Many-flavor Schwinger model at finite chemical potential

[arXiv:1307.4969]

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Outline

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Introduction

- Toy model: two-dimensional QED on a $l \times \beta$ torus (l: circumference of spatial circle; β: inverse temperature) with f flavors of massless Dirac fermions
- Flavor-dependent chemical potentials $\mu_1, \ldots, \mu_f$
- Same gauge coupling for all f fermion flavors
- Size l used to set all scales; dimensionless parameters:

$$e = l\, e_{\text{phys}}\,, \qquad \mu = l\, \mu_{\text{phys}}/2\pi\,, \qquad \tau = l/\beta$$

- Physics at zero temperature governed by toron variables $-\frac{1}{2} \leq h_{1,2} < \frac{1}{2}$ in Hodge decomposition of the $\text{U}(1)$ gauge field on the torus:

$$A_1(x_1,x_2) = \frac{2\pi h_1}{l} + \partial_1\chi(x_1,x_2) - \partial_2\phi(x_1,x_2) - \frac{2\pi k}{l\beta}x_2\,,$$
$$A_2(x_1,x_2) = \frac{2\pi h_2}{\beta} + \partial_2\chi(x_1,x_2) + \partial_1\phi(x_1,x_2)$$

χ: generates gauge transformations,
ϕ: periodic function on torus with no zero momentum mode,
k: integer-valued topological charge
(fermion determinant of a massless Dirac fermion is zero unless $k = 0$)

Factorization into bosonic (ϕ) partiton function and toronic (h) partition function
$Z(\mu_1,\ldots,\mu_f,\tau,e) = Z_b(\tau,e)Z_t(\mu_1,\ldots,\mu_f,\tau)$

$$Z_t = \int_{-\frac{1}{2}}^{\frac{1}{2}} dh_{1,2} \prod_{j=1}^{f} \sum_{n_j,m_j=-\infty}^{\infty} \exp\left[-\pi\tau\left[\left(n_j+h_2-i\frac{\mu_j}{\tau}\right)^2+\left(m_j+h_2-i\frac{\mu_j}{\tau}\right)^2\right]+2\pi i h_1\left(n_j-m_j\right)\right]$$

After suitable variable changes, integration over toron fields $h_{1,2}$ leads to representation of Z_t in the form of a $(2f-2)$-dimensional theta function

$$Z_t \propto \sum_{\boldsymbol{k}=-\infty}^{\infty} \exp\left[-\frac{\pi}{\tau}\left(\boldsymbol{k}^t\Omega^{-1}\boldsymbol{k}-2\boldsymbol{k}^t\boldsymbol{z}\right)\right] \propto \sum_{\boldsymbol{n}=-\infty}^{\infty} \exp\left[-\pi\tau\left(\boldsymbol{n}+\frac{i}{\tau}\boldsymbol{z}\right)^t\Omega\left(\boldsymbol{n}+\frac{i}{\tau}\boldsymbol{z}\right)\right]$$

$$\boldsymbol{z}^t = \begin{pmatrix}\bar{\mu}_2 & \bar{\mu}_3 & \cdots & \bar{\mu}_f & -\bar{\mu}_2 & -\bar{\mu}_3 & \cdots & -\bar{\mu}_{f-1} & 0\end{pmatrix} \qquad \bar{\mu}_i \equiv \mu_1-\mu_i$$

$$\Omega^{-1} = \begin{pmatrix}
2 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 1\\
1 & 2 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 1\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 1 & \cdots & 2 & 1 & 0 & 0 & \cdots 0 & 0 & 1\\
1 & 1 & \cdots & 1 & 2 & 0 & 0 & \cdots 0 & 0 & 1\\
0 & 0 & \cdots & 0 & 0 & 2 & 1 & \cdots & 1 & 1\\
0 & 0 & \cdots & 0 & 0 & 1 & 2 & \cdots & 1 & 1\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 2 & 1\\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 2-\frac{2}{f}
\end{pmatrix}$$

Introduction

- Particle number densities $N_i = \frac{\tau}{4\pi}\frac{\partial}{\partial \mu_i}\ln Z_t$ and $\bar{N}_i \equiv N_1 - N_i$
- No dependence on $\mu_1 + \ldots + \mu_f$, therefore $N_1 + \ldots + N_f = 0$ for all τ as expected (integration over toron fields projects on state with net zero charge).
- Infinite-τ limit: $(2f-2)$-dimensional infinite sum dominated by $\boldsymbol{n} = 0$, resulting in $\lim_{\tau\to\infty}\bar{N}_i = \bar{\mu}_i$
- Z_t at $\tau = 0$ determined by vectors $\boldsymbol{k} \in \mathbb{Z}^{2f-2}$ minimizing $\boldsymbol{k}^t\Omega^{-1}\boldsymbol{k} - 2\boldsymbol{k}^t\boldsymbol{z}$ (number densities $\bar{N}_i$ at $\tau = 0$ as certain linear combinations of k_j)
- Phases with different $\bar{N}$ are separated by first-order phase transitions
- Degenerate minima (with different corresponding $\bar{N}_i$'s): coexistence of multiple phases at zero temperature
- Quasi-periodicty (with $m_1, \ldots, m_{f-2}, \frac{f}{2}m_{f-1} \in \mathbb{Z}$) for all τ:

$$\bar{\mu}'_{k+1} = \bar{\mu}_{k+1} + m_k - \frac{f}{2}m_{f-1} + \sum_{i=1}^{f-1} m_i \qquad 1 \le k \le f-1,$$

$$\bar{N}_{k+1}(\boldsymbol{\mu}', \tau) = \bar{N}_{k+1}(\boldsymbol{\mu}, \tau) + m_k - \frac{f}{2}m_{f-1} + \sum_{i=1}^{f-1} m_i$$

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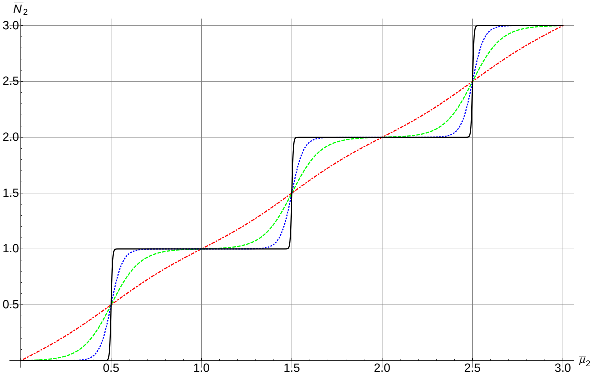
$f = 2$

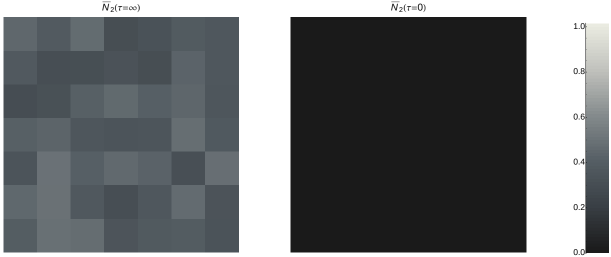
Theta-function representation reduces to

$$\bar{N}_2 = \frac{\sum_{k=-\infty}^{\infty} k e^{-\frac{\pi}{\tau}\left(k-\bar{\mu}_2\right)^2}}{\sum_{k=-\infty}^{\infty} e^{-\frac{\pi}{\tau}\left(k-\bar{\mu}_2\right)^2}}, \qquad \bar{\mu}_2 = \mu_1 - \mu_2\,, \quad \bar{N}_2 = N_1 - N_2$$

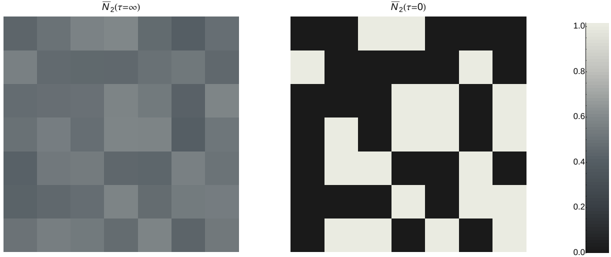


Reproducing results of arXiv:1210.3072 (R. Narayanan)

Fluctuations around $\bar{N}_2 = 2/5$ at large τ result in uniform $\bar{N}_2 = 0$ at $\tau = 0$:



Fluctuations around $\bar{N}_2 = 1/2$ at large τ result in two coexisting phases at $\tau = 0$:

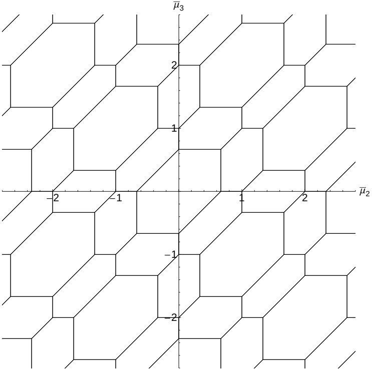


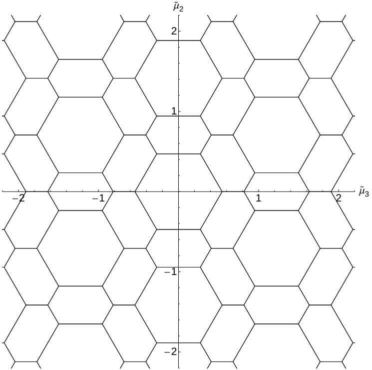
Outline

$f = 3$

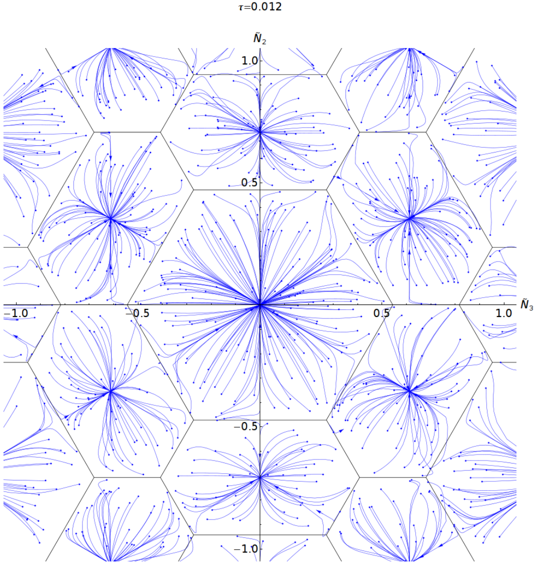
Phase structure at $\tau = 0$ in coordinates

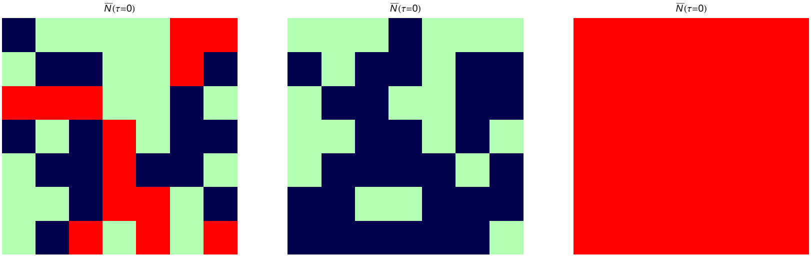
$$\begin{pmatrix} \bar{\mu}_1 \\ \bar{\mu}_2 \\ \bar{\mu}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \qquad \begin{pmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \\ \tilde{\mu}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$





Evolution with decreasing τ





- Left: Fluctuations around $\bar{N} \equiv (\bar{N}_2, \bar{N}_3) = (\frac{2}{3}, \frac{2}{3})$ at large τ result in coexistence of three phases with $\bar{N} = (0,0)$ (red), $\bar{N} = (\frac{1}{2}, 1)$ (blue), $\bar{N} = (1, \frac{1}{2})$ (green) at $\tau = 0$.
- Center: $\bar{N} = (\frac{3}{4}, \frac{3}{4})$ at high τ results in two coexisting phases ($\bar{N} = (\frac{1}{2}, 1)$ and $\bar{N} = (1, \frac{1}{2})$) at $\tau = 0$.
- Right: $\bar{N} = (\frac{1}{2}, \frac{1}{2})$ at high τ results in a single phase at $\tau = 0$.

Outline

$f = 4$

- Convenient coordinates:

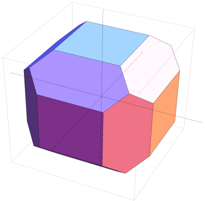
$$\begin{pmatrix}\tilde{\mu}_1\\ \tilde{\mu}_2\\ \tilde{\mu}_3\\ \tilde{\mu}_4\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\\ \mu_4\end{pmatrix},$$

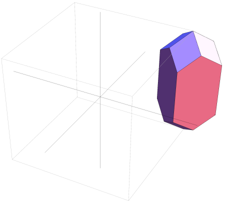
- Phase structure is periodic under shifts

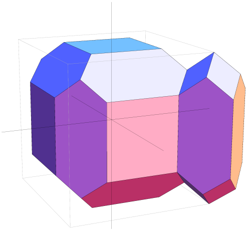
$$\begin{pmatrix}\tilde{\mu}_2\\ \tilde{\mu}_3\\ \tilde{\mu}_4\end{pmatrix} \to \begin{pmatrix}\tilde{\mu}_2\\ \tilde{\mu}_3\\ \tilde{\mu}_4\end{pmatrix} + l_1\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + l_2\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} + l_3\begin{pmatrix}0\\ 1\\ 0\end{pmatrix} \qquad l_{1,2,3} \in \mathbb{Z}$$

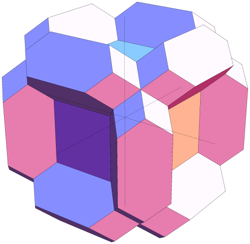
- At $\tau = 0$, $\tilde{\mu}_{2,3,4}$ space is divided into two types of three-dimensional cells (characterized by identical $\tilde{N}_{2,3,4}$ inside each cell)
- Different types of vertices: four and six phases can coexist
- At all edges: three phases can coexist

Cell types for $f = 4$

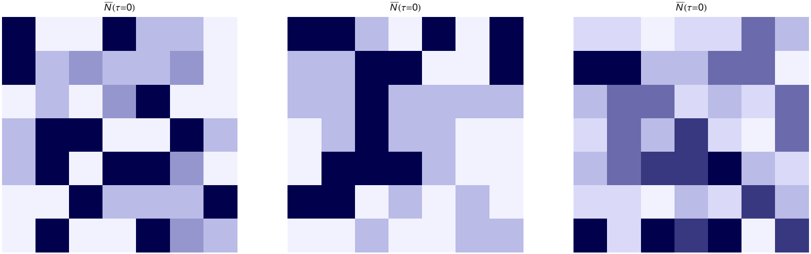








Coexisting phases at $\tau = 0$ for $f = 4$



- Left: four phases coexist at $\tau = 0$ for $\tilde{\mu}_{2,3,4} = (\frac{3}{8}, \frac{3}{8}, \frac{3}{8})$
- Center: three phases coexist at $\tau = 0$ for $\tilde{\mu}_{2,3,4} = (\frac{7}{16}, \frac{7}{16}, \frac{7}{16})$
- Right: six phases coexist at $\tau = 0$ for $\tilde{\mu}_{2,3,4} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

Outline

Phase structure for large f

- For $f = 3$, $f = 4$: $\bar{\mu}_i$ coordinates of all vertices are multiples of $\frac{1}{f}$
- Two special vertices for all f:
  - f coexisting phases at $\bar{\mu}_i = 1 - \frac{1}{f}$ for all $2 \leq i \leq f$
  - $\binom{f}{2}$ coexisting phases at $\bar{\mu}_i = 1$ for all $2 \leq i \leq f$
- For $f = 5$, up to $\binom{5}{2}$ coexisting phases
- For $f = 6$, up to $\binom{6}{3}$ coexisting phases (for example at $\bar{\mu}_{2,\ldots,6} = (1, \frac{1}{2}, 0, 0, 0)$)
- For $f = 8$, up to $\binom{8}{4}$ coexisting phases (for example at $\bar{\mu}_{2,\ldots,8} = (1, 1, 1, 1, 1, 1, 0)$)
- Conjecture: maximal number of coexisting phases is given by $\binom{f}{\lfloor f/2 \rfloor}$, increasing exponentially for large f

Outline

Chiral 1111-2 model

- Four left-handed Weyl fermions with charge 1 and chemical potentials $\mu_{1,2,3,4}$
- One right-handed Weyl fermion with charge 2 and chemical potential μ_5
- Anomalies cancel, general condition: $\sum_i q_{L,i}^2 = \sum_i q_{R,i}^2$
- Transformation for left-handed fermions:

$$\begin{pmatrix} \tilde{\mu}_1 \\ \tilde{\mu}_2 \\ \tilde{\mu}_3 \\ \tilde{\mu}_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix},$$

- Integration over toron-fields results in completely factorized partition function

$$Z_t \propto e^{\frac{\pi}{2\tau}(\tilde{\mu}_1 - \mu_5)^2} \prod_{i=2}^{4} \left(\sum_{r=-\infty}^{\infty} e^{-\frac{\pi}{\tau}\left[(r-\tilde{\mu}_i)^2 - \tilde{\mu}_i^2\right]} \right)$$

- No dependence on $\tilde{\mu}_1 + \mu_5 = (\mu_1 + \mu_2 + \mu_3 + \mu_4 + 2\mu_5)/2$, trivial dependence on $\tilde{\mu}_1 - \tilde{\mu}_5$
- Zero-temperature phase structure in $\tilde{\mu}_{2,3,4}$ space: cubic cells (up to 8 coexisting phases at $\tau = 0$)
- Work in progress: generalizations, e.g., to model with one right-handed fermion with charge q and q^2 left-handed fermions with charge 1

Outline

Summary

- Multiflavor QED with flavor-dependent chemical potential on a two-dimensional torus exhibits rich phase structure at zero temperature
- Infinite number of phases, separated by first-order phase transitions
- Toron variables completely dominante dependence on chemical potential; resulting partition function has representation in form of multidimensional theta function
- We explicitly determined the phase structure for $f = 3$ (two or three coexisting phases) and $f = 4$ (two, three, four or six coexisting phases)
- Based on additional exploratory investigation of $f = 5, 6, 8$: conjecture that up to $\binom{f}{\lfloor f/2 \rfloor}$ phases can coexist in a theory with f flavors