Relativistic, model-independent
3-particle quantization condition:
(2) Threshold expansion

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Based on unpublished work with Max Hansen
Relativistic, model-independent 3-particle quantization condition:
(2) Explication & utility

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Outline

- New features compared to 2-particle case
- How to truncate and make practical
- Important check: threshold expansion compared to results from NR EFT
- Closing comments
New features compared to 2-particle case
3-particle quant. condition

- For given $P$, adjust total energy $E$ until:

$$\det[F_{\text{three}}^{-1} + iM_{df,3\rightarrow 3}] = 0$$

$$F_{\text{three}} \equiv \frac{1}{2\omega L^3} \left[ (2/3) iF - \frac{1}{[iF]^{-1} - [1 - iMiG]^{-1} iM} \right]$$

- Entries are infinite dim. matrices with “indices”

[“spectator” momentum: $k=2\pi n/L$] x [2-particle CM angular momentum: $l,m$]

- $M=M_{2\rightarrow 2}$ and $M_{df,3\rightarrow 3}$ are on-shell amplitudes (analytically continued if below threshold)

- $F$ and $G$ are kinematical, finite-volume factors
2-particle quant. condition

• For given \( \mathbf{P} \), adjust total energy \( E \) until:

\[
\det \left( F^{-1} + i\mathcal{M} \right) = 0
\]

Form of result given by [Kim, Sachrajda & SRS] ; equivalent to earlier results of [Luscher; Rummukainen & Gottlieb]

• Entries are infinite dim. matrices with “indices”

[2-particle CM angular momentum: \( l,m \)]

• \( \mathcal{M} = \mathcal{M}_{2 \rightarrow 2} \) is on-shell amplitude (analytically continued if below threshold)

• \( F \) is kinematical, finite-volume factor
Comparison

• Overall forms are (superficially) similar

• $F$ is essentially the same finite-volume kinematical factor in both cases (with trivial spectator momentum dependence in the 3-particle case)

• Differences for 3 particles:
  • Enlarged matrix index space
  • Need to introduce divergence-free $3 \rightarrow 3$ amplitude [see Max’s talk]
  • Presence of “switch factor” $G$
  • Necessarily includes subthreshold $2 \rightarrow 2$ scattering [see Max’s talk]
Enlarged index space

[2-particle CM angular momentum: \(l,m\)]

[\text{“spectator” momentum: } k=2\pi n/L \times 2\text{-particle CM angular momentum: } l,m\]

- Reflects larger on-shell phase space
- Finite volume restricts index space for 3 particles
- This restriction to quantized \(k\) essential to obtain the correct result if third particle is non-interacting [see Max’s talk]
Presence of switch-factor $G$

- Enters because of subtraction of divergent part of $3 \rightarrow 3$ amplitude

\[ i\mathcal{M}_{3\rightarrow3,\text{unsym}}^{(2)} \]

\[ i\mathcal{M}_{d,3\rightarrow3}^{(2, \text{unsym})} \equiv i\mathcal{M}_{3\rightarrow3}^{(2, \text{unsym})} - i\mathcal{M} \frac{i}{2\omega(E - 3\omega)} i\mathcal{M} \]

- Obtain $G$ when add back in subtracted part

\[ +i\mathcal{M} iG i\mathcal{M} \]

- Arises when switch from $2 \rightarrow 2$ scatterings of one pair to a different pair
  - Switches which particle is spectator in coordinate system
Relation to dimer approach

- Roles of F and G are almost symmetrical

Previous form:

\[ F_{\text{three}} \equiv \frac{if}{2\omega L^3} \left[ \frac{2}{3} - \frac{1}{1 - [1 - iMiG]^{-1}iMiF} \right] \]

May allow relation to dimer approach of [Briceno & Davoudi, arXiv:1212.3398] to be worked out

"Dimer form":

\[ -F_{\text{three}} \equiv \frac{if}{6\omega L^3} + \frac{1}{2\omega L^3} \frac{if}{1 - iDiG} \frac{1}{1 - iMiF} \frac{iM}{iDiF} \]

\[ iD \equiv \frac{1}{1 - iMiF} iM \]

\[ = iM + iMiFiM + iMiFiMiFiM + \ldots \]
How to truncate & make practical
Truncation in 2 particle case

\[ \det \left( F^{-1} + i\mathcal{M} \right) = 0 \]

- Entries are infinite dim. matrices with “indices” [CM angular momentum: \( l,m \)]

- If \( \mathcal{M} \) (which is diagonal in \( l,m \)) vanishes for \( l > l_{\text{max}} \) then can show that need only keep \( l \leq l_{\text{max}} \) in \( F \) (which is not diagonal) and so have finite matrix condition which can be inverted to find \( \mathcal{M}(E) \) from energy levels.
Truncation in 3 particle case

\[
\text{det}\left[F_{\text{three}}^{-1} + iM_{df,3\rightarrow3}\right] = 0
\]

\[
F_{\text{three}} \equiv \frac{1}{2\omega L^3} \left[ (2/3)iF - \frac{1}{[iF]^{-1} - [1 - iMiG]^{-1} iM} \right]
\]

\[
iF_{k,k'} = \delta_{k,k'} \frac{1}{2} \left[ \frac{1}{L^3} \sum_{\bar{a}} - \int_{\bar{a}} \right] \frac{i \cdot 4\pi Y(\hat{a}^*)Y^*(\hat{a}^*)}{2\omega a 2\omega_{P-k-a}(E - \omega_k - \omega_a - \omega_{P-k-a} + i\epsilon)}
\]

• Matrix “indices” are [Spectator mom. \(k=2\pi n/L\)] x [CM angular mom. \(l,m\)]

• For fixed \(E\) & \(P\), as \(|k|\) increases, remaining two-particle system drops below threshold, so \(F\) becomes exponentially suppressed (since sum and integral do not hit pole)

• Thus \(k\) index is naturally truncated (with, say, \(N\) terms required)

• \(l\) is truncated if both \(M\) and \(M_{df,3\rightarrow3}\) vanish for \(l > l_{\text{max}}\)
Truncation in 3 particle case

\[
\det\left[F_{\text{three}}^{-1} + iM_{df,3\rightarrow3}\right] = 0
\]

• Thus can truncate the quantization condition to that for an \([N(2l_{\text{max}}+1)]^2\) block

• Given prior knowledge of \(M\) (from 2 particle analysis) each energy level \(E_i\) of the 3 particle system gives information on \(M_{df,3\rightarrow3}\) at the corresponding 3-particle CM energy \(E_i^*\)

• Could proceed by parameterizing \(M_{df,3\rightarrow3}\) by a number of parameters (e.g. one!), in which case one would need at least that many levels at given energy to determine parameters

• Given \(M\) and \(M_{df,3\rightarrow3}\) one can reconstruct \(M_{3\rightarrow3}\)

\[
\begin{align*}
    iM_{df,3\rightarrow3} &\equiv iM_{3\rightarrow3} \\
    &= \left[iM \frac{i}{2\omega(E-3\omega)}iM + \int iM \frac{i}{2\omega(E-3\omega)} \frac{1}{2\omega}iM \frac{i}{2\omega(E-3\omega)}iM + \cdots \right]
\end{align*}
\]
Important check: threshold expansion
Threshold expansion

- Given complexity of derivation & new features of result, it is clearly important to check it to the extent possible

- Can do so for \( P = 0 \) and near threshold: \( E = 3m + \Delta E \), with \( \Delta E \sim 1/L^3 + \ldots \)

- In other words, study energy shift of three particles (almost) at rest

- Dominant effects \((L^{-3}, L^{-4}, L^{-5})\) involve 2-particle interactions, but 3-particle interaction enters at \( L^{-6} \)

- For large \( L \), particles are non-relativistic \((\Delta E \ll m)\) and can use NREFT methods

- This has been done previously by [Beane, Detmold & Savage, 0707.1670] and [Tan, 0709.2530]
NR EFT results

[Beane, Detmold & Savage, 0707.1670]

2 particles

\[
E_0(2, L) = \frac{4\pi a}{ML^3} \left\{ 1 - \left( \frac{a}{\pi L} \right) I + \left( \frac{a}{\pi L} \right)^2 [I^2 - J] \right. \\
+ \left( \frac{a}{\pi L} \right)^3 \left[ -I^3 + 3IJ - K \right] \right. \\
+ \frac{8\pi^2 a^3}{ML^6} r + \mathcal{O}(L^{-7}), \tag{11}
\]

3 particles

\[
E_0(3, L) = \frac{12\pi a}{ML^3} \left\{ 1 - \left( \frac{a}{\pi L} \right) I + \left( \frac{a}{\pi L} \right)^2 [I^2 + J] \right. \\
+ \left( \frac{a}{\pi L} \right)^3 \left[ -I^3 + IJ + 15K - 8(2Q + R) \right] \right. \\
+ \frac{64\pi a^4}{ML^6} (3\sqrt{3} - 4\pi) \log(\mu L) + \frac{24\pi^2 a^3}{ML^6} r \\
+ \frac{1}{L^6} \eta_3(\mu) + \mathcal{O}(L^{-7}), \tag{12}
\]

- 2-particle result agrees with [Luscher]
- Scattering length \( a \) is in nuclear physics convention
- \( r \) is effective range
- \( I, J, K \) are zeta-functions

- 3 particle result through \( L^{-4} \) is 3x(2-particle result) from number of pairs
- Not true at \( L^{-5}, L^{-6} \) where additional finite-volume functions \( Q, R \) enter
- \( \eta_3(\mu) \) is 3-particle contact potential, which requires renormalization
NR EFT results

[Beane, Detmold & Savage, 0707.1670]

2 particles

\[ E_0(2, L) = \frac{4\pi a}{ML^3} \left\{ 1 - \left( \frac{a}{\pi L} \right) I + \left( \frac{a}{\pi L} \right)^2 \left[ I^2 - \mathcal{J} \right] \right. \\
\left. + \left( \frac{a}{\pi L} \right)^3 \left[ -I^3 + 3I\mathcal{J} - \mathcal{K} \right] \right\} + \frac{8\pi^2 a^3}{ML^6} r + \mathcal{O}(L^{-7}), \quad (11) \]

3 particles

\[ E_0(3, L) = \frac{12\pi a}{ML^3} \left\{ 1 - \left( \frac{a}{\pi L} \right) I + \left( \frac{a}{\pi L} \right)^2 \left[ I^2 + \mathcal{J} \right] \right. \\
\left. + \left( \frac{a}{\pi L} \right)^3 \left[ -I^3 + I\mathcal{J} + 15\mathcal{K} - 8(Q + R) \right] \right\} + \frac{64\pi a^4}{ML^6} (3\sqrt{3} - 4\pi) \log(\mu L) + \frac{24\pi^2 a^3}{ML^6} r \\
\left. + \frac{1}{L^6} \eta_3(\mu) + \mathcal{O}(L^{-7}), \quad (12) \right. \]

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Tan has 36 instead of 24, but a different definition of \( \eta_3 \)
NR EFT results

\[ E_0(3, L) = \frac{12\pi a}{ML^3} \left[ 1 - \left( \frac{a}{\pi L} \right) I + \left( \frac{a}{\pi L} \right)^2 \left[ I^2 + J \right] \right. \]
\[ \left. + \left( \frac{a}{\pi L} \right)^3 \left[ -I^3 + IJ + 15K - 8(2Q + R) \right] \right] \]
\[ + \frac{64\pi a^4}{ML^6} (3\sqrt{3} - 4\pi) \log(\mu L) + \frac{24\pi^2 a^3}{ML^6} r \]
\[ + \frac{1}{L^6} \eta_3(\mu) + \mathcal{O}(L^{-7}), \quad (12) \]

**zeta-functions**

\[ I = Z_{00}(1, 0) = \sum_{\vec{n} \neq 0} \frac{1}{\vec{n}^2} - 4\pi A, \quad J = Z_{00}(2, 0) = \sum_{\vec{n} \neq 0} \frac{1}{(\vec{n}^2)^2}, \quad K = Z_{00}(3, 0) = \sum_{\vec{n} \neq 0} \frac{1}{(\vec{n}^2)^3} \]

**additional finite-volume quantities**

\[ \hat{Q} = \sum_{i \neq 0} \sum_{j \neq 0} \frac{1}{|i|^2 |j|^2 (|i|^2 + |j|^2 + |i + j|^2)} \rightarrow Q + \frac{4}{3} \pi^4 \log(\mu L) - \frac{2\pi^4}{3(d-3)} \]
\[ \hat{R} = \sum_{j \neq 0} \frac{1}{|j|^4} \left[ \sum_{i} \frac{1}{|i|^2 + |j|^2 + |i + j|^2} - \frac{1}{2} \int d^d i \frac{1}{|i|^2} \right] \rightarrow R - 2\sqrt{3} \pi^3 \log(\mu L) + \frac{\sqrt{3} \pi^3}{d-3} \]
Expanding our result

\[
\text{det}[1 + F_{\text{three}} i M_{df,3\rightarrow3}] = 0
\]

\[
F_{\text{three}} = -\frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{M^{-1} + F + G} \right]
\]

- Take \(M\) to be purely s-wave and \(M_{df,3\rightarrow3}\) to be a constant (i.e. \(l_{\text{max}}=0\))

- \(F_{\text{three}}, F, G\) are then truncated to matrices in spectator momentum space

- Can show that \([F_{\text{three}}]_{0,0}\) dominates other matrix elements by at least \(L^2\), so quantization condition becomes

\[
[F_{\text{three}}]_{0,0} = -i M_{df,3\rightarrow3}
\]

- \(F\) is \(O(L^0)\), so to cancel the \(1/L^3\) in \(F_{\text{three}}\) need \([M^{-1}+F+G]^{-1} \sim L^3\)

- Roughly speaking this requires the cancellation of \(L^0, L^{-1}\) & \(L^{-2}\) terms in \([M^{-1}+F+G]\), which requires tuning \(E\) and determines the \(L^{-3}, L^{-4}\) & \(L^{-5}\) in \(\Delta E\)

- The \(L^{-6}\) term in \(\Delta E\) is then determined by the quantization condition
Examples of expansions

\( (F_R)_{k,k} = \frac{1}{2} \left[ \frac{1}{L^3} \sum_a -P \int_a \right] \frac{1}{2\omega_a 2\omega_{-k-a}(E - \omega_k - \omega_a - \omega_{-k-a})} \)

Real part (imag. part cancels with \( \mathcal{M} \))

- **NR expansion**: 
  \[ [F_R]_{0,0} = \frac{1}{8\omega_q} \left[ \frac{1}{q^2 L^3} - \frac{I}{4\pi^2 L} - \frac{q^2 L^3 J}{(4\pi^2 L)^2} - \frac{(q^2 L^3)^2 K}{(4\pi^2 L)^3} + \ldots \right] \]
  \( q \) is momentum of each of non-spectator pair

  \[ [F_R]_{k,k} = -\frac{1}{16\pi^2 mL} \sum_{\tilde{n}_a} \frac{1}{\tilde{n}^2_k + \tilde{n}^2_a + (\tilde{n}_k + \tilde{n}_a)^2} \]
  contributes to \( \mathcal{R} \)

\[ G_{k,p} = \frac{1}{2\omega_p L^3 2\omega_{p+k}(E - \omega_p - \omega_k - \omega_{p+k})} \]

- **NR expansion**: 
  \[ G_{0,0} = \frac{1}{4m^2 \Delta E L^3} \]
  \( \sim L^0 \)

  \[ G_{0,k} = -\frac{1}{16\pi^2 mL \tilde{n}} \left[ 1 + \frac{m\Delta E L^3}{4\pi^2 L\tilde{n}^2} + \ldots \right] \]
  \( \sim L^{-1} \)

  \[ G_{k,p} = -\frac{1}{16\pi^2 mL \tilde{n}^2_k + \tilde{n}^2_p + (\tilde{n}_k + \tilde{n}_p)^2} \]
  \( \sim L^{-1} \)

Need spectator-momentum matrix structure of \( F \) & \( G \) to evaluate \([F_{\text{three}}]_{0,0}\)
Our threshold expansion

\[ E = 3m + \frac{12\pi a}{mL^3} \left[ 1 - \left( \frac{a}{\pi L} \right) \mathcal{I} + \left( \frac{a}{\pi L} \right)^2 \left[ \mathcal{I}^2 + \mathcal{J} \right] + \left( \frac{a}{\pi L} \right)^3 \left[ -\mathcal{I}^3 + \mathcal{I}\mathcal{J} + 15\mathcal{K} - 16\mathcal{Q} - 8\mathcal{R} \right] \right] \]

\[ + \frac{72a^3\pi^2 r}{mL^6} + \frac{36a^2\pi^2}{m^3L^6} + \frac{\tilde{a}_6}{L^6} + \mathcal{O}(1/L^7) \]
Our threshold expansion

\[ E = 3m + \frac{12\pi a}{mL^3} \left[ 1 - \left( \frac{a}{\pi L} \right) \mathcal{I} + \left( \frac{a}{\pi L} \right)^2 [\mathcal{I}^2 + \mathcal{J}] + \left( \frac{a}{\pi L} \right)^3 [-\mathcal{I}^3 + \mathcal{I} \mathcal{J} + 15\mathcal{K} - 16\mathcal{Q} - 8\mathcal{R}] \right] \]

\[ + \frac{72\alpha^3 \pi^2 r}{mL^6} + \frac{36\alpha^2 \pi^2}{m^3 L^6} + \frac{\tilde{a}_6}{L^6} + \mathcal{O}(1/L^7) \]

agrees with [Beane et al.] and [Tan] modulo definitions of \( Q \) & \( R \)

\[ Q \equiv -2048L^3 m^3 \pi^6 \sum_{\vec{k} \neq 0, \vec{p} \neq 0} G_{0, k} G_{k, p} G_{p, 0} \]

\[ = \sum_{\vec{n}_k \neq 0, \vec{n}_p \neq 0} \frac{1}{\vec{n}_k^2 \vec{n}_p^2 [\vec{n}_k^2 + \vec{n}_p^2 + (\vec{n}_k + \vec{n}_p)^2]} + \mathcal{O}(1/L) \]

\( \hat{Q} \) of [Beane et al.]

Log divergent after NR expansion, so requires regulation as in [Beane et al.]

UV convergent!

Similar situation for \( R \)
Our threshold expansion

\[ E = 3m + \frac{12\pi a}{mL^3} \left[ 1 - \left( \frac{a}{\pi L} \right) I + \left( \frac{a}{\pi L} \right)^2 \left[ I^2 + J \right] + \left( \frac{a}{\pi L} \right)^3 \left[ -I^3 + IJ + 15K - 16Q - 8R \right] \right] \]

+ \frac{72a^3\pi^2 r}{mL^6} + \frac{36a^2\pi^2}{m^3L^6} + \frac{\tilde{a}_6}{L^6} + O(1/L^7)

G required to get correct factors in these terms
Our threshold expansion

\[ E = 3m + \frac{12\pi a}{mL^3} \left[ 1 - \left( \frac{a}{\pi L} \right) \mathcal{I} + \left( \frac{a}{\pi L} \right)^2 [\mathcal{I}^2 + \mathcal{J}] + \left( \frac{a}{\pi L} \right)^3 [-\mathcal{I}^3 + \mathcal{I}\mathcal{J} + 15\mathcal{K} - 16\mathcal{Q} - 8\mathcal{R}] \right] \]

\[ + \frac{72a^3\pi^2 r}{mL^6} + \frac{36a^2\pi^2}{m^3L^6} + \frac{\tilde{a}_6}{L^6} + \mathcal{O}(1/L^7) \]

\[ \tilde{a}_6 = -\frac{M_{df,3 \to 3,\Lambda,00}(E = 3m, \vec{P} = 0)}{48m^3} \]

Physical, finite quantity, with no \( \mu \) dependence
Directly related to scattering amplitudes
In [Beane et al.] this term is

\[ \frac{1}{L^6} \eta_3(\mu) \]

[Beane et al.] have 24, [Tan] has 36, we have 72

[Beane et al.] and [Tan] do not have this term
Interpretation of “differences”

\[ + \frac{64 \pi a^4}{ML^6} (3\sqrt{3} - 4\pi) \log(\mu L) + \frac{24 \pi^2 a^3}{ML^5} r + \frac{1}{L^6} \eta_3(\mu) \quad \text{VS.} \quad + \frac{72 a^3 \pi^2 r}{mL^6} + \frac{36 a^2 \pi^2}{m^3 L^6} + \frac{\ddot{a}_6}{L^6} \]

[Beane et al.] [Hansen & SRS]

- We do not know a priori the relation between \( M_{df,3 \rightarrow 3} \) and \( \eta_3 \)

- \( M_{df,3 \rightarrow 3} \) is physical, while \( \eta_3 \) is a short-distance parameter, indirectly related to physical quantities

- We can view this comparison as providing the relation between \( M_{df,3 \rightarrow 3} \) and \( \eta_3 \) if we equate the two expressions

- As far as we can see, there is nothing forbidding this relation to include the finite \( a^2 \) and \( a^3 r \) terms

  - Indeed, a similar finite difference is required to match [Beane et al.] with [Tan]

- It would clearly be good to check this purported relation in another context
Closing comments

- Having a formalism is only the first step, especially as it is complicated.

- Threshold expansion check gives us confidence in the expression & shows how it can be used in practice.

- We plan further studies of its practical utility using simple forms for the scattering amplitudes.

- We also plan to compare in more detail with [Polejaeva & Rusetsky], [Briceno & Davoudi] & [HAL QCD] (see following talk by Sinya Aoki).
Closing comments

- Having a formalism is only the first step, especially as it is complicated

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Thank you!
Any questions?
Backup Slides
Comment on derivation

- Our derivation is rather involved (40+ pages)
- Including $3 \rightarrow 3$ Bethe-Salpeter kernel is easy
- Difficulty comes from multiple $2 \rightarrow 2$ interactions: analyze according to number of “switches”
- Leads to unsymmetrized, divergent contributions to $3 \rightarrow 3$ amplitudes
- Symmetrization occurs only after combining terms with different numbers of switches---i.e. all orders summation
- Removing divergences leads to switch factors $G$
- There is probably a better approach....
Comment on derivation

\[ C_L(E, \vec{P}) = + \quad + \quad + \quad \cdots \]

0 switches

1 switch

2 switches

\[ + \quad + \quad + \quad + \quad \cdots \]

\[ + \quad + \quad + \quad + \quad + \quad \cdots \]

\[ + \quad + \quad + \quad + \quad + \quad \cdots \]