Simulating the Random Surface Representation of Abelian Gauge Theories

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LATTICE 13

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MAINZ, GERMANY

based on [T.K., Ulli Wolff, Nucl.Phys. B871 (2013) 145]

Ising Gauge Theory

First and simplest lattice gauge theory

[Wegner, J.Math.Phys. 12 (1971) 2259]

- D-dimensional hypercubic lattice of size $L_0 \times L_1 \times \ldots \times L_{D-1}$
- Gauge field $\sigma(x,\mu) = \pm 1$ defined on the links
- Plaquette action

$$S[\sigma] = -\beta \sum_{x,\mu<\nu} \sigma(x,\mu)\sigma(x+\hat{\mu},\nu)\sigma(x+\hat{\nu},\mu)\sigma(x,\nu)$$

- Observables

$$\begin{aligned} \langle O[\sigma] \rangle &= \frac{1}{Z} \sum_{\{\sigma\}} O[\sigma]\, e^{-S[\sigma]} \\ \langle 1 \rangle &= 1 \qquad\qquad \Rightarrow Z \end{aligned}$$

A general observable $O[j] = \prod_{x,\mu} \sigma(x,\mu)^{j(x,\mu)}$ with $j(x,\mu) \in \{0,1\}$.

Non-vanishing: closed loops or pairs of Polyakov lines

Strong Coupling Expansion

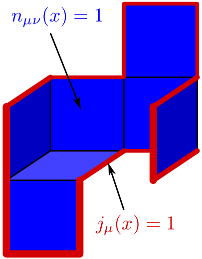
$$Z[j] = \sum_{\{\sigma\}} e^{-S[\sigma]} \prod_{x,\mu} \sigma(x,\mu)^{j(x,\mu)}$$

Observables: $\langle O[j] \rangle = Z[j]/Z[0]$
On every plaquette use:

$$\begin{aligned} e^{\beta\sigma_p} &= \cosh(\beta) + \sigma_p \sinh(\beta) \\ &= \cosh(\beta) \sum_{n\in\{0,1\}} t^n \sigma_p^n, \qquad t = \tanh(\beta) \end{aligned}$$

$$\begin{aligned} Z[j] &\propto \sum_{\{n\}} \left[\prod_{x,\mu<\nu} t^{n_{\mu\nu}(x)} \right] \prod_{x,\mu} \left[\sum_{\sigma} \sigma^{j(x,\mu) + \sum_{\nu>\mu} (n_{\mu\nu}(x) + n_{\mu\nu}(x-\hat{\nu}))} \right] \\ &\propto \sum_{\{n\}} \delta(\partial n + j)\, t^{\sum_{x,\mu<\nu} n_{\mu\nu}(x)} \end{aligned}$$

Strong Coupling Expansion

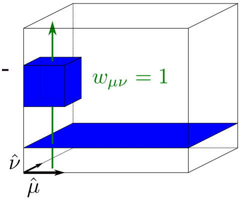


The constraint $\delta(\partial n + j)$:

- Links with $j = 0$ touch an even number of $n = 1$ plaquettes
- Links with $j = 1$ touch an odd number of $n = 1$ plaquettes

Topological sectors defined by "wrapping numbers"

$$w_{\mu\nu} = \sum_{x|_{x_\mu = x_\nu = 0}} n_{\mu\nu}(x) \pmod 2$$



Twisted boundary conditions can be incorporated by inclusion of the phase Φ_γ into the graph weight

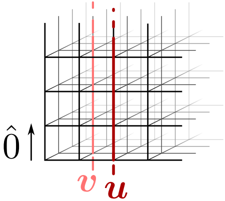
$$\Phi_\gamma[n] = \prod_{\mu<\nu} (\gamma_{\mu\nu})^{w_{\mu\nu}[n]}$$

where $\gamma_{\mu\nu}$ is the twist in the μ/ν plane

Enlarged Ensemble

- Polyakov line at spatial position $\boldsymbol{u}$:
$$\pi(\boldsymbol{u}) = \prod_{u_0=0}^{L_0-1} \sigma(u,0)$$
- Defects $j^{(\boldsymbol{u},\boldsymbol{v})}$, such that $O[j^{(\boldsymbol{u},\boldsymbol{v})}] = \pi(\boldsymbol{u})\pi(\boldsymbol{v})$
- Simulate enlarged ensemble

$$\begin{aligned}
\mathcal{Z} &= \sum_{\boldsymbol{u},\boldsymbol{v}} \rho^{-1}(\boldsymbol{u}-\boldsymbol{v})\, Z[j^{(\boldsymbol{u},\boldsymbol{v})}] \\
&= \sum_{\{n\},\boldsymbol{u},\boldsymbol{v}} \rho^{-1}(\boldsymbol{u}-\boldsymbol{v})\, t^{\sum_{x,\mu,\nu} n_{\mu\nu}(x)}\, \delta(\partial n + j^{(\boldsymbol{u},\boldsymbol{v})})
\end{aligned}$$



- Observables

$$\langle\langle O[n,\boldsymbol{u},\boldsymbol{v}]\rangle\rangle = \frac{1}{\mathcal{Z}} \sum_{\{n\},\boldsymbol{u},\boldsymbol{v}} O[n,\boldsymbol{u},\boldsymbol{v}]\, \rho^{-1}(\boldsymbol{u}-\boldsymbol{v})\, t^{\sum_{x,\mu,\nu} n_{\mu\nu}(x)}\, \delta(\partial n + j^{(\boldsymbol{u},\boldsymbol{v})})$$

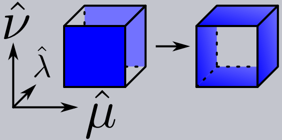
Observables

- Polyakov correlator: $G(\boldsymbol{x}) = \langle \pi(\boldsymbol{x})\pi(\boldsymbol{0})\rangle = \rho(\boldsymbol{x})\frac{\langle\langle \delta_{\boldsymbol{x},\boldsymbol{u}-\boldsymbol{v}}\rangle\rangle}{\langle\langle \delta_{\boldsymbol{u},\boldsymbol{v}}\rangle\rangle}$
  $\rightarrow$ simple counters!
- Abbreviation for "vacuum observables" $\langle\langle O[n]\rangle\rangle_0 \equiv \frac{\langle\langle O[n]\ \delta_{\boldsymbol{u},\boldsymbol{v}}\rangle\rangle}{\langle\langle \delta_{\boldsymbol{u},\boldsymbol{v}}\rangle\rangle}$
- Average plaquette $\langle \sigma_p \rangle = \frac{\partial \log Z}{\partial \beta} = t + \frac{(t^{-1} - t)}{N_p} \left\langle\left\langle \sum_{x,\mu<\nu} n_{\mu\nu}(x) \right\rangle\right\rangle_0$
  $\rightarrow$ Related to the total surface of vacuum graphs
- Topological observables $\frac{Z_\gamma}{Z} = \langle\langle \prod_{\mu<\nu} (\gamma_{\mu\nu})^{w_{\mu\nu}[n]} \rangle\rangle_0$
  - Do not need renormalization
  - Can be obtained from the same simulation
  - Very sensitive to β
  - Higher derivatives $\partial/\partial\beta$ also feasible

Simulation: Cube Updates

A defect-conserving update ($D > 2$)

- Proposal
  - Choose a $3D$ cube, i.e. x and $\mu < \nu < \lambda$ (randomly or in an ordered fashion)
  - The six plaquettes associated with this cube are $\mathcal{P} = \{(x,\mu,\nu),(x+\hat{\lambda},\mu,\nu),(x,\mu,\lambda),$ $(x+\hat{\nu},\mu,\lambda),(x,\nu,\lambda),(x+\hat{\mu},\nu,\lambda)\}$
  - propose to flip these six plaquettes: $n(p) \to 1 - n(p)$ for all $p \in \mathcal{P}$
- Acceptance with Metropolis probability $\min[1,q]$ $q = \prod_{p\in\mathcal{P}} t^{1-2n(p)}$



- Ancient! [T.Sterling, J.Greensite, Nucl.Phys.B220 (1983)]
- Does not change ∂n. I.e is not ergodic
- Does not change the wrapping number $w_{\mu\nu}$ $\to$ Fluctuating boundary conditions only
- In $D = 3$: related to a spin-flip in the dual Ising model $\to$ critical slowing down with $z \approx 2$ expected.

Simulation: Cube-Cluster Update

Instead of sweeps of cube-flips one can build clusters of cubes

Cluster Algorithm

- Determine for every plaquette a bond $b(p) \in \{0, 1\}$. The value 1 has probability
  $P(b(p) = 1) = 1 - t^{1-n(p)}$
- Choose a $3D$ cube c_0 randomly: $x, \mu < \nu < \lambda$
- Determine the maximal set C of cubes, so that for each cube $c \in C$
  - There exists a path from c to c_0 that crosses only plaquettes with $b(p) = 1$
  - The entire path lies in the $3D$ subspace spanned by $\hat{\mu}, \hat{\nu}, \hat{\lambda}$
- For every $c \in C$, flip $n(p) \to 1 - n(p)$ on all six plaquettes
  $\to$ Amounts to flipping n on the cluster boundary

- Eliminates critical slowing down in $D = 3$ with fluctuating b.c.
- Is a valid algorithm in $D > 3$, but is it efficient?

Simulation: Polyakov Shift Updates

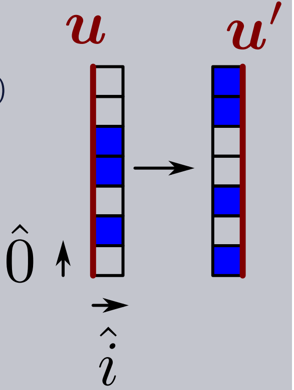
Move the Polyakov lines while preserving the constraints

Worm Algorithm

- Calculate auxiliary field

$$k(\boldsymbol{x}, i) = \sum_{x_0=0}^{L_0-1} n_{0i}((x_0, \boldsymbol{x})), \qquad k(\boldsymbol{x}, -i) \equiv k(\boldsymbol{x} - \hat{i}, i)$$

- If $\boldsymbol{u} = \boldsymbol{v}$, move both to a random position
- Propose to move $\boldsymbol{u}$ to one of its $2(D-1)$ spatial neighbors $\boldsymbol{u} \to \boldsymbol{u}' = \boldsymbol{u} \pm \hat{i}$, together with a flip of all n in the “ladder”
- Accept proposal with probability $\min\left[1, t^{L_0 - 2k(\boldsymbol{u}, \pm i)} \frac{\rho(\boldsymbol{u}-\boldsymbol{v})}{\rho(\boldsymbol{u}'-\boldsymbol{v})}\right]$



- Together with cube-flips: ergodic
- Can change the wrapping numbers w_{0i}

Simulation: Non-Rejecting Polyakov Shift Updates

- Potential problem: High rejection rate in worm updates when L_0 is large
- Can be cured, if we are only interested in vacuum observables $\langle\langle O \rangle\rangle_0$

  [Q.Liu, Y.Deng, T.Garoni, Nucl.Phys. B846 (2011)]

Rejection Free Update

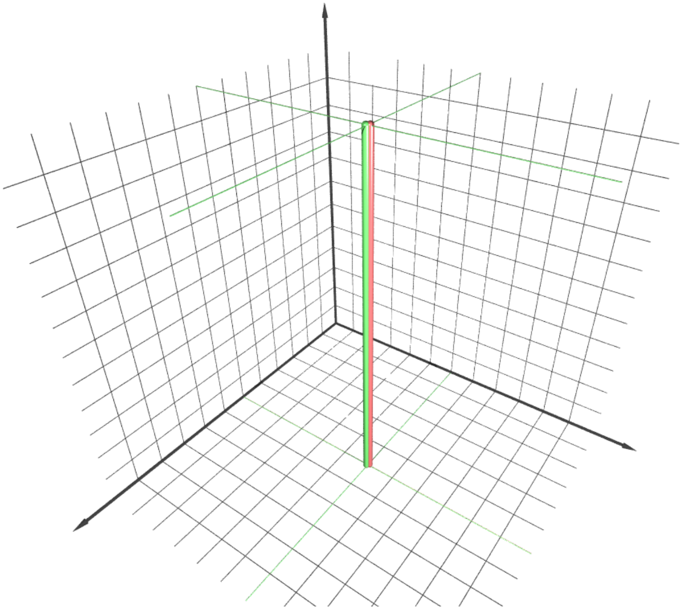
- If $\boldsymbol{u} = \boldsymbol{v}$: move $\boldsymbol{u}$ according to the usual worm algorithm
- If $\boldsymbol{u} \neq \boldsymbol{v}$:
  - Calculate usual worm acceptance probabilities for all directions i

    $\tilde{p}_{i,\pm} = \min\left[1, t^{L_0 - 2k(\boldsymbol{u},\pm i)} \, \frac{\rho(\boldsymbol{u}-\boldsymbol{v})}{\rho(\boldsymbol{u}'-\boldsymbol{v})}\right]$
  - Calculate $A = \frac{1}{2(D-1)} \sum_i (\tilde{p}_{i,+} + \tilde{p}_{i,-})$
  - Move $\boldsymbol{u}$ to position $\boldsymbol{u} + \hat{i}$ with probability $p_{i,\pm} = \frac{\tilde{p}_{i,\pm}}{2(D-1)A}$

    $\sum p_{i,\pm} = 1 \rightarrow$ some move is always made
- The sampled ensemble changes to

$$\mathcal{Z}_{\mathrm{nr}} = \sum_{\{n\},\boldsymbol{u},\boldsymbol{v}} t^{\sum_p n(p)} \, \rho^{-1}(\boldsymbol{u}-\boldsymbol{v}) \, \delta(\partial n + j^{(\boldsymbol{u},\boldsymbol{v})}) \; [\delta_{\boldsymbol{u},\boldsymbol{v}} + (1 - \delta_{\boldsymbol{u},\boldsymbol{v}})A]$$

Ising Gauge Theory Simulation Example



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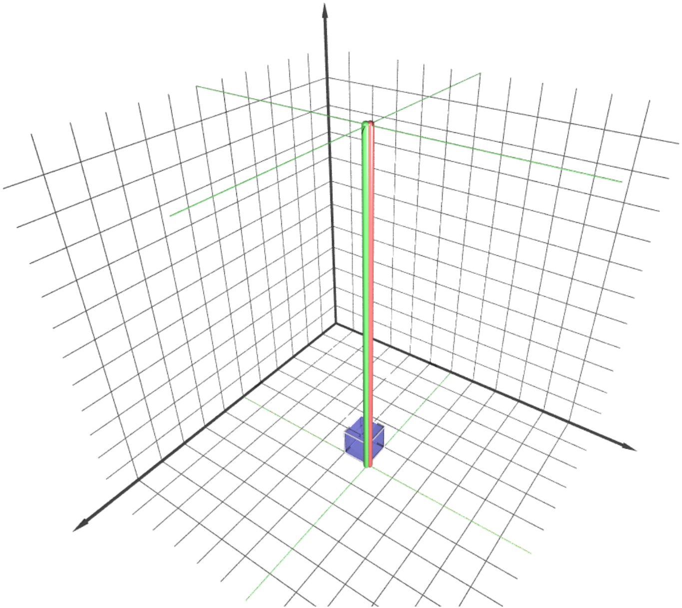

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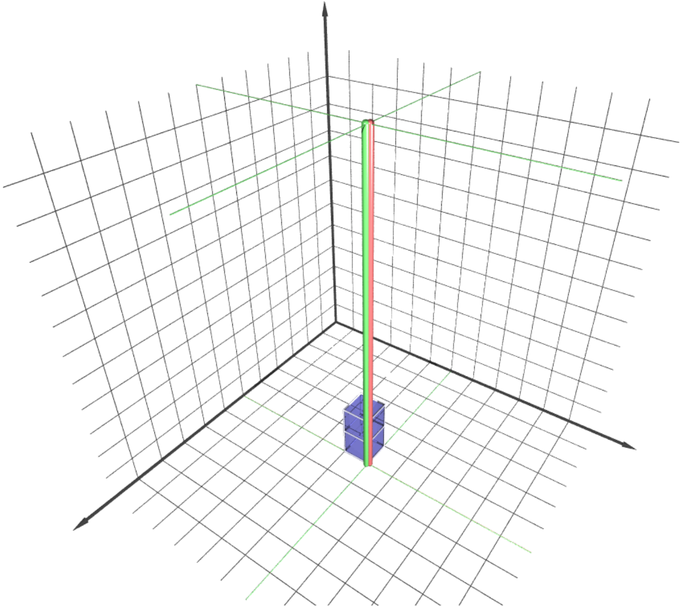

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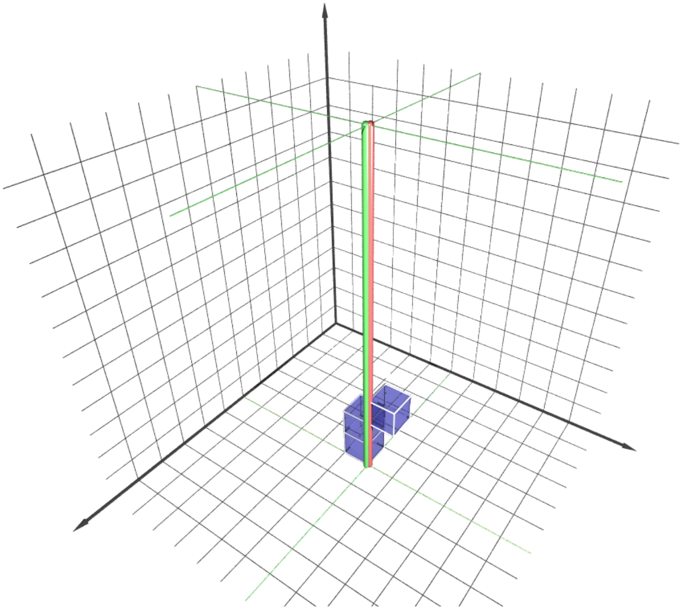

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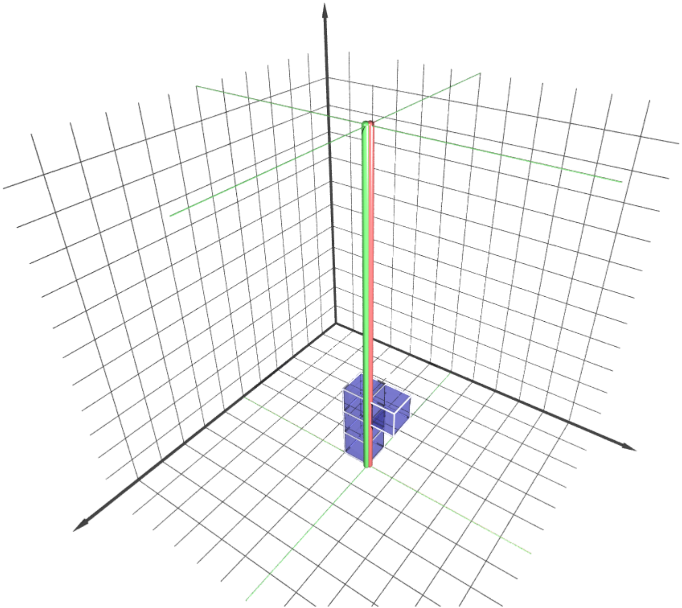

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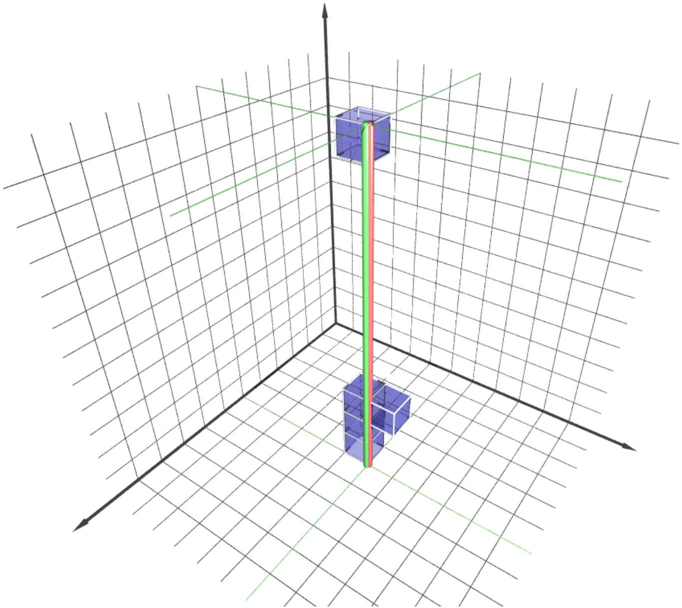

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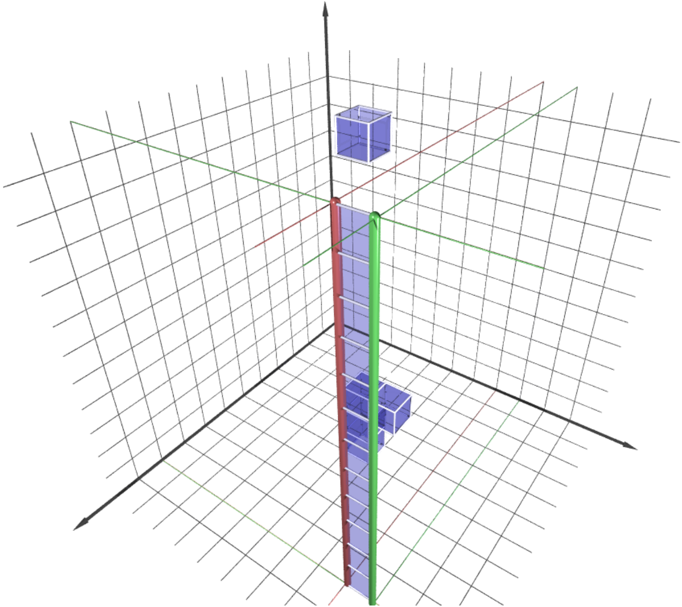

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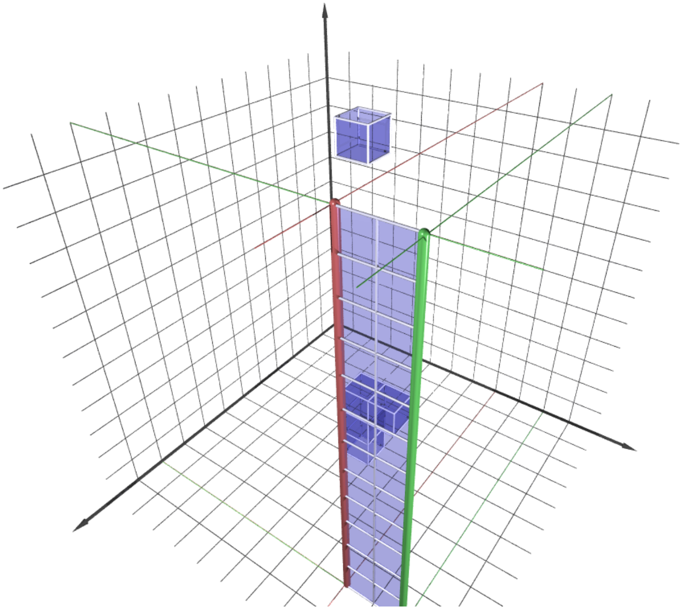

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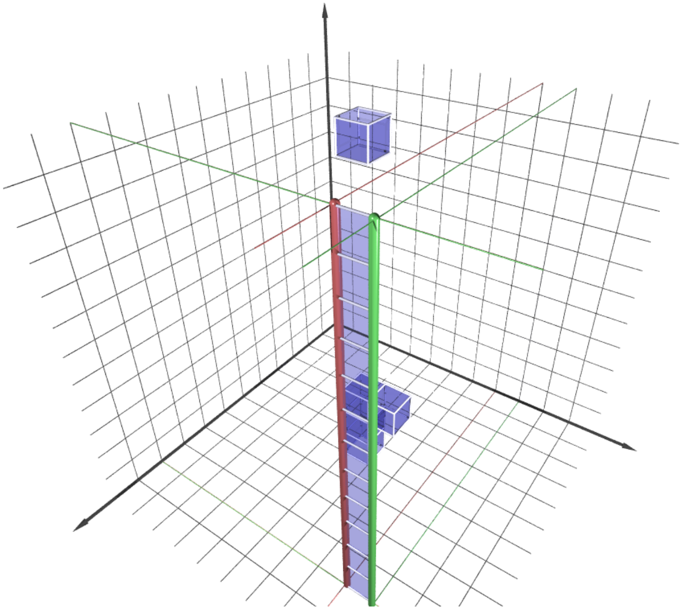

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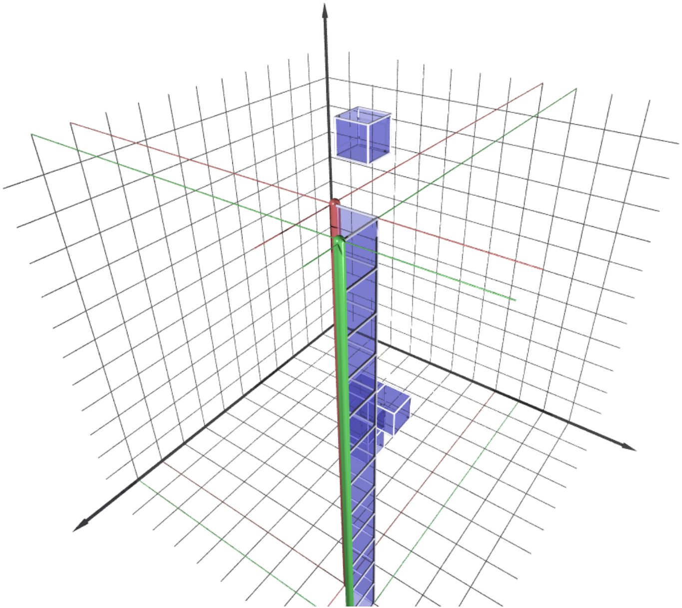

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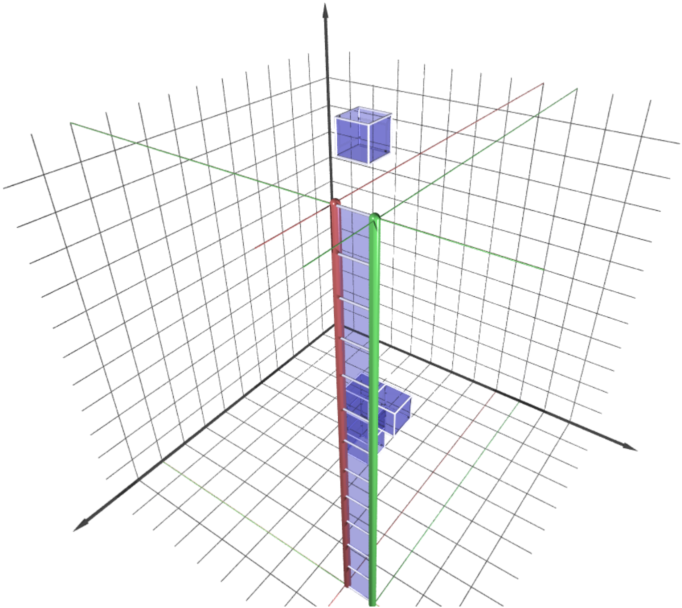

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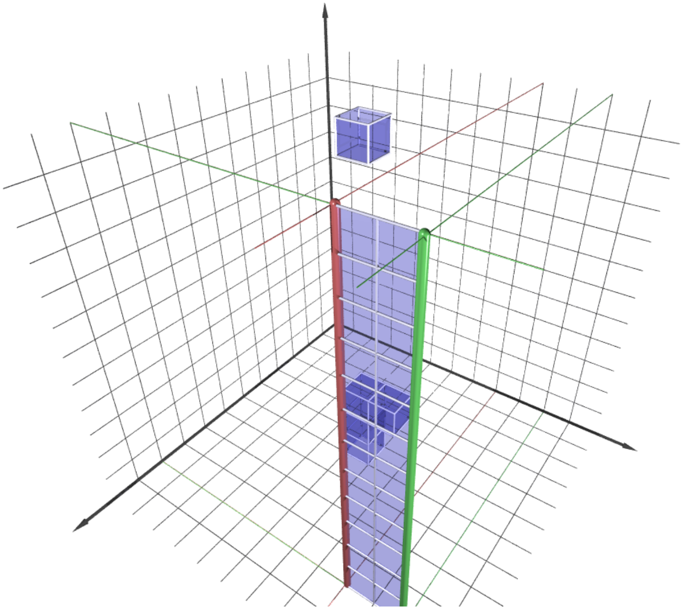

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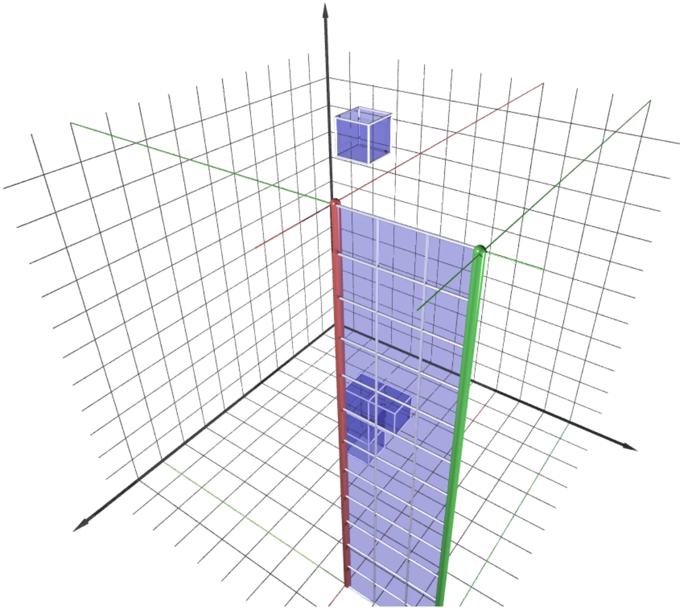

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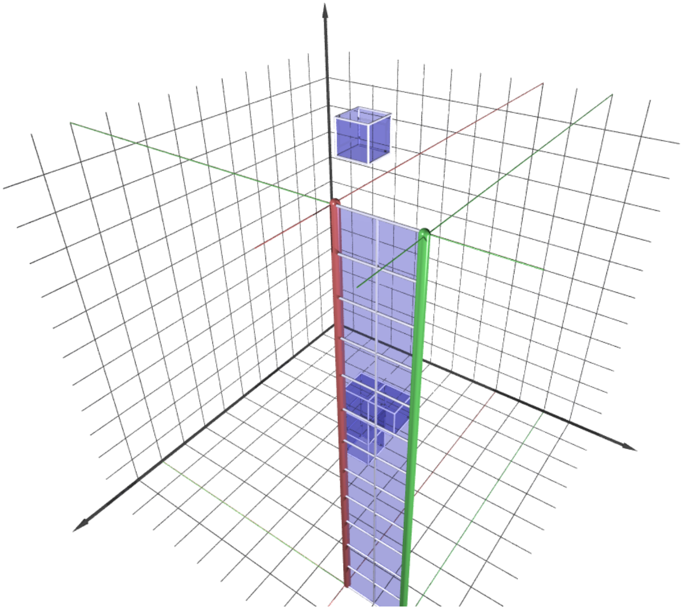

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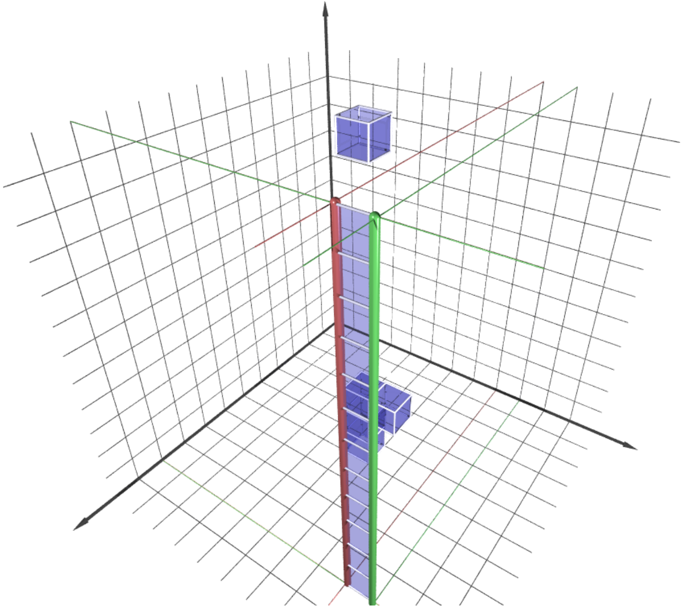

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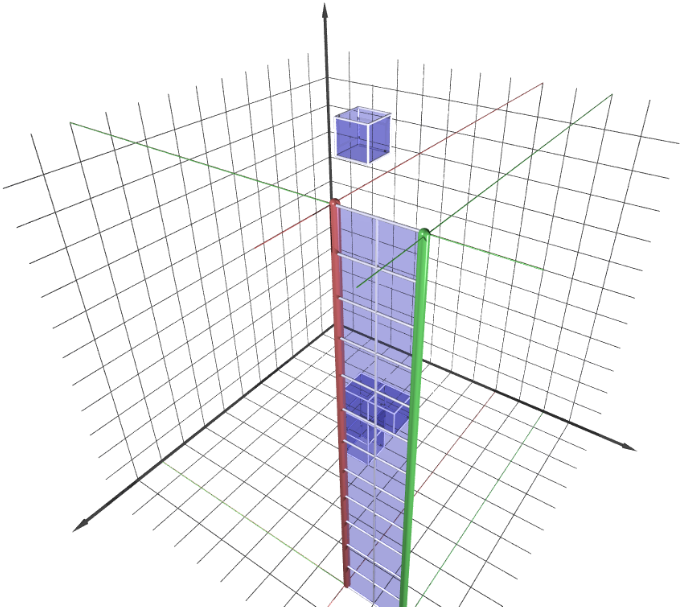

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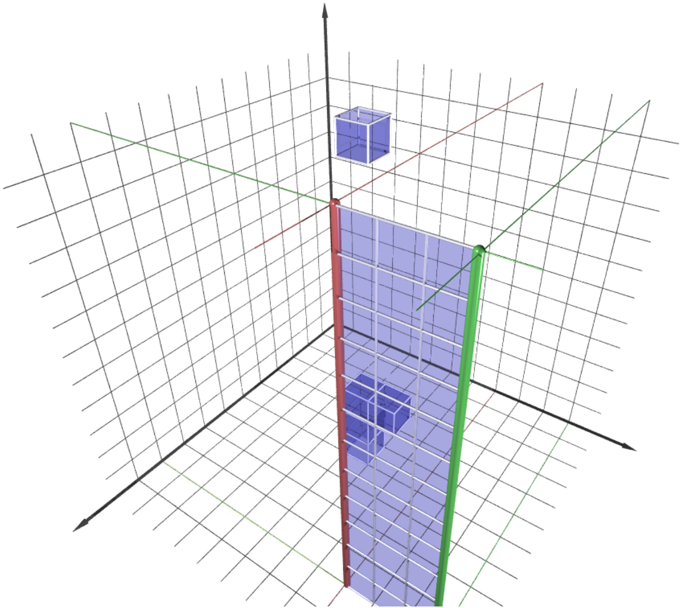

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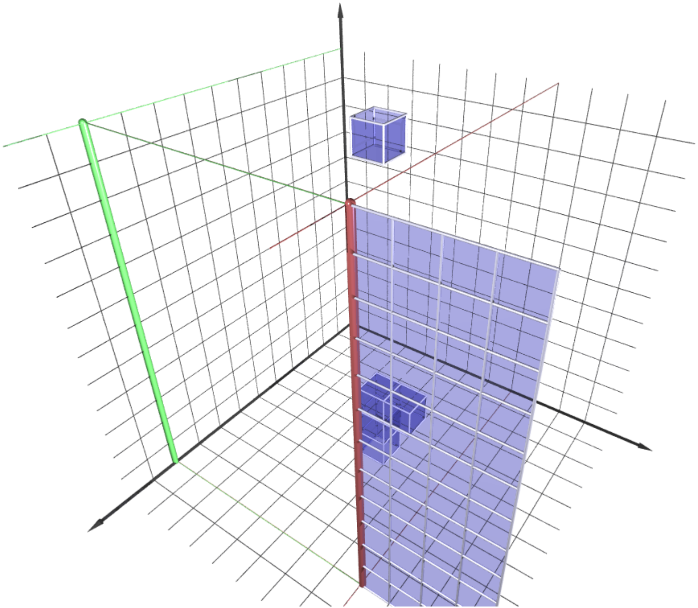

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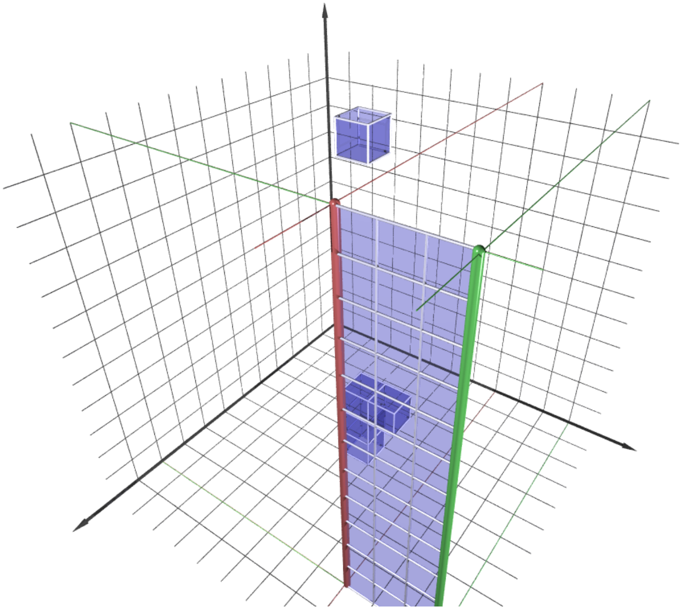

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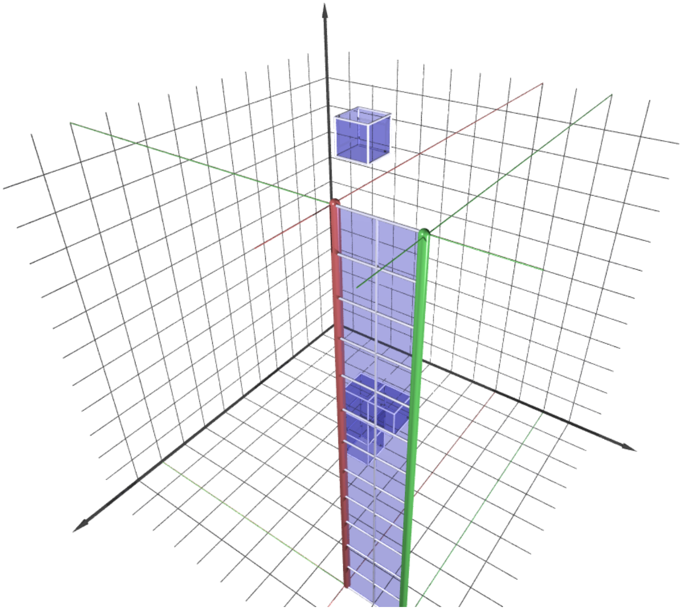

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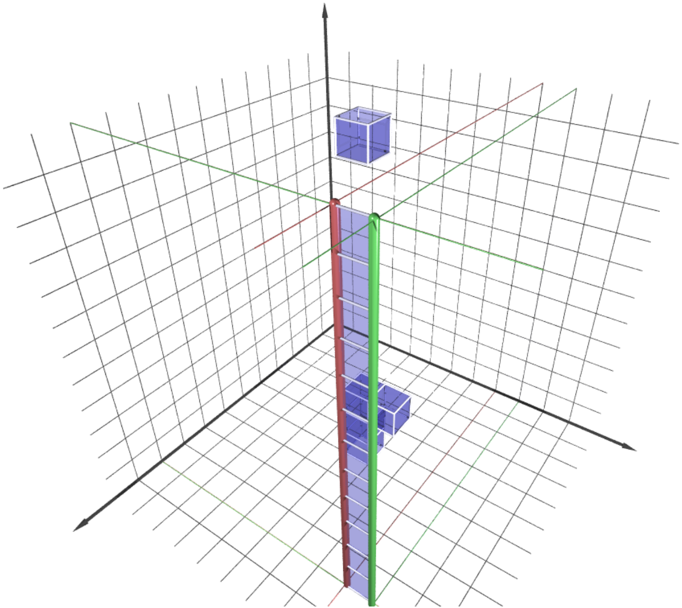

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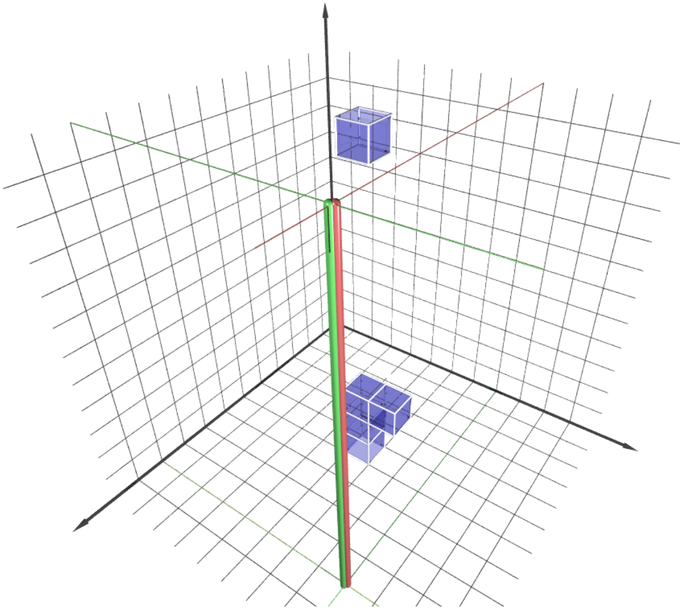

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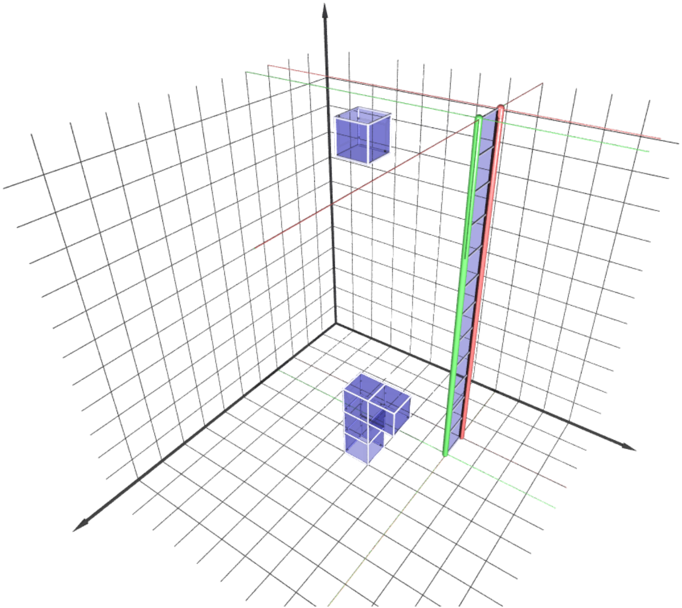

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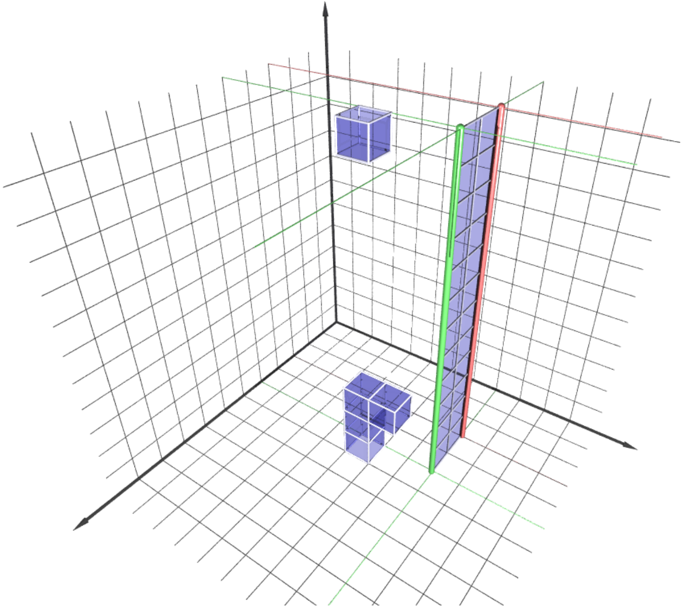

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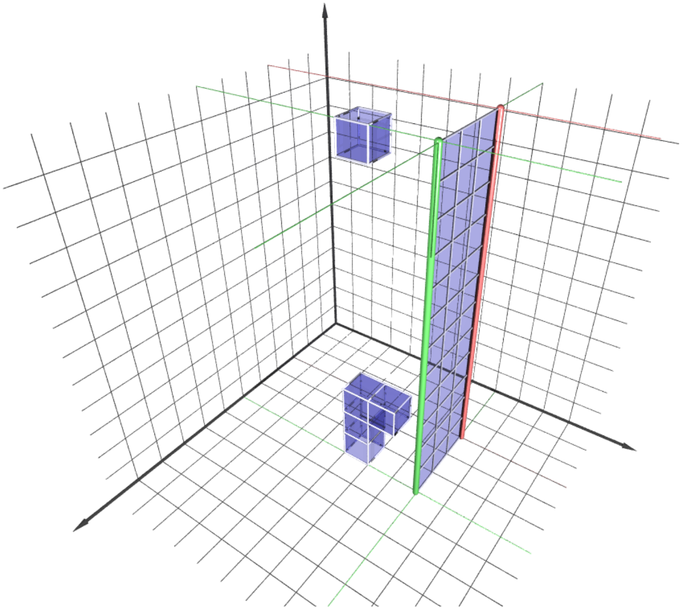

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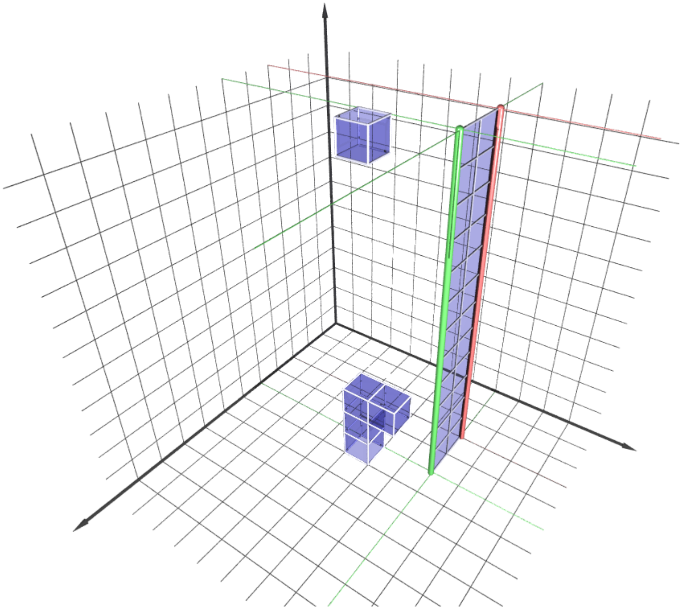

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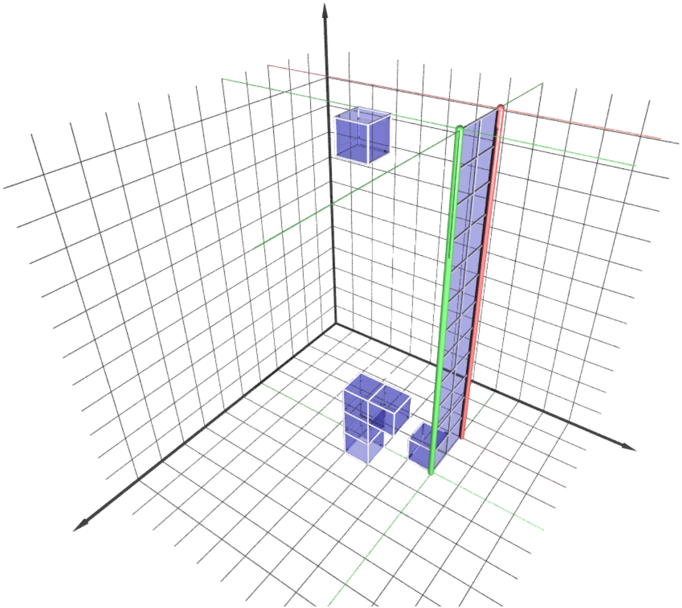

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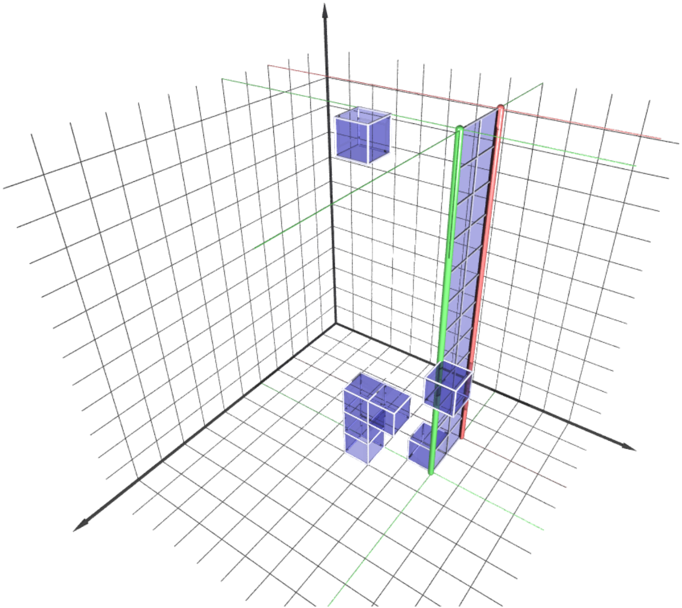

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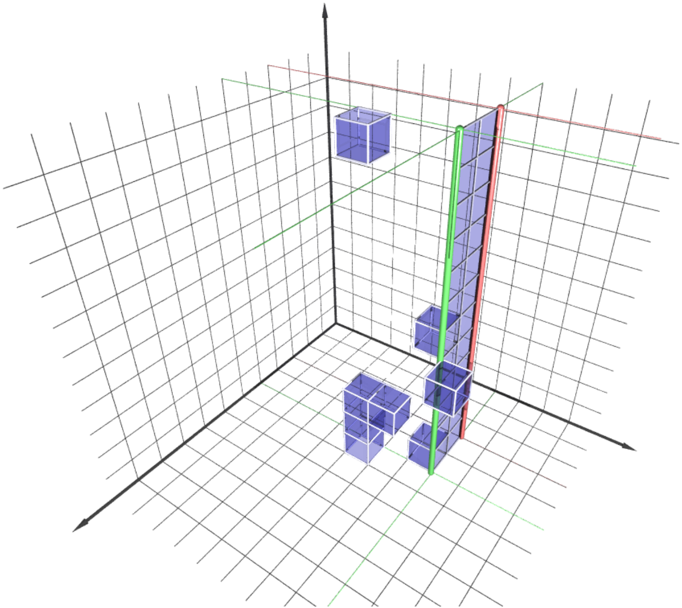

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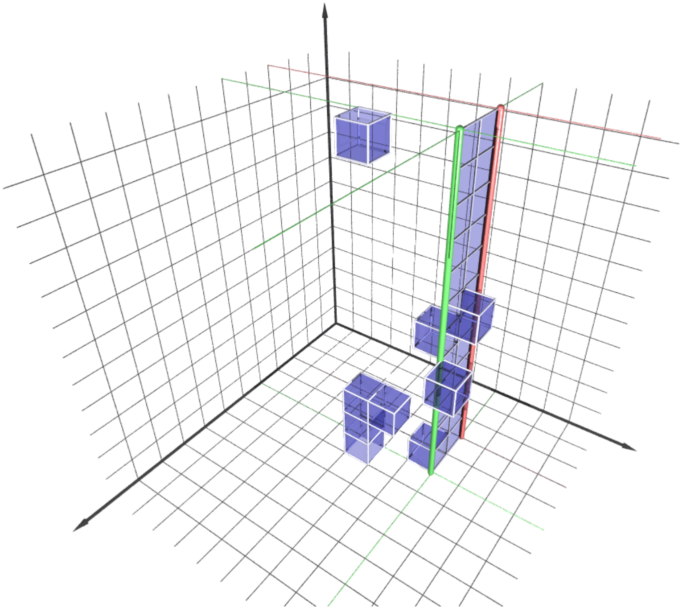

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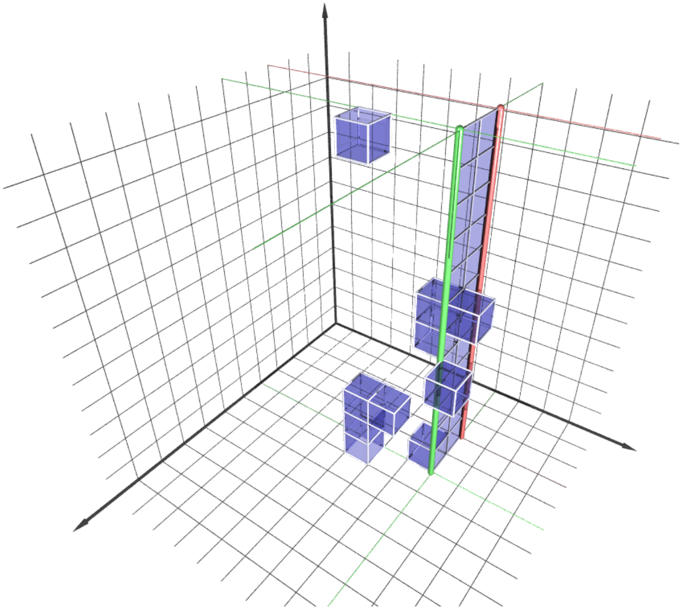

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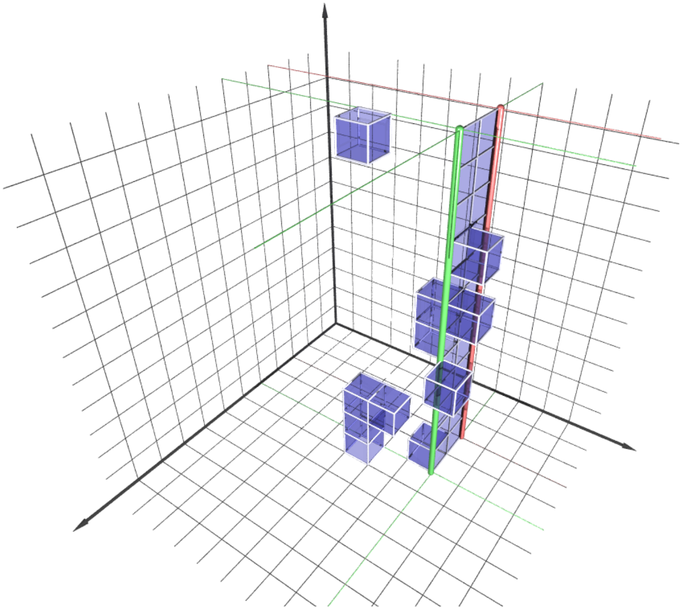

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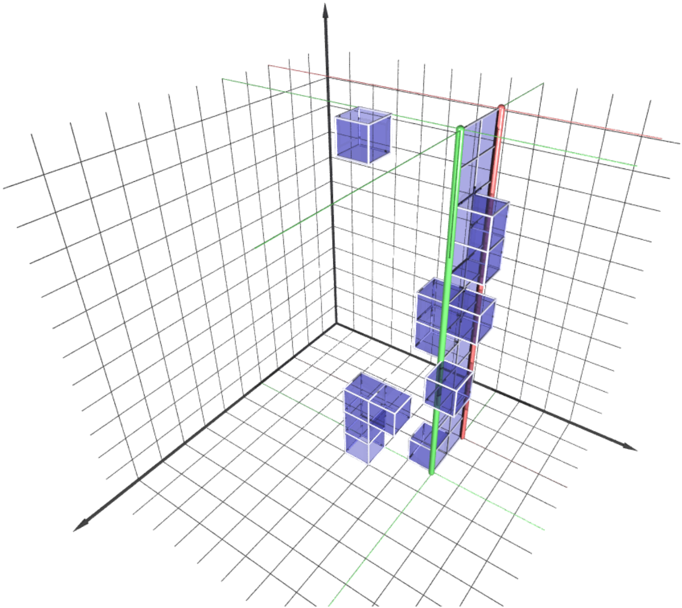

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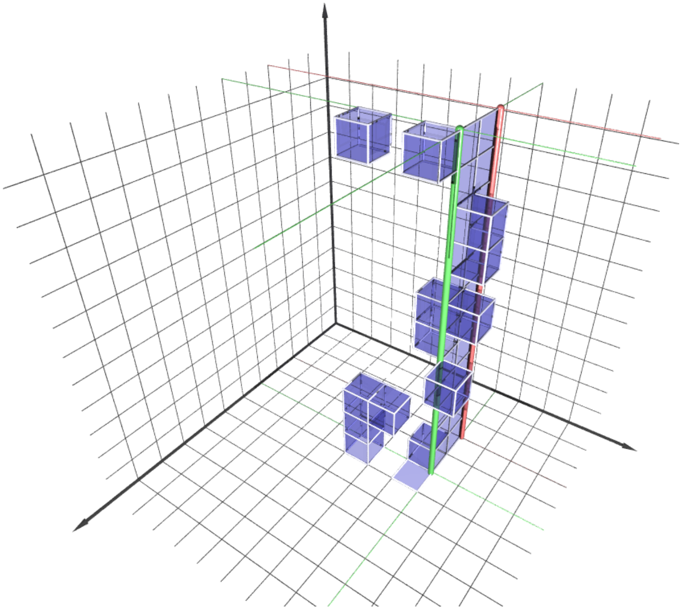

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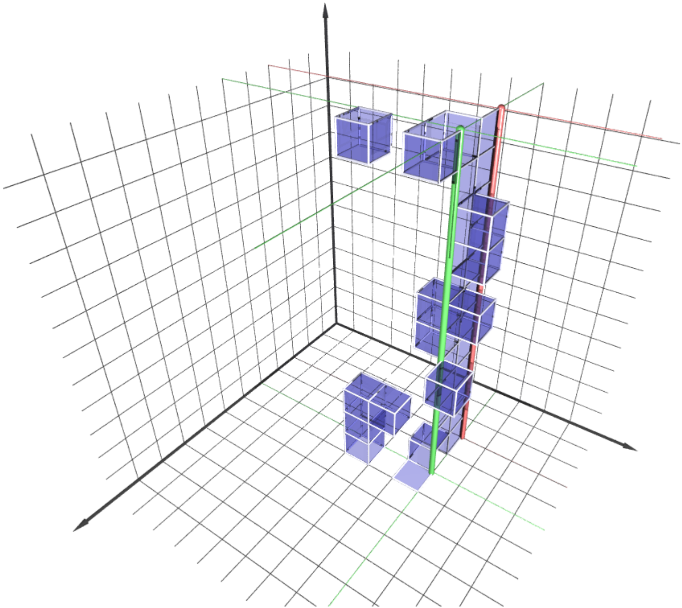

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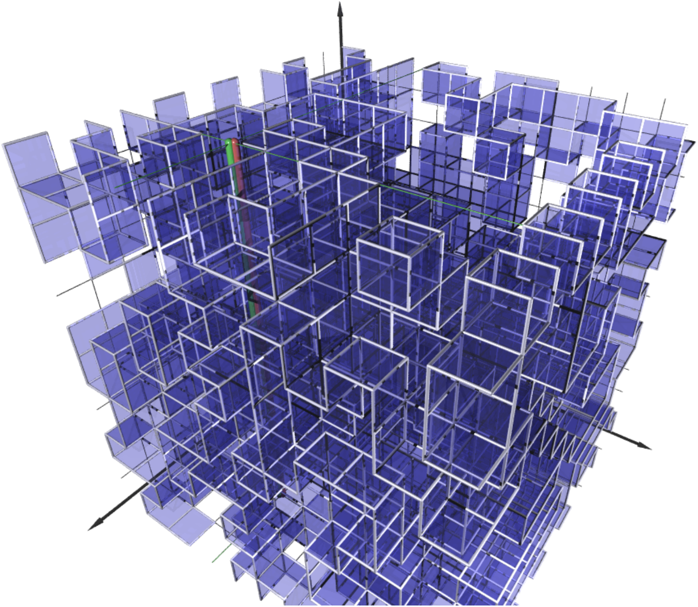

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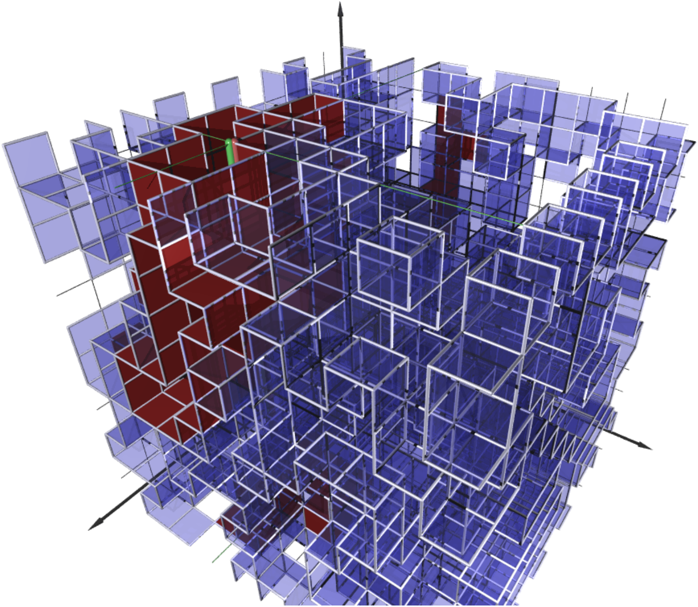

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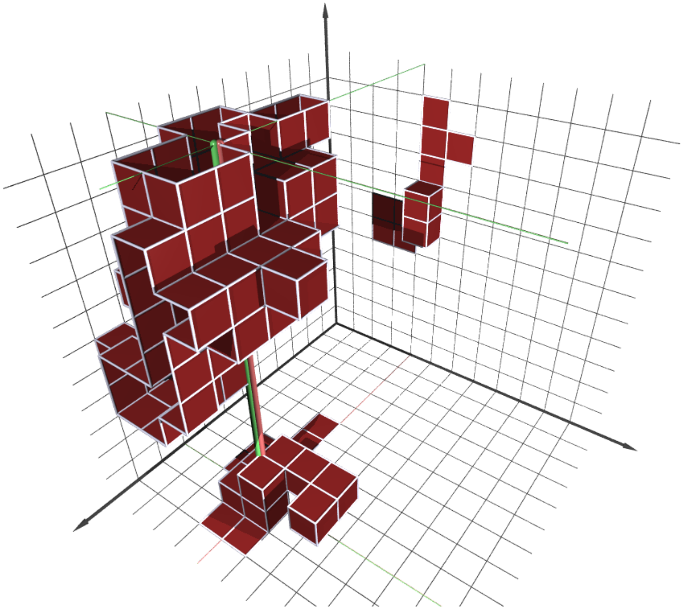

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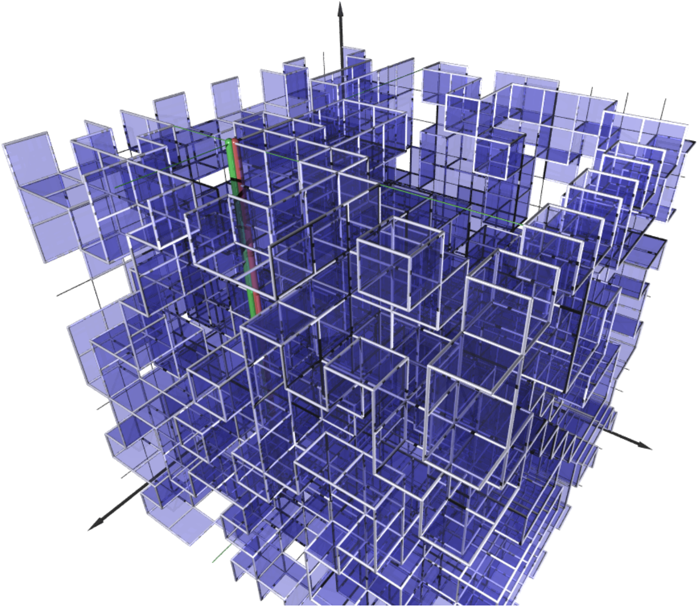
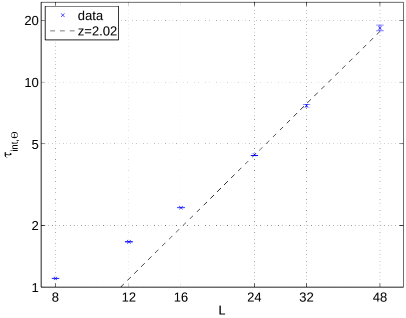


Critical Slowing Down

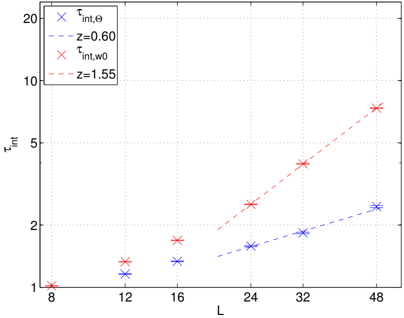
$\beta = 0.76141346 \approx \beta_c$ known from the dual Ising model

[M.Hasenbusch, Phys.Rev.B85 174421 (2012)], [Y.Deng, H.Blöte, Phys.Rev. E68 036125 (2003)]

Cubes + NR-Shifts



Cubes + Cluster + NR-Shifts



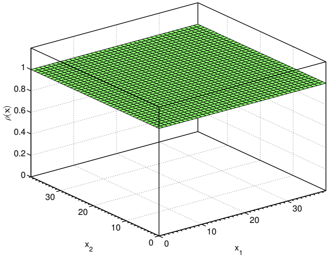
- At critical point: $\tau_{\rm int} \sim L^z$
- Θ is the total surface in $u = v$ configurations.
  w_0 is the fraction of zero-wrapping number configurations

Simulations in the Confined Phase

- $\beta < \beta_c$, anisotropic lattice $L_0 \times L \times L$
- We measure the correlator $G(\boldsymbol{x}) = \langle \pi(\boldsymbol{x})\pi(\mathbf{0}) \rangle$
- Project to zero momentum: $C(x_1) = \sum_{x_2} G(\boldsymbol{x})$
- $C(x_1) = \sum_n |v_n|^2 \, \exp(-E_n x_1)$
- We are mainly interested in E_0 as a function of L_0
  $\Rightarrow$ effective string theory
- Infinite volume, continuum:
  [M.Lüscher, P.Weisz, JHEP 0407, 014 (2004)]
  $$G(\boldsymbol{x}) = \sum_{n \geq 0} |v_n|^2 \, 2r \left(\frac{E_n}{2\pi r}\right)^{\frac{D-1}{2}} K_{\frac{1}{2}(D-3)}(E_n r)$$
- Inspired by this formula we set the weight function to
  $$\rho(\boldsymbol{x}) \propto \sum_{\boldsymbol{k} \in \mathbb{Z}} K_0 \left(\hat{M} \sqrt{(x_1 - k_1 L)^2 + (x_2 - k_2 L)^2} \right)$$
  with one free algorithmic parameter $\hat{M}$

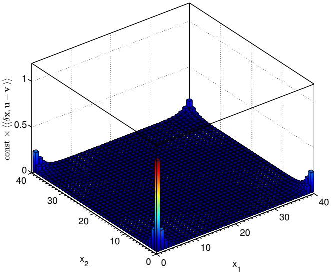
Polyakov Correlators: Reweighting

$\rho(\boldsymbol{x}) = 1$

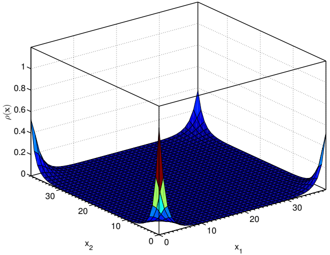


$\Rightarrow$

Histogram

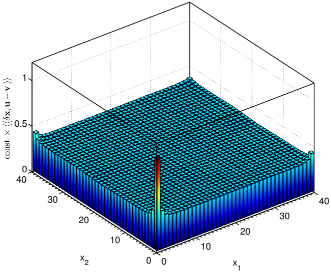


$\rho(\boldsymbol{x})$ with $\hat{M} = 0.4825$



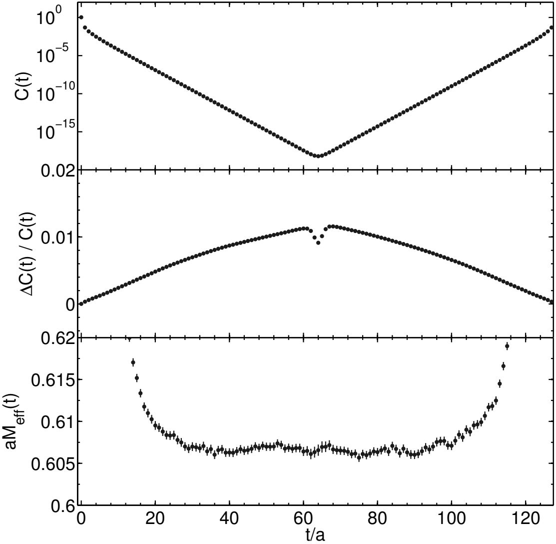
$\Rightarrow$

Histogram



Closed String Mass Gap

- Our most demanding run
  $L_0 = 56$, $L = 128$
- $M_{eff}(t) = \operatorname{acosh}\left[\frac{C(t-1)+C(t+1)}{2C(t)}\right]$
- $E_0 = 0.60661(24)$



Conclusions and Outlook

- Algorithm
  - Random surface formulation of Ising gauge theory with positive, factorizable weights
  - Simulation with local and cluster algorithms
  - Straightforward adjustment to e.g. $U(1)$ gauge theory
  - Critical slowing down not eliminated
  - But: other beneficial worm properties salvaged
    - ⋆ Interesting observables that are hard to measure in a traditional simulation
    - ⋆ Modified importance sampling
      $\Rightarrow$ Polyakov correlators with (exponentially) reduced errors
- Physics
  - Effective string theory picture of confinement $\rightarrow$ Ulli Wolff's talk