Extensions of the HAL QCD approach to inelastic and multi-particle scatterings in lattice QCD

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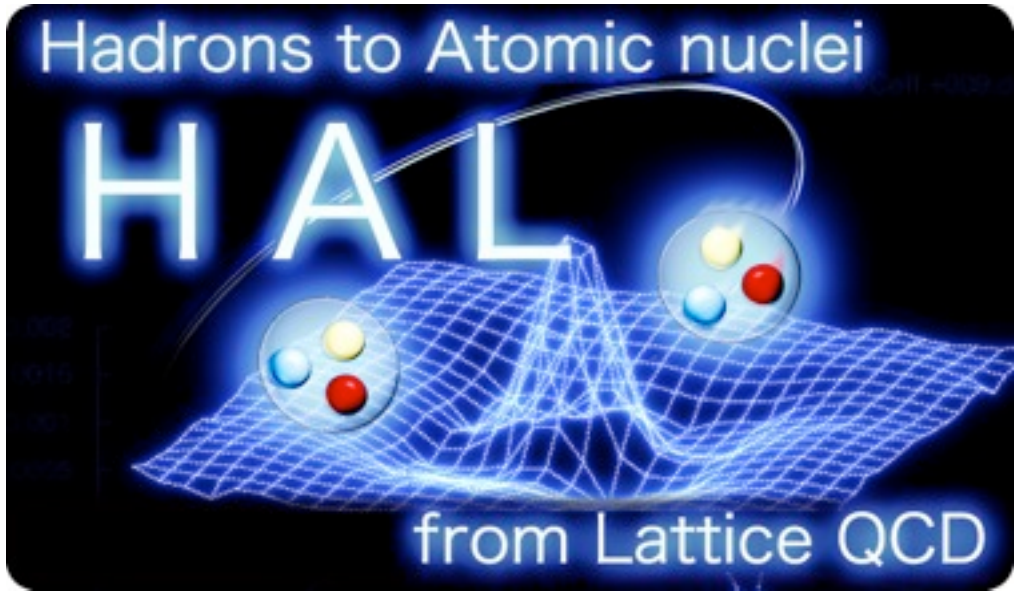

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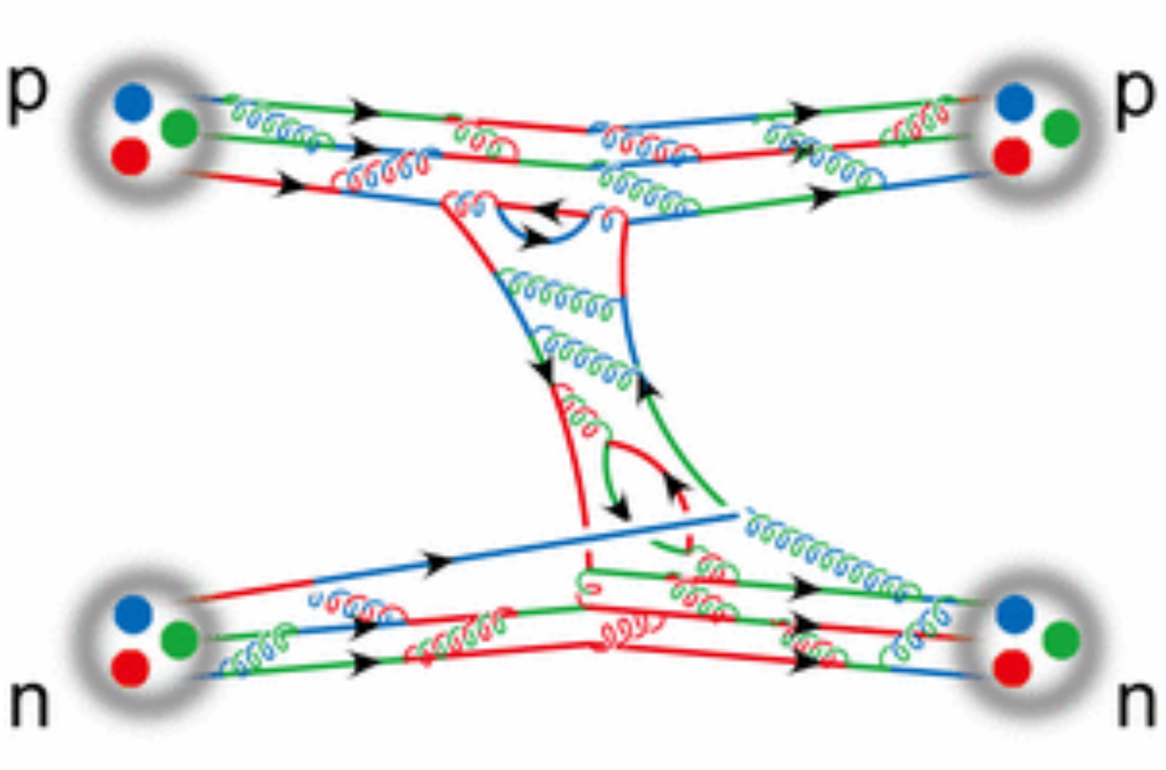
1. Introduction

HAL QCD approach to Nuclear Force



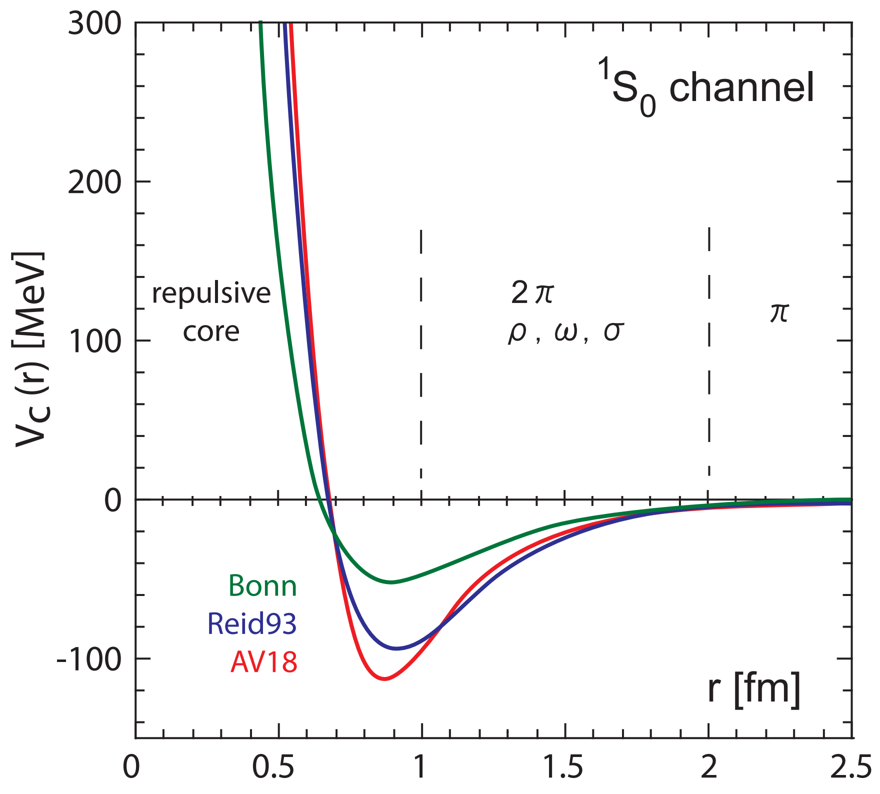
Potentials in QCD ?

"Potentials" themselves can NOT be directly measured.

scheme dependent, ambiguities in inelastic region   cf. running coupling in QCD

potentials, but not unique

experimental data of scattering phase shifts



"Potentials" are still useful tools to extract observables such as scattering phase shift.

One may adopt a convenient definition of potentials as long as they reproduce correct physics of QCD.

HAL QCD strategy

Aoki, Hatsuda & Ishii, PTP123(2010)89.

Step 1

define (Equal-time) Nambu-Bethe-Salpeter (NBS) Wave function

$$\varphi_{\mathbf{k}}(\mathbf{r}) = \langle 0 | N(\mathbf{x}+\mathbf{r},0) N(\mathbf{x},0) | NN, W_k \rangle$$

$W_k = 2\sqrt{\mathbf{k}^2 + m_N^2}$ energy

$N(x) = \varepsilon_{abc} q^a(x) q^b(x) q^c(x)$: local operator

Key Property 1

Lin et al., 2001; CP-PACS, 2004/2005

$$\varphi_{\mathbf{k}}(\mathbf{r}) \simeq \sum_{l,m} C_l \frac{\sin(kr - l\pi/2 + \delta_l(k))}{kr} Y_{ml}(\Omega_{\mathbf{r}})$$

$r = |\mathbf{r}| \to \infty$

$\delta_l(k)$ scattering phase shift (phase of the S-matrix by unitarity) in QCD.

Step 2

define non-local but energy-independent "potential" as

$$[\epsilon_k - H_0]\,\varphi_{\mathbf{k}}(\mathbf{x}) = \int d^3y\, \underline{U(\mathbf{x},\mathbf{y})}\varphi_{\mathbf{k}}(\mathbf{y})$$

$\mu = m_N/2$

reduced mass

$$\epsilon_k = \frac{\mathbf{k}^2}{2\mu} \qquad H_0 = \frac{-\nabla^2}{2\mu}$$

Key Property 2

A non-local but energy-independent potential exists.

Proof

$$U(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k},\mathbf{k}'}^{W_k, W_{k'} \le W_{\rm th}} [\epsilon_k - H_0]\,\varphi_{\mathbf{k}}(\mathbf{x})\eta^{-1}_{\mathbf{k},\mathbf{k}'}\varphi^{\dagger}_{\mathbf{k}'}(\mathbf{y})$$

inner product

$\eta^{-1}_{\mathbf{k},\mathbf{k}'}$: inverse of $\eta_{\mathbf{k},\mathbf{k}'} = (\varphi_{\mathbf{k}}, \varphi_{\mathbf{k}'})$

For ${}^{\forall}W_{\mathbf{p}} < W_{\rm th} = 2m_N + m_\pi$ (threshold energy)

$$\int d^3y\, U(\mathbf{x},\mathbf{y})\phi_{\mathbf{p}}(\mathbf{y}) = \sum_{\mathbf{k},\mathbf{k}'} [\epsilon_k - H_0]\,\varphi_{\mathbf{k}}(x)\eta^{-1}_{\mathbf{k},\mathbf{k}'}\eta_{\mathbf{k}',\mathbf{p}} = [\epsilon_p - H_0]\,\varphi_{\mathbf{p}}(x)$$

Step 3 expand the non-local potential in terms of derivative as

$$U(\mathbf{x},\mathbf{y}) = V(\mathbf{x},\nabla)\delta^3(\mathbf{x}-\mathbf{y})$$

Step 4 extract the local potential. At LO, for example, we simply have

$$V_{\mathrm{LO}}(\mathbf{x}) = \frac{[\epsilon_k - H_0]\varphi_{\mathbf{k}}(\mathbf{x})}{\varphi_{\mathbf{k}}(\mathbf{x})}$$

Step 5 solve the Schroedinger Eq. in the infinite volume with this potential.

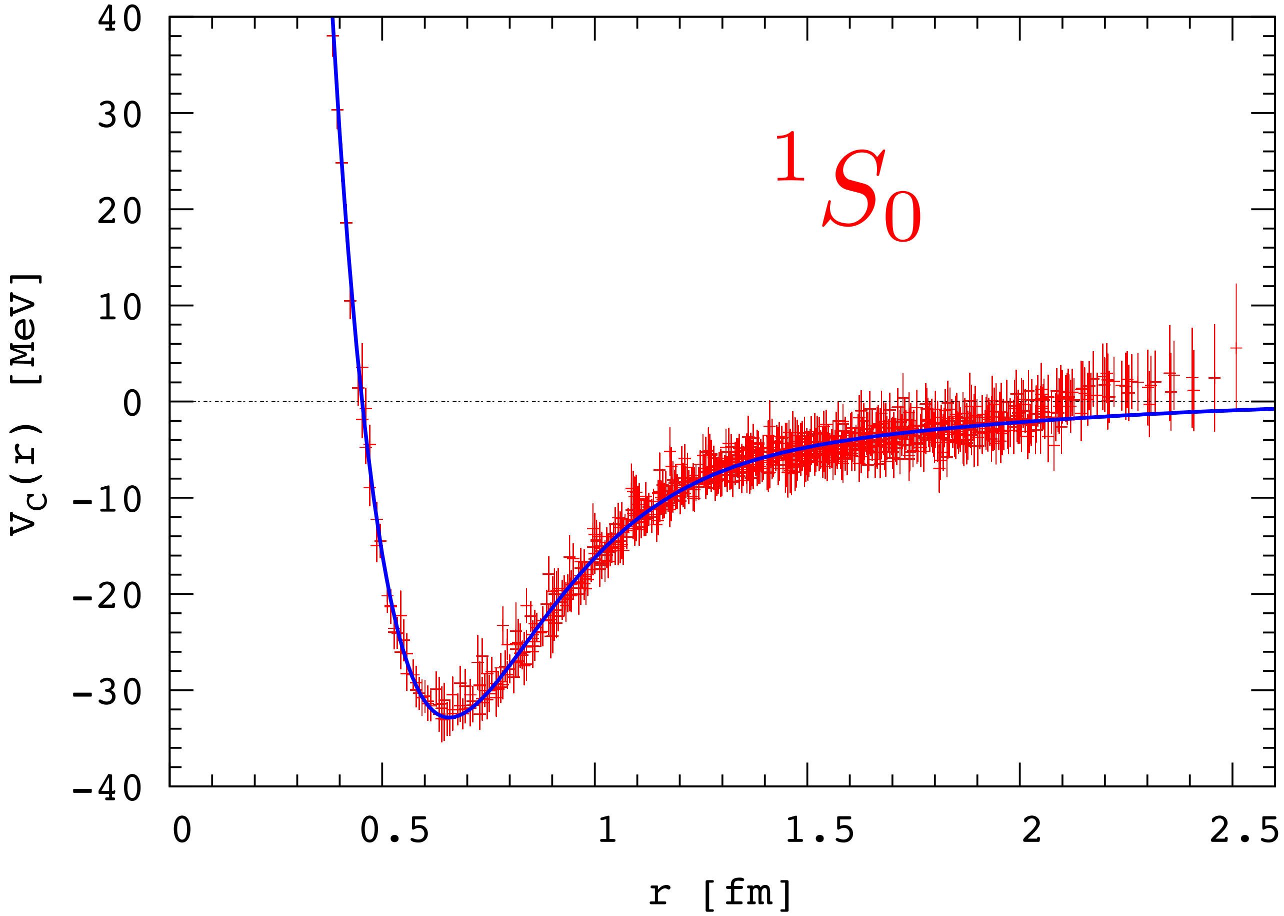
→ phase shifts and binding energy below inelastic threshold

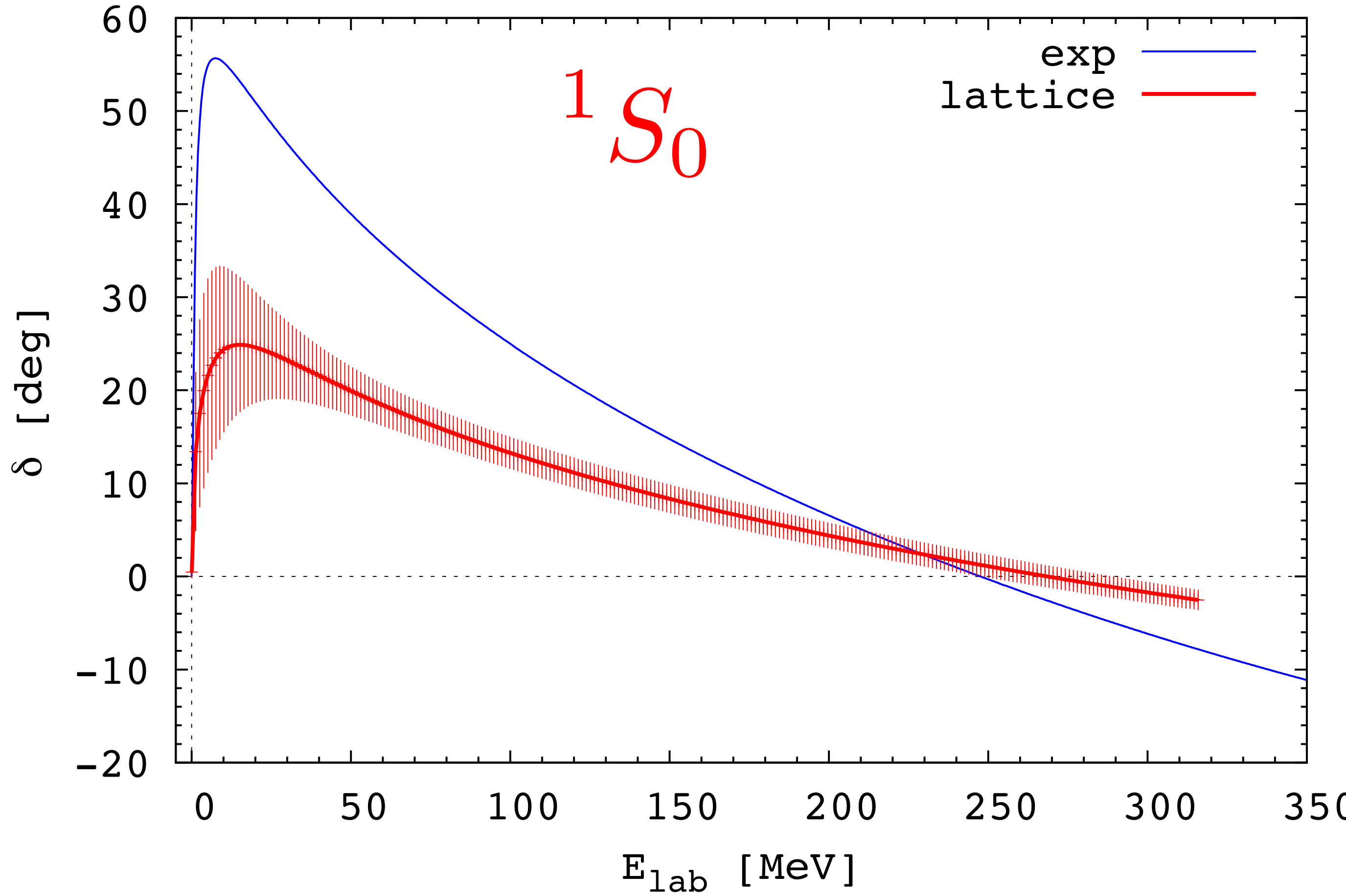
Example

2+1 flavor QCD a=0.09fm, L=2.9fm $m_\pi \simeq 700$ MeV

Ishii *et al.* (HALQCD), PLB712(2012) 437.

NN potential → phase shift





Qualitative features of NN potential are reproduced.

→ It has a reasonable shape. The strength is weaker due to the heavier quark mass.

Need calculations at physical quark mass.

In order to extend the HAL QCD method to inelastic and/or multi-particle scatterings, we have to show

Key Property 1

Asymptotic behaviors of NBS wave functions for more than 2 particles

Key Property 2

Existence of energy independent potentials above inelastic thresholds

2. NBS wave functions for multi-particles

Key Property 1

For simplicity,
(1) we consider scalar particles with “flavors”
(2) we assume no bound state exists.

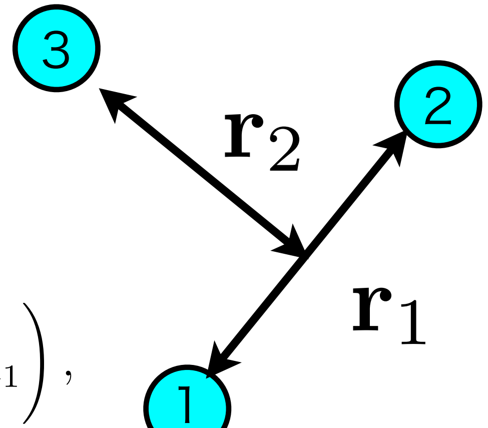
Sinya Aoki, et al., arXiv.1303.2210 [hep-lat], to appear in PRD

Unitarity constraint

$$T^\dagger - T = iT^\dagger T.$$

parametrization

$$_0\langle[\boldsymbol{p}^A]_n|T|[\boldsymbol{p}^B]_n\rangle_0 \equiv \delta(E^A - E^B)\delta^{(3)}(\boldsymbol{P}^A - \boldsymbol{P}^B)T([\boldsymbol{q}^A]_n, [\boldsymbol{q}^B]_n)$$



(modified) Jacobi coordinates and momenta

$$\boldsymbol{r}_k = \sqrt{\frac{k}{k+1}} \times \boldsymbol{r}^J_k, \qquad \boldsymbol{q}_k = \sqrt{\frac{k+1}{k}} \times \boldsymbol{q}^J_k \qquad \boldsymbol{r}^J_k = \frac{1}{k}\sum_{i=1}^{k}\boldsymbol{x}_i - \boldsymbol{x}_{k+1}, \quad \boldsymbol{q}^J_k = \frac{k}{k+1}\left(\frac{1}{k}\sum_{i=1}^{k}\boldsymbol{p}_i - \boldsymbol{p}_{k+1}\right),$$

$$\begin{aligned} T([\boldsymbol{q}^A]_n, [\boldsymbol{q}^B]_n) &\equiv T(\boldsymbol{Q}_A, \boldsymbol{Q}_B) \\ &= \sum_{[L],[K]} T_{[L][K]}(Q_A, Q_B) Y_{[L]}(\Omega_{\boldsymbol{Q}_A})\overline{Y_{[K]}(\Omega_{\boldsymbol{Q}_B})} \end{aligned}$$

$$\boldsymbol{Q}_X = (\boldsymbol{q}^X{}_1, \boldsymbol{q}^X{}_2, \cdots, \boldsymbol{q}^X{}_{n-1})$$

momentum in D=3(n-1) dim.

hyper-spherical harmonic function

$$\hat{L}^2 Y_{[L]}(\Omega_{\boldsymbol{s}}) = L(L+D-2)Y_{[L]}(\Omega_{\boldsymbol{s}})$$

solution to the unitarity constraint with non-relativistic approximation

$$T_{[L][K]}(Q,Q) = \sum_{[N]} U_{[L][N]}(Q) T_{[N]}(Q) U^{\dagger}_{[N][K]}(Q),$$

↑

$$T_{[L]}(Q) = -\frac{2n^{3/2}}{mQ^{3n-5}} e^{i\delta_{[L]}(Q)} \sin\delta_{[L]}(Q),$$

"phase shift" $\delta_{[L]}(Q)$

Lippmann-Schwinger equation in QFT

$$|\alpha\rangle_{\rm in} = |\alpha\rangle_0 + \int d\beta \frac{|\beta\rangle_0 T_{\beta\alpha}}{E_\alpha - E_\beta + i\varepsilon},$$

$$T_{\beta\alpha} = {}_0\langle\beta|V|\alpha\rangle_{\rm in},$$

off-shell

$$ {}_0\langle\beta|T|\alpha\rangle_0 = 2\pi\delta(E_\alpha - E_\beta) T_{\alpha\beta}.$$

on-shell off-shell

$$(H_0 + V)|\alpha\rangle_{\rm in} = E_\alpha|\alpha\rangle_{\rm in},$$ full

$$H_0|\alpha\rangle_0 = E_\alpha|\alpha\rangle_0.$$ free

NBS wave functions

n-scalar fields with different flavors

$$\Psi_\alpha^n([\boldsymbol{x}]) = {}_{\rm in}\langle 0|\varphi^n([\boldsymbol{x}],0)|\alpha\rangle_{\rm in},$$

$$\varphi^n([\boldsymbol{x}],t) = T\{\prod_{i=1}^n \varphi_i(\boldsymbol{x}_i,t)\},$$

$$\Psi_\alpha^n([\boldsymbol{x}]) = \frac{1}{Z_\alpha}{}_0\langle 0|\varphi^n([\boldsymbol{x}],0)|\alpha\rangle_0 + \int d\beta \frac{1}{Z_\beta}\frac{{}_0\langle 0|\varphi^n([\boldsymbol{x}],0)|\beta\rangle_0 T_{\beta\alpha}}{E_\alpha - E_\beta + i\varepsilon}.$$

$$ {}_0\langle 0|\varphi^n([\boldsymbol{x}],0)|[\boldsymbol{k}]_n\rangle_0 = \left(\frac{1}{\sqrt{(2\pi)^3}}\right)^n \prod_{i=1}^n \frac{1}{\sqrt{2E_{k_i}}} e^{i\boldsymbol{k}_i\boldsymbol{x}_i}$$

D-dimensional hyper-coordinates

$$\Psi^n(\boldsymbol{R}, \boldsymbol{Q}_A) = C\left[e^{i\boldsymbol{Q}_A\cdot R} + \frac{2m}{2\pi n^{3/2}}\int d^DQ\,\frac{e^{i\boldsymbol{Q}\cdot R}}{Q_A^2 - Q^2 + i\varepsilon}T(\boldsymbol{Q}, \boldsymbol{Q}_A)\right]$$

Expansion in terms of hyper-spherical harmonic function

$$e^{i\boldsymbol{Q}\cdot\boldsymbol{R}} = (D-2)!!\frac{2\pi^{D/2}}{\Gamma(D/2)}\sum_{[L]} i^L\, \underline{j_L^D(QR)}\, Y_{[L]}(\Omega_{\boldsymbol{R}})\,\overline{Y_{[L]}(\Omega_{\boldsymbol{Q}})},$$

hyper-spherical Bessel function

$$\Psi^n(\boldsymbol{R}, \boldsymbol{Q}_A) = \sum_{[L],[K]} \Psi^n_{[L],[K]}(R, Q_A) Y_{[L]}(\Omega_{\boldsymbol{R}})\overline{Y_{[K]}(\Omega_{\boldsymbol{Q}_A})},$$

Asymptotic behavior of NBS wave functions $R \to \infty$

$$\Psi^n_{[L],[K]}(R, Q_A) \simeq Ci^L\frac{(2\pi)^{D/2}}{(Q_AR)^{\frac{D-1}{2}}}\sum_{[N]} U_{[L][N]}(Q_A)e^{i\delta_{[N]}(Q_A)}U^\dagger_{[N][K]}(Q_A)$$

$$\times\sqrt{\frac{2}{\pi}}\;\underline{\sin\left(Q_AR - \Delta_L + \delta_{[N]}(Q_A)\right)}$$

$$\Delta_L = \frac{2L_D - 1}{4}\pi.$$

scattering wave with “phase shift” !

3. Energy-independent potential above inelastic thresholds

Key Property 2

Sinya Aoki, et al., Phys. Rev. D87 (2013) 034512.

Let us consider $NN \to NN, NN\pi$

energy $W \in \Delta_1$

$W_{\rm th}^2 = 2m_N + 2m_\pi$

$\Delta_1 = [W_{\rm th}^1, W_{\rm th}^2)$

$W_{\rm th}^1 = 2m_N + m_\pi$

$\Delta_0 = [W_{\rm th}^0, W_{\rm th}^1)$

$W_{\rm th}^0 = 2m_N$

2 operators		2 states
$N(x)N(y)$	$\times$	$\vert NN, W, c_0\rangle$
$N(x)N(y)\pi(z)$		$\vert NN+\pi, W, c_1\rangle$

other quantum numbers (c_0, c_1)

4 NBS wave functions

$$
\begin{aligned}
Z_N \varphi^{00}_{W,c_0}(\boldsymbol{x}_0) &= \langle 0 | T\{N(\boldsymbol{x},0)N(\boldsymbol{x}+\boldsymbol{x}_0,0)\} | NN, W, c_0\rangle_{\rm in},\\
Z_N Z_\pi^{1/2} \varphi^{10}_{W,c_0}(\boldsymbol{x}_0,\boldsymbol{x}_1) &= \langle 0 | T\{N(\boldsymbol{x},0)N(\boldsymbol{x}+\boldsymbol{x}_0,0)\pi(\boldsymbol{x}+\boldsymbol{x}_1,0)\} | NN, W, c_0\rangle_{\rm in},\\
Z_N \varphi^{01}_{W,c_1}(\boldsymbol{x}_0) &= \langle 0 | T\{N(\boldsymbol{x},0)N(\boldsymbol{x}+\boldsymbol{x}_0,0)\} | NN+\pi, W, c_1\rangle_{\rm in},\\
Z_N Z_\pi^{1/2} \varphi^{11}_{W,c_1}(\boldsymbol{x}_0,\boldsymbol{x}_1) &= \langle 0 | T\{N(\boldsymbol{x},0)N(\boldsymbol{x}+\boldsymbol{x}_0,0)\pi(\boldsymbol{x}+\boldsymbol{x}_1,0)\} | NN+\pi, W, c_1\rangle_{\rm in},
\end{aligned}
$$

$\varphi^{ij}_{W,c_j}([\mathbf{x}]_i)$ $i(j)$: number of π's in the operator(state) $[\boldsymbol{x}]_0 = \boldsymbol{x}_0$ $[\boldsymbol{x}]_1 = \boldsymbol{x}_0, \boldsymbol{x}_1$

coupled channel equation

$$(E_W^k - H_0^k)\varphi_{W,c_i}^{ki} = \sum_{l=0,1} \int \prod_{n=0}^{l} d^3y_n \, U^{kl}([\boldsymbol{x}]_k, [\boldsymbol{y}]_l)\varphi_{W,c_i}^{li}([\boldsymbol{y}]_l), \quad k, i \in (0,1)$$

$$E_W^n = \frac{\boldsymbol{p}_1^2}{2m_N} + \frac{\boldsymbol{p}_2^2}{2m_N} + \sum_{i=1}^{n} \frac{\boldsymbol{k}_i^2}{2m_\pi}$$

$$W = \sqrt{m_N^2 + \boldsymbol{p}_1^2} + \sqrt{m_N^2 + \boldsymbol{p}_2^2} + \sum_{i=1}^{n} \sqrt{m_\pi^2 + \boldsymbol{k}_i^2}$$

kinetic energy

non-relativistic approx. for n=1

total energy

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 + \textstyle\sum_{i=1}^{n} \boldsymbol{k}_i = 0.$$

Proof of existence for U

Define a vector of NBS wave functions as

state index

$$\varphi_{W,c_i}^{i} \equiv \left(\varphi_{W,c_i}^{0i}([\boldsymbol{x}]_0), \varphi_{W,c_i}^{1i}([\boldsymbol{x}]_1)\right)^T, \quad i = 0, 1, \qquad W \in \Delta_1$$

$$\varphi_{W,c_0}^{0} \equiv \left(\varphi_{W,c_0}^{00}([\boldsymbol{x}]_0), \varphi_{W,c_0}^{10}([\boldsymbol{x}]_1)\right)^T, \qquad W \in \Delta_0$$

Norm kernel

$$\mathcal{N}^{ij}_{W_1 c_i, W_2 d_j} = \left(\varphi^i_{W_1,c_i}, \varphi^j_{W_2,d_j}\right) \equiv \sum_{k=0,1} \int \prod_{l=0}^{k} d^3x_l \, \overline{\varphi^{ki}_{W_1,c_i}([\boldsymbol{x}]_k)} \varphi^{kj}_{W_2,d_j}([\boldsymbol{x}]_k).$$

Inverse

$$\sum_{W \in \Delta_0 + \Delta_1} \sum_{h \in I(W), e_h} (\mathcal{N}^{-1})^{ih}_{W_1 c_i, W e_h} \mathcal{N}^{hj}_{W e_h, W_2 d_j} = \delta^{ij} \delta_{W_1, W_2} \delta_{c_i, d_j}$$

Structure

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}^{00}(\Delta_0, \Delta_0), & \mathcal{N}^{00}(\Delta_0, \Delta_1), & \mathcal{N}^{01}(\Delta_0, \Delta_1) \\ \mathcal{N}^{00}(\Delta_1, \Delta_0), & \mathcal{N}^{00}(\Delta_1, \Delta_1), & \mathcal{N}^{01}(\Delta_1, \Delta_1) \\ \mathcal{N}^{10}\underline{(\Delta_1, \Delta_0)}, & \mathcal{N}^{10}(\Delta_1, \Delta_1), & \mathcal{N}^{\underline{11}}(\Delta_1, \Delta_1) \end{pmatrix}$$

energy state

bra, ket

$$\langle [\boldsymbol{x}]_k | \varphi^i_{W,c_i} \rangle = \varphi^{ki}_{W,c_i}([\boldsymbol{x}]_k),$$

$$\langle \psi^i_{W,c_i} | [\boldsymbol{x}]_k \rangle = \sum_{W_1 \in \Delta_0 \cup \Delta_1} \sum_{j \in I(W_1), d_j} (\mathcal{N}^{-1})^{ij}_{W c_i, W_1 d_j} \overline{\varphi^{kj}_{W_1,d_j}([\boldsymbol{x}]_k)}$$

orthogonality

$$\langle \psi^i_{W_1,c_i} | \varphi^j_{W_2,d_j} \rangle = \sum_{k=0,1} \int \prod_{l=0}^{k} d^3x_l \, \langle \psi^i_{W_1,c_i} | [\boldsymbol{x}]_k \rangle \langle [\boldsymbol{x}]_k | \varphi^j_{W_2,d_j} \rangle = (\mathcal{N}^{-1} \cdot \mathcal{N})^{ij}_{W_1 c_i, W_2 d_j}$$

$$= \delta^{ij} \delta_{W_1,W_2} \delta_{c_i,d_j}.$$

Abstract operators

$$\langle [\boldsymbol{x}]_k | (E_W - H_0) | [\boldsymbol{y}]_l \rangle \equiv \delta_{kl} (E_W^k - H_0^k) \prod_{n=0}^{k} \delta^{(3)} (\boldsymbol{x}_n - \boldsymbol{y}_n)$$

$$\langle [\boldsymbol{x}]_k | U | [\boldsymbol{y}]_l \rangle \equiv U^{kl}([\boldsymbol{x}]_k, [\boldsymbol{y}]_l),$$

Abstract coupled channel equation

$$(E_W - H_0)|\varphi_{W,c_i}^i\rangle = U|\varphi_{W,c_i}^i\rangle.$$

construction of non-local coupled channel potential

$$U = \sum_{W \in \Delta_0 \cup \Delta_1} \sum_{i \in I(W)} \sum_{c_i} (E_W - H_0)|\varphi_{W,c_i}^i\rangle\langle\psi_{W,c_i}^i|,$$

$\because$

$$U|\varphi_{W,c_i}^i\rangle = \sum_{W_1 \in \Delta_0 \cup \Delta_1} \sum_{j \in I(W_1)} \sum_{d_j} (E_W - H_0)|\varphi_{W_1,d_j}^j\rangle\langle\psi_{W_1,d_j}^j|\varphi_{W,c_i}^i\rangle = (E_W - H_0)|\varphi_{W,c_i}^i\rangle$$

Energy independent (coupled channel) potential exists above the inelastic threshold.

The construction of U can easily be generalized to $NN + n\pi \longrightarrow NN + k\pi$

or to $\Lambda\Lambda \longrightarrow \Lambda\Lambda, N\Xi, \Sigma\Sigma$

QCD at given energy

$W_{\rm total}$

Quantum mechanics with coupled channel potentials for stable particles

$N, \bar{N}, \pi, \cdots$

$\Delta, \rho, \cdots$ resonance $N\pi, \pi\pi, \cdots$

deuteron, H,... bound-state ? $NN, \Lambda\Lambda, \cdots$

$D, H, \cdots$

4. Conclusion

- HAL QCD approach is a promising method to extract hadronic interactions in lattice QCD.
  - LS force(K. Murano, 10C), Antisymmetric LS(N. Ishii, 10C)
  - D-D,D-K(Y. Ikeda, 6G), Omega-Omega(M. Yamada, 10C)
  - Comparison of HAL and Luescher: NN(B. Charon,3G), pipi(T. Kurth, 3G)
- Extensions of the HAL QCD method to inelastic/multi-particle scatterings
  - Asymptotic behavior of the NBS wave functions
  - Existence of non-local but energy-independent coupled channel potentials
  - 3 nucleon force (T.Doi, 1F), coupled channel(K. Sasaki, 10C)
- A treatment of bound-states ?