$q\bar{q}$-potential

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Motivation

- Physical problem
  - defining the static potential in the octet (adjoint) $q\bar{q}$-channel

    C. Borgs, E. Seiler, Commun. Math. Phys. **91** (1983) 329

    S. Necco, R. Sommer, Phys. Lett. B **523** (2001) 135

    O. Philipsen, Phys. Lett. B **535** (2002) 138

    O. Kaczmarek, F. Karsch, P. Petreczky, F. Zantow, Phys. Lett. B **543** (2002) 41

    O. Jahn, O. Philipsen, Phys. Rev. D **70** (2004) 074504

    O. Philipsen, M. Wagner, arXiv:1305.5957 [hep-lat]

    ... plus many other papers ...
  - useful hints for exotic quark state analysis
  - perturbative vs. non-perturbative $q\bar{q}$-interaction energies
- A challenging theoretical problem that has to do with
  - Faddeev–Popov procedure
  - effective global colour bleaching in lattice simulations
  - $q\bar{q}$-source and gluon colour indices entanglement
- We provide
  - a theoretical basis to rigorously frame the discussion
  - a workable solution to the problems listed above

Plan of the Talk

- The temporal gauge
  - Formulation [GCR, M. Testa, Nucl. Phys. B **163** (1980) 109 & **176** (1980) 477
    M. Lüscher, R. Narayanan, P. Weisz, U. Wolff, Nucl. Phys. B **384** (1992) 168]
  - Introducing external colour sources
- Present talk based on GCR, M. Testa, Phys. Rev. D **87** (2013) 085014
  - Reexamine the Faddeev–Popov procedure
  - Explain the role of global SU(N_c) rotations $\rightarrow$ colour bleaching
  - Energy eigenstate classification in terms of SU(N_c) irrep's ...
    ... allows disentangling source and gluon colour indices
- Extracting the $q\bar{q}$-potentials from the Feynman kernel
  - Partial trace
  - Vanishing boundary gauge fields
  - Character expansion
- Conclusions & Outlook

The $q\bar{q}$-system in the temporal gauge - I

- The Feynman kernel with $q\bar{q}$-sources in $A_0 = 0$ gauge
  - $K(\mathbf{A_2}, s_2, r_2; \mathbf{A_1}, s_1, r_1; T) =$
    $$= \int_{\mathcal{G}_0} \mathcal{D}\mu(h)\, [U_h(\mathbf{x}_q)]_{s_2 s_1} [U_h(\mathbf{x}_{\bar{q}})]^*_{r_2 r_1} \widetilde{K}(\mathbf{A_2}^{U_h}, \mathbf{A_1}; T)$$
  - $$\widetilde{K}(\mathbf{A}_2, \mathbf{A}_1; T) = \int_{\mathbf{A}(\mathbf{x},T_1)=\mathbf{A}_1(\mathbf{x})}^{\mathbf{A}(\mathbf{x},T_2)=\mathbf{A}_2(\mathbf{x})} \mathcal{D}\mathbf{A} \exp\left[-S_{YM}(\mathbf{A}, A_0 = 0)\right]$$
  - $\mathcal{D}\mathbf{A} = \prod_{\mathbf{x}, T_1 < t < T_2} d\mathbf{A}(\mathbf{x}, t)$
  - $\mathcal{D}\mu(h) =$ invariant Haar measure over the group $\mathcal{G}_0$
  - $\mathcal{G}_0 =$ group of (topologically trivial) time-independent gauge transformations that tend to the identity at spatial infinity
  - $\mathcal{G}_0$-integration projects $\widetilde{K}$ over the desired source sector
- $K(\mathbf{A_2}, s_2, r_2; \mathbf{A_1}, s_1, r_1; T)$ identical in $A_0 = 0$ and Coulomb gauge

The $q\bar{q}$-system in the temporal gauge - II

- $K(\mathbf{A_2}, s_2, r_2; \mathbf{A_1}, s_1, r_1; T) =$

$$= \int_{\mathcal{G}_0} \mathcal{D}\mu(h) \Big[\exp\left[i\lambda^a h^a(\mathbf{x}_q)\right] \Big]_{s_2 s_1} \Big[\exp\left[-i\lambda^a h^a(\mathbf{x}_{\bar{q}})\right] \Big]_{r_2 r_1} \widetilde{K}(\mathbf{A_2}^{U_h}, \mathbf{A_1}; T)$$

- Spectral decomposition

$$K(\mathbf{A}_2, s_2, r_2; \mathbf{A}_1, s_1, r_1; T) = \sum_k e^{-E_k T} \psi_k(\mathbf{A}_2, s_2, r_2)\, \psi_k^\star(\mathbf{A}_1, s_1, r_1)$$

- $\psi_k(\mathbf{A}, s, r)$ eigenstate of the Hamiltonian with eigenvalue E_k

$$\mathcal{H}\psi_k(\mathbf{A}, s, r) = E_k \psi_k(\mathbf{A}, s, r)$$

- transforming under $U_w(\mathbf{x}) \in \mathcal{G}_0$ as

$$\psi_k(\mathbf{A}^{U_w}, s, r) = \sum_{s', r'} \left[e^{-i\lambda^a w^a(\mathbf{x}_q)} \right]_{s\, s'} \left[e^{i\lambda^a w^a(\mathbf{x}_{\bar{q}})} \right]_{r'\, r} \psi_k(\mathbf{A}, s', r')$$

- This yields the Gauss' law in the presence of $q\bar{q}$ external sources

The $q\bar{q}$-system in the temporal gauge - III

- $\sum_{s,r} \psi^*(\mathbf{A}, s, r)\phi(\mathbf{A}, s, r)$ is invariant under $\mathcal{G}_0$ gauge transformations

$$(\psi, \phi) \sim \int \mathcal{D}\mathbf{A} \sum_{s,r} \psi^*(\mathbf{A}, s, r)\phi(\mathbf{A}, s, r) = \infty$$

  - Scalar product must be defined via the FP procedure

    - $(\psi, \phi) = \int \mathcal{D}\mu_F(\mathbf{A}) \sum_{s,r} \psi^*(\mathbf{A}, s, r)\phi(\mathbf{A}, s, r)$

- $\mathcal{D}\mu_F(\mathbf{A}) = \Delta_F(\mathbf{A}) \prod_{\mathbf{x}} \delta[F(\mathbf{A})]\, d\mathbf{A}(\mathbf{x})$
- $1 = \Delta_F(\mathbf{A}) \int_{\mathcal{G}_0} \mathcal{D}\mu(h)\delta[F(\mathbf{A}^{U_h})]$

  - Independent of the gauge functional $F(\mathbf{A})$ (typically $F(\mathbf{A}) \to \nabla\mathbf{A}$)
  - It effectively completely fixes the gauge

- The color trace of the Feynman kernel is also gauge invariant
- For the full trace, one gets (d_k = degeneracy of the energy level E_k)

$$\int \mathcal{D}\mu_F(\mathbf{A}) \sum_{s,r} K(\mathbf{A}, s, r; \mathbf{A}, s, r; T) = \sum_k d_k e^{-E_k T}$$

- Gauge invariance of l.h.s. implies gauge invariance of E_k

Technical problems

1. Faddeev–Popov procedure

$$1 = \Delta_F(A) \int_{\mathcal{G}} \mathcal{D}\Omega \; \delta[F(A^\Omega)]$$

- For admissible gauge fields, the equation

$$F(A^\Omega) = 0$$

  must have a (locally) unique solution for Ω

2. Colour bleaching (in the temporal gauge)

- Only colour singlet states survive, if $\mathcal{G}$ includes global rotations

$$\text{effective kernel} \rightarrow \overline{K} = \int_{\mathrm{SU(N_c)}} \mathcal{D}V \, K(\mathbf{A}_2^V, \mathbf{A}_1; T)$$

1. (almost) automatic in perturbation theory
2. (very) difficult to avoid/prevent in actual simulations

A more conceptual (and difficult) problem

- SU(N_c)-transformations of $\mathcal{H}$-eigenstates ($\psi(\mathbf{A}; s, r) \equiv \psi(\mathbf{A})$)

  $$\bullet\ \mathcal{U}(V)\psi(\mathbf{A}) = V\psi(\mathbf{A}^V)V^\dagger$$

- $[\mathcal{U}(V), K]$, $V \in$ SU(N_c) $\rightarrow \mathcal{H}$-eigenstates belong to SU(N_c) irrep's
- $\mathcal{H}$-eigenstates in the $q\bar{q}$-sector can be parametrized as

  $$\bullet\ \psi(\mathbf{A}) = \phi(\mathbf{A})I + \phi_a(\mathbf{A})\lambda^a \equiv \phi(\mathbf{A})I + \phi_a(\mathbf{A})\lambda^a$$

- They transform according to the formula

  $$\bullet\ \mathcal{U}(V)\psi(\mathbf{A}) = V\psi(\mathbf{A}^V)V^\dagger = \phi(\mathbf{A}^V)I + \phi_a(\mathbf{A}^V)V\lambda^a V^\dagger$$

1. The global colour transformation of ψ has two contributions
   - "colour-spin" contribution from the action of V on source indices
   - "orbital" contribution from the transformation $\mathbf{A} \rightarrow \mathbf{A}^V$
2. "Colour indices entanglement"
   - orbital and source indices are non-trivially entangled
   - how do we identify, e.g. the adjoint $q\bar{q}$ static potential?
3. Need a classification of $\mathcal{H}$-eigenstates in terms of SU(N_c) irrep's

Energy eigenstate classification - I

- Energy eigenstates belong to $SU(N_c)$ irrep's
- $\mathcal{U}(V)\psi(\mathbf{A}) \equiv \psi^V(\mathbf{A})$ must span a unique irrep for any value of $\mathbf{A}$
- For $\mathbf{A} = \mathbf{0}$ the orbital contribution is absent and we get

$$\psi^V(\mathbf{0}) = V\psi(\mathbf{0})V^\dagger = \phi(\mathbf{0})I + \phi_a(\mathbf{0})V\lambda^a V^\dagger$$

The two terms in the r.h.s. cannot be simultaneously non-vanishing (otherwise $\psi^V(\mathbf{0})$ would belong to the reducible $I \oplus [N_c^2 - 1]$ rep.)

- Thus we have the following three alternatives
  1) $\phi_a(\mathbf{0}) = 0$ and $\phi(\mathbf{0}) \neq 0$
  2) $\phi(\mathbf{0}) = 0$ and $\phi_a(\mathbf{0}) \neq 0$ (for some a)
  3) $\phi(\mathbf{0}) = \phi_a(\mathbf{0}) = 0$

in correspondence to different types of irrep's

Energy eigenstate classification - II

1. colour-spin singlet $\otimes$ orbital singlet states $(\phi_a(\mathbf{0}) = 0$ & $\underline{\phi(\mathbf{0}) \neq 0})$

$$\psi^{[S]}_{[S]}(\mathbf{A}) = \phi(\mathbf{A})I \quad \text{with} \quad \phi(\mathbf{A}^V) = \phi(\mathbf{A})$$

2. colour-spin adjoint $\otimes$ orbital singlet states $(\phi(\mathbf{0}) = 0$ & $\underline{\phi_a(\mathbf{0}) \neq 0})$

$$\psi^{[S]}_{[Ad]}(\mathbf{A}) = \lambda^a \phi_a(\mathbf{A}) \quad \text{with} \quad \phi_a(\mathbf{A}^V) = \phi_a(\mathbf{A})$$

3. colour-spin singlet $\otimes$ orbital $[\alpha]$ states $(\phi^{[\alpha]}_m(\mathbf{0}) = 0)$

$$\psi^{[\alpha]}_{m\,[S]}(\mathbf{A}) = \phi^{[\alpha]}_m(\mathbf{A})I \quad \text{with} \quad \phi^{[\alpha]}_m(\mathbf{A}^V) = R^{[\alpha]}_{mm'}(V)\phi^{[\alpha]}_{m'}(\mathbf{A})$$

4. colour-spin adjoint $\otimes$ orbital $[\beta]$ states $\to$ irrep. $[\alpha']$ $(\phi_{ak}(\mathbf{0}) = 0)$

$$\psi^{[\alpha]}_{m\,[Ad]}(\mathbf{A}) = \lambda^a \phi_{ak}(\mathbf{A}) \quad \text{with} \quad \phi_{ak}(\mathbf{A}^V) = R^{[\beta]}_{kk'}(V)\phi_{ak'}(\mathbf{A})$$

singlet and adjoint $q\bar{q}$-potentials extracted from channels (1) and (2)

But ...

... only (1) & (4) (overall singlets) survive global colour rotations, $\mathbf{A}^V$

1. colour-spin singlet $\otimes$ orbital singlet states ($\phi_a(\mathbf{0}) = 0$ & $\phi(\mathbf{0}) \neq 0$)

$$\psi^{[S]}_{[S]}(\mathbf{A}) = \phi(\mathbf{A}) I \quad \text{with} \quad \phi(\mathbf{A}^V) = \phi(\mathbf{A})$$

2.

3.

4. colour-spin adjoint $\otimes$ orbital $[\beta]$ states $\to [\alpha'] = I$ ($\phi_{ak}(\mathbf{0}) = 0$)

$$\psi^{[\alpha]}_{m\,[Ad]}(\mathbf{A}) = \lambda^a \phi_{ak}(\mathbf{A}) \quad \text{with} \quad \phi_{ak}(\mathbf{A}^V) = R^{[\beta]}_{kk'}(V)\phi_{ak'}(\mathbf{A})$$

Extracting the $q\bar{q}$ static potentials ...

- How can we in practice extract the $q\bar{q}$ static potentials?
- A number of possible ways ...
  - ... from the partially traced kernel

$$K_{s_2 r_2; s_1 r_1}(T) \equiv \int \mathcal{D}\mu_F(\mathbf{A}) K(\mathbf{A}, s_2, r_2; \mathbf{A}, s_1, r_1; T)$$

  - ... from the Feynman kernel with homogeneous b.c.'s

$$K(\mathbf{0}, s_2, r_2; \mathbf{0}, s_1, r_1; T)$$

  - ... from the use of "character expansion formulae"

$$\overline{K}^{[\gamma]}_{s_2, r_2; s_1, r_1}(T) \equiv$$

$$\equiv \int_{\mathrm{SU}(N_c)} \mathcal{D}V (\chi^{[\gamma]}(V))^* V_{s_2 s_3} V^*_{r_2 r_3} \int \mathcal{D}\mu_F(\mathbf{A}) K(\mathbf{A}^V, s_3, r_3; \mathbf{A}, s_1, r_1; T)$$

- What we get depends on how we deal with global colour rotations

- ... from the partially traced kernel, upon projecting over $\delta\delta$ and $\lambda\lambda$

$$K_{s_2 r_2; s_1 r_1}(T) \equiv \int \mathcal{D}\mu_F(\mathbf{A}) K(\mathbf{A}, s_2, r_2; \mathbf{A}, s_1, r_1; T)$$

- No integration over $V \in SU(N_c)$ in $\mathbf{A}^V$ $\rightarrow$ we get what we want

- $$\sum_{s_2 r_2 s_1 r_1} \frac{1}{N_c} \delta_{r_2 s_2} \delta_{s_1 r_1} K_{s_2 r_2; s_1 r_1}(T) \stackrel{T\to\infty}{\rightarrow} \underline{\underline{e^{-E^{[S]}T}}} + \ldots + D_{[\alpha]} e^{-E^{[\alpha]}T} + \ldots$$

- $$\sum_{s_2 r_2 s_1 r_1} \sum_{a} \lambda^a_{r_2 s_2} \lambda^a_{s_1 r_1} K_{s_2 r_2; s_1 r_1}(T) \stackrel{T\to\infty}{\rightarrow} (N_c^2 - 1)\, \underline{\underline{e^{-E^{[Ad]}T}}} + \ldots + D_{[\alpha']} e^{-E^{[\alpha']}T} + \ldots$$

- Otherwise $\rightarrow$ only global singlets ($\overline{K} = V$-averaged kernel)

- $$\sum_{s_2 r_2 s_1 r_1} \frac{1}{N_c} \delta_{r_2 s_2} \delta_{s_1 r_1} \overline{K}_{s_2 r_2; s_1 r_1}(T) \stackrel{T\to\infty}{\rightarrow} \underline{\underline{e^{-E^{[S]}T}}} + \ldots$$

- $$\sum_{s_2 r_2 s_1 r_1} \sum_{a} \lambda^a_{r_2 s_2} \lambda^a_{s_1 r_1} \overline{K}_{s_2 r_2; s_1 r_1}(T) \stackrel{T\to\infty}{\rightarrow} D_{[\alpha']}\, \underline{\underline{e^{-E^{[\alpha']}T}}} + \ldots$$

where $[\alpha'] = [N_c^2 - 1]_{source} \otimes [\beta]_{orbital} = I$

- ... from the Feynman kernel with homogeneous b.c.'s

$$K(\mathbf{0}, s_2, r_2; \mathbf{0}, s_1, r_1; T)$$

- No integration over global colour rotations → we get what we want

- $K(\mathbf{0}, s_2, r_2; \mathbf{0}, s_1, r_1; T) =$

$$= \left|\phi(\mathbf{0})\right|^2 \frac{1}{N_c} \delta_{s_2 r_2} \delta_{r_1 s_1} \, \underline{\underline{e^{-E^{[S]}T}}} + \sum_a \left|\phi_a(\mathbf{0})\right|^2 \sum_b \lambda^b_{s_2 r_2} \lambda^b_{r_1 s_1} \, \underline{\underline{e^{-E^{[Ad]}T}}}$$

  - Conjecture is that only these two terms are actually present
  - $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{0}$ kills orbital gluon excitations (state classification)
  - The conjecture has been verified to hold in perturbation theory

- Otherwise → only global singlets survive

- $\overline{K}(\mathbf{0}, s_2, r_2; \mathbf{0}, s_1, r_1; T) = \left|\phi(\mathbf{0})\right|^2 \frac{1}{N_c} \delta_{s_2 r_2} \delta_{r_1 s_1} \, \underline{\underline{e^{-E^{[S]}T}}}$

- Turning a nuisance into a benefit ...
  ... use "character expansion formulae"
  - in the absence of sources
    - $$\int \mathcal{D}\mu_F(\mathbf{A}) K(\mathbf{A}^V, \mathbf{A}; T) = \sum_{\alpha} \chi^{[\alpha]}(V)\, e^{-E^{[\alpha]}T}$$
    - $$\int_{\mathrm{SU(N_c)}} \mathcal{D}V (\chi^{[\gamma]}(V))^* \int \mathcal{D}\mu_F(\mathbf{A}) K(\mathbf{A}^V, \mathbf{A}; T) = \sum_{\alpha} \delta_{[\alpha],[\gamma]}\, e^{-E^{[\gamma]}T}$$
  - in the presence of sources one can filter global $SU(N_c)$ irrep's
    - $$\overline{K}^{[\gamma]}_{s_2,r_2;s_1,r_1}(T) \equiv$$
      $$\equiv \int_{\mathrm{SU}(N_c)} \mathcal{D}V (\chi^{[\gamma]}(V))^* V_{s_2 s_3} V^*_{r_2 r_3} \int \mathcal{D}\mu_F(\mathbf{A}) K(\mathbf{A}^V, s_3, r_3; \mathbf{A}, s_1, r_1; T)$$
- Resulting formulae are a bit ...
  - ... too complicated to be discussed here
    [I refer the interested people to the published paper
    GCR, M. Testa, Phys. Rev. D **87** (2013) 085014]
  - ... too noisy for simulation purposes (see, however, poster)

Conclusions & outlook

- Conclusions
  - Difficulties in defining singlet and adjoint $q\bar{q}$-potentials
    - Uniqueness of the Faddeev–Popov procedure
    - Global colour rotations
    - Entanglement of source and orbital colour indices
    - Classification of colour irrep's
  - Extracting $q\bar{q}$-potentials from the $A_0 = 0$ Feynman kernel
    - partially traced
    - homogeneous boundary conditions
    - character filtering
- Outlook
  - Numerical simulations are under way
    - See the poster by Guerrieri, Petrarca, Rubeo, Testa

Thank you for your Attention