On one-loop corrections to matching conditions of Lattice HQET including $1/m_b$ terms

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Motivations

Lattice calculations enter in two estimations of $V_{ub}$

For processes with $b \to u$ transitions $\Gamma \sim |V_{ub}|^2$:
- exclusive semi-leptonic $B \to \pi l \nu$ (involves $f_+(q^2)$)
- exclusive leptonic $B \to \tau \nu$ (involves $f_B$)

$\Rightarrow$ high precision lattice calculations needed.

Hierarchy of scales

- $L > 4/m_\pi \approx 6\ \text{fm}$ to suppress finite-size effects for light quarks
- $a < (2m_B)^{-1}$ to control discretization errors for the heavy quark

$\Rightarrow$ solution: HQET on the lattice.
HQET at order \( \Lambda_{QCD}/m_b \)

\[
\mathcal{L}_{HQET} = \mathcal{L}_{stat} + \mathcal{O}(1) \omega_{\text{kin}} \mathcal{L}_{\text{kin}} + \mathcal{O}(1/m) \omega_{\text{spin}} \mathcal{L}_{\text{spin}}
\]

with \( \mathcal{L}_{stat} = \bar{\psi}_h D_0 \psi_h \).

HQET operators: Example of \( A_0 \)

\[
Z_{A_0}^{QCD} (A_i^{QCD})_0 = Z_{A_0}^{QCD} \left[ \bar{\psi}_I \gamma_0 \gamma_5 \psi_b \right]
\]

\[
\downarrow
\]

\[
Z_{A_0}^{HQET} (A_i^{HQET})_0 = Z_{A_0}^{HQET} \left[ \bar{\psi}_I \gamma_0 \gamma_5 \psi_h + a c_{A_{0,1}} \bar{\psi}_I \gamma_5 \psi^k (\nabla^S_k - \bar{\nabla}^S_k) \psi_h + a c_{A_{0,2}} \bar{\psi}_I \gamma_5 \psi^k (\nabla^S_k + \bar{\nabla}^S_k) \psi_h + \ldots \right]
\]

\[\Rightarrow\] in total 19 HQET parameters are needed.
Finite volume scheme

In order to match HQET and QCD:

- we use a set of observables \( \{\phi_i\} \) defined in a finite volume,
- we evaluate them in QCD and HQET to obtain \( \{\phi_i^{\text{QCD}}(L, z = mL, a)\} \) and \( \{\phi_i^{\text{HQET}}(L, a)\} \),
- by equating the corresponding quantities

\[
\phi_i^{\text{QCD}}(L, z = mL, a = 0) \overset{!}{=} \phi_i^{\text{HQET}}(L, a) = \eta(L, a)_i + \varphi(L, a)_{ij} \omega_j(z, a)
\]

we can extract the HQET parameters \( \omega_j(z, a) \).

\[\Rightarrow\] non-perturbative definition (matching) and evolution of HQET parameters
One-loop lattice perturbation theory for HQET

Why?

Any HQET quantity is a truncated expansion in $1/z$ of the corresponding QCD quantity, hence one would like to make sure that matching doesn’t introduce artificially large $1/z^2$ corrections

- already checked at tree-level in $\bar{g}^2$
- in this work: verification at one-loop

How?

PASTOR [written by D. Hesse]: automatic tool for generation and calculation of lattice Feynman diagrams in the Schrödinger functional framework.

input: discretized action, correlation function, $L/a$, $z = \bar{m}L$, $a$

output: Feynman rules, Feynman diagrams, numerical contributions of each diagram
Schrödinger functional correlation functions

- boundary-to-boundary: \( F_1(\theta) = \sum_{u,v,y,z} \langle \bar{\zeta}'(u) \gamma_5 \zeta_h(v) \bar{\zeta}_h(y) \gamma_5 \zeta_l(z) \rangle \)

- current insertions: \( f_{A_0}(\theta, x_0) = \sum_{u,v} \langle \bar{\zeta}_h(u) \gamma_5 \zeta_l(v) A_0(x_0) \rangle \)

Additional parameters: \( \psi_h(x + L \hat{k}) = e^{i\theta_k} \psi_h(x) \), \( \psi_l(x + L \hat{k}) = e^{i\theta_k} \psi_l(x) \).

Observable for \( c_{A_0,2} \)

\[
\phi_{QCD}^5 = \log \frac{f_{A_0}(\theta^i_1 = \theta_1, \theta^i_h = \theta_2)}{f_{A_0}(\theta^i_1 = \theta_1, \theta^i_h = \theta_3)}
\]
Continuum extrapolations of $\phi_{QCD}^5$ at one loop

$\phi_{QCD}^5(a,z)$ at one loop

Naive expectation: tree-level $\sim 1/z \rightarrow$ one-loop $\sim \frac{1}{4\pi} \frac{1}{z}$. 

$z=5$ 
$z=7$ 
$z=9$ 
$z=11$
The generic structure of a matching condition for an observable which doesn’t need to be renormalized is
\[
\phi^{\text{QCD}}(z, a = 0) = \phi^{\text{stat}} + \sum_t \omega_t(z) \phi_t^{1/m} + \mathcal{O}(1/z^2).
\]

The \(z\)-dependence of the one-loop correction can be studied using
\((\hat{\omega}_t = \bar{m}\omega_t, \hat{\phi}_t = L\phi_t)\)

\[
R = \frac{\phi_{\text{QCD}}^{(1)}(\theta, z) - \phi^{\text{stat}}_{\text{QCD}}^{(1)}(\theta)}{\phi_{\text{QCD}}^{(0)}(\theta, z) - \phi^{\text{stat}}_{\text{QCD}}^{(0)}(\theta)} = \frac{\sum_t \hat{\omega}_t^{(0)}(z) \hat{\phi}_t^{(1)}(\theta)}{\sum_t \hat{\omega}_t^{(0)}(z) \hat{\phi}_t^{(0)}(\theta)} + \frac{\sum_t \hat{\omega}_t^{(1)}(z) \hat{\phi}_t^{(0)}(\theta)}{\sum_t \hat{\omega}_t^{(0)}(z) \hat{\phi}_t^{(0)}(\theta)} = \alpha(\theta) + \gamma(\theta) \log(z) + \mathcal{O}(1/z)
\]

When \(R\) is plotted on a linear-log plot, the ratio \(R\) measures:
- \(1/z^2\) corrections: deviations from a linear behaviour,
- coefficient of the subleading logarithm: slope of the data.
**Results**

$\phi_{QCD}^5(z)$ at one-loop

<table>
<thead>
<tr>
<th>$1/z$</th>
<th>0.0</th>
<th>0.05</th>
<th>0.1</th>
<th>0.15</th>
<th>0.2</th>
<th>0.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio $R_5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.0/0.5$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$0.0/1.0$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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<td>0.0</td>
</tr>
<tr>
<td>$0.5/1.0$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

(a) Fitting ansatz:

$f(z) = \beta_0/z + \beta_1 \log z/z$.

(b) Fitting ansätze:

$f(z) = \alpha + \gamma \log z$ and $f'(z) = \alpha' + \gamma' \log z + \delta'/z$.

Conclusion: $\frac{f(4)-f'(4)}{f(4)} \sim 0.003$. 
The observable used is

\[ \phi^\text{QCD}_6 = \log \frac{f_{A_0}(\theta, \theta)}{\sqrt{F_1(\theta)}} \]

Matching condition

We match the renormalized observables at static order

\[ (\phi^\text{QCD}_6)^\text{ren}(L, z) = (\phi^\text{stat}_6)^\text{ren}(L, z) + \log C^\text{match}_{A_0} (\bar{g}^2(z)) + \mathcal{O}(1/z) \]

and expand both sides in the coupling \( \bar{g}^2(z) \)

\[ (\phi^\text{QCD}_6)^{(0)}(z) = (\phi^\text{stat}_6)^{(0)} + \mathcal{O}(1/z), \]

\[ (\phi^\text{QCD}_6)^{(\text{ren}, 1)}(z) = (\phi^\text{stat}_6)^{(\text{bare}, 1)} - \gamma_0 \log(a \bar{m}) + B_{A_0} + \mathcal{O}(1/z). \]
Subtracting $\gamma_0 \log z$ from both sides yields

$\left( \phi_6^{QCD} \right)^{(\text{ren},1)}(z) - \gamma_0 \log z = \left( \phi_6^{\text{stat}} \right)^{(\text{bare},1)} - \gamma_0 \log \left( \frac{a}{L} \right) + B_{A_0} + \mathcal{O}(1/z)$

$= \left( \phi_6^{\text{stat}} \right)^{(\text{ren},1)} + \mathcal{O}(1/z)$.

In order to make visible the $1/m_b^2$ corrections we plot the quantity $Q$ defined as

$Q = z \left[ \left( \phi_6^{QCD} \right)^{(\text{ren},1)}(z) - \gamma_0 \log z \right] - z \left[ \left( \phi_6^{\text{stat}} \right)^{(\text{ren},1)} \right]$

$= z \left[ \mathcal{O}(1/z) + \mathcal{O}(1/z^2) \right] = \alpha_0 + \alpha_1 \log(z) + \mathcal{O}(1/z)$

When $Q$ is plotted on a linear-log plot:

- the $1/z^2$ corrections are seen as deviations from a linear behaviour
- the coefficient of the subleading logarithm as the slope of the data
(a) Fitting ansatz: 
\[ f(z) = \beta_0 + \beta_1/z + \beta_2 \log z/z. \]

(b) Fitting ansätze: 
\[ f(z) = \alpha_0 + \alpha_1 \log z \]
and \[ f'(z) = \alpha'_0 + \alpha'_1 \log z + \alpha'_2/z. \]

Conclusion: 
\[ \frac{f(4) - f'(4)}{f(4)} \sim 0.01. \]
Conclusions

- lattice HQET is a prototype of an effective theory where one can perform a non-perturbative matching.

- We have a setup for checking contamination with $1/z^2$ terms of the matching conditions.

- We can further optimize the choice of kinematics for the matching conditions.

- We presented results for 2 generic matching conditions; results for all 19 matching conditions fixing $\mathcal{L}_{HQET}$, vector and axial currents at order $1/m_b$ are ready.