{\frac{-3b(T_E, \mu_E)}{10c}}.$$ 

thermodynamic potential around the critical end point

$$\Omega(T, \mu, \sigma) = \Omega(T_E, \mu_E, \sigma_0) + A_1 \hat{\sigma} + A_2 \hat{\sigma}^2 + A_3 \hat{\sigma}^3 + A_4 \hat{\sigma}^4,$$

$$A_1 = (C_a \sigma_0 + C_b \sigma_0^3) \tilde{t}_E + (D_a \sigma_0 + D_b \sigma_0^3) \tilde{\mu}_E$$

$$A_2 = \frac{1}{2} (C_a + 3C_b \sigma_0^3) \tilde{t}_E + \frac{1}{2} (D_a + 3D_b \sigma_0^3) \tilde{\mu}_E$$

$$A_3 = (C_b \tilde{t}_E + D_b \tilde{\mu}_E) \sigma_0$$

$$A_4 = -\frac{b(T_E, \mu_E)}{2} + \frac{1}{4} (C_b \tilde{t}_E + D_b \tilde{\mu}_E),$$

$$\tilde{t}_E = T - T_E, \quad \tilde{\mu}_E = \mu - \mu_E$$
stability condition

\[ A_4 > 0 \quad \Rightarrow \quad \tilde{\mu}_E > \frac{2b(T_E, \mu_E) - C_b\tilde{t}_E}{D_b}. \]

typical behaviors of \( \Omega \) at temperature around the CEP.

first order phase transition for \( \tilde{t}_E < 0 \)
critical point at \( \tilde{t}_E = 0 \)

\( \tilde{t}_E = -0.2 \)

\( \tilde{\mu}_E \equiv \mu - \mu_E = 0.052, 0.054, 0.0555, 0.058, 0.06 \)

\( \tilde{t}_E = 0 \)

\( \tilde{\mu}_E = -0.15, -0.1, -0.05, 0, 0.1 \)
crossover for $\tilde{t}_E > 0$

\[ \tilde{t}_E = 0.2 \]

\[ \tilde{\mu}_E = -0.08, -0.04, 0.0222 \text{ (crossover)}, -0.005. \]
3. Singularities in the complex \( \mu \) plane

\[
\frac{\partial \Omega}{\partial \sigma} = 0, \quad \frac{\partial^2 \Omega}{\partial \sigma^2} = 0 \quad (*)
\]  

\[
A_1 + 2A_2 \sigma + 3A_3 \sigma^2 + 4A_4 \sigma^3 = 0, \quad 2A_2 + 6A_3 \sigma + 12A_4 \sigma^2 = 0.
\]

Example: for \( x_m = 0.2 \) and \( \tilde{t}_E = 0.2 \)

solutions to \((*)\)  
\[
\tilde{\mu}_E = -0.3857 \text{ (i)}, \quad -0.0538 \text{ (ii)}, \quad -0.0222 \pm 0.00254 i \text{ (iii)}.
\]

The stability condition of \( \Omega \to \tilde{\mu}_E > -0.3801 \) for \( \tilde{t}_E = 0.2 \)

singularity is located at \( \tilde{\mu}_E = -0.0538 \)

\( \text{Re} \tilde{\mu}_E^* = -0.0222 \pm 0.00254i \)

\[(*)\]

at \( \sigma = -0.07602 \)

\( \tilde{\mu}_E = -0.0222 \)
Susceptibility peak is attained at $\text{Re} \tilde{\mu}^*_E$

$$\tilde{t}_E = 0.1$$

Chiral susceptibility at $\tilde{t}_E = 0.1, 0.2, 0.3$
phase diagram

\[ \tilde{t}_E = T - T_E \]

- CEP: \( \tilde{t}_E = 0, \tilde{\mu}_E = 0 \)
- 1st order: \( \tilde{t}_E < 0, \tilde{\mu}_E > 0 \)
- \( \tilde{t}_E > 0, \tilde{\mu}_E < 0 \)

\[ \tilde{\mu}_E = \mu - \mu_E \]

\[ x_m = m^{1/5} = 0.2 \]
Locations of singularities type (iii) for $\tilde{t}_E > 0$

\[\text{Im } \tilde{\mu}_E = \text{Im } (\mu - \mu_E)\]

\[\text{Re } \tilde{\mu}_E = \text{Re } (\mu - \mu_E)\]

Re $\tilde{\mu}_E = 0$, Im $\tilde{\mu}_E = 0$
3. Stokes lines

reflect the analytic structure around the branch points in the vicinity of the CEP

The Stokes line is understood as the curve to which the Lee-Yang zeros accumulate.

Lee-Yang theorem:
zeros on the imaginary $h$ axis (d-dimensions)

$$\rho = e^{-2h}$$
Stokes lines in the complex $\tilde{\mu}_E$ plane

(I) $\tilde{t}_E > 0$

$\tilde{\mu}_E \equiv \rho e^{i\theta}$

$\Omega(T, \mu, \sigma)$

$\tilde{\mu}_E > 0 \quad (\theta = 0)$

$\tilde{\mu}_E < 0 \quad (\theta = \pi)$
locations of the global minimum of \( \Omega \)

\[ \text{Analytical continuation (} \theta \neq 0, \pi \text{)} \quad \tilde{\mu}_E \equiv \rho \, e^{i\theta} \]

vary \( \theta \) 0 to \( \pi \)

consider three cases

(i) \( \rho > \rho^* \)

(ii) \( \rho < \rho^* \)

(iii) \( \rho = \rho^* \)
Summary

By using an effective theory based on a mean field theory, we studied

- singularities in the complex $\mu$ plane
- extrema of real part of the effective potential
- the Stokes lines

- the location of the crossover is regarded as a real part of the singularity $\tilde{\mu}_E^*$

- the Stokes lines are located by explicitly looking at $\text{Re } \Omega$ as a function of the complex order parameter $\sigma$

- baryon number distribution and complex $\mu$, Morita et. al. (2013)