Localised distributions in complex Langevin dynamics

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Lattice 2013, Mainz (Germany), 31/July/2013

Introduction

- Complex Langevin dynamics provides a way to simulate theories with complex actions ⟶ no importance sampling, no sign problem!

- It opens the way to QCD simulations at $\mu_B \neq 0$

- The method was introduced in 1983 by G.Parisi and J.R.Klauder but shortly after it was clear that correct results **are not** garanteed

- We do not have a FULL UNDERSTANDING of the problem yet!

- A combination of analytical and numerical results, also on simple models, can help us!

- Recently the importance of the properties of the probability distribution (generated by the Langevin process) in the complexified configuration space has been clarified: the distribution has to drop very rapidly (in particular in the imaginary direction) ⟶ this can be formalised in a criterion for correctness [G. Aarts, F. A. James, E. Seiler and I. -O. Stamatescu, Eur. Phys. J. C 71 (2011) 1756]

The goal of this work

- Here we study the probability distribution (by brute force and solving the Fokker-Planck Equation, FPE) and then we relate the results to the criterion for correctness

- We have a complete characterisation of the dynamics by studying:

  - Classical flow

  - Criterion for correctness

  - Explicit solution of the FPE

- We show moreover that:

  - If the distribution has support only on a strip of the complexified configuration space, then correct results are obtained !

The model + CL

- The toy model:
$$Z = \int_{-\infty}^{\infty} dx\, e^{-S}, \qquad S = \frac{1}{2}\sigma x^2 + \frac{1}{4}\lambda x^4, \qquad \sigma \in \mathbb{C}, \lambda \in \mathbb{R}$$

- Analytic solution:
$$Z = \sqrt{\frac{4\xi}{\sigma}} e^{\xi} K_{-\frac{1}{4}}(\xi) \quad \xi = \sigma^2/(8\lambda) \quad \Rightarrow \langle x^n \rangle$$

- Complex Langevin (CL) equation: $\dot{z} = -\partial_z S(z) + \eta$

- Complexification: $z = x + iy, \qquad \eta = \eta_R + i\eta_I, \qquad \sigma = A + iB$

- CL is now: $\dot{x} = K_x(x,y) + \eta_R, \qquad \dot{y} = K_y(x,y) + \eta_I$

- Drift:
$$K_x \equiv -\mathrm{Re}\,\partial_z S(z) = -Ax + By - \lambda x\left(x^2 - 3y^2\right)$$
$$K_y \equiv -\mathrm{Im}\,\partial_z S(z) = -Ay - Bx - \lambda y\left(3x^2 - y^2\right)$$

- Noise:
$$\langle \eta_R(t)\eta_R(t')\rangle = 2N_R\delta(t-t'), \quad \langle \eta_I(t)\eta_I(t')\rangle = 2N_I\delta(t-t'), \quad N_R - N_I = 1$$

Criterion for correctness

- Averaging over the noise we can determine the expectation values $\langle O \rangle_\eta$

- The probability distribution $P(x, y; t)$ describes how the configuration space is sampled; its evolution in time is given by the Fokker-Planck Equation:

$$\dot{P}(x, y; t) = L^T P(x, y; t), \qquad L^T = \partial_x \left(N_R \partial_x - K_x \right) + \partial_y \left(N_I \partial_y - K_y \right)$$

- The expectation value is given by:

$$\langle O \rangle_{P(t)} = \int dx dy \, P(x, y; t) O(x + iy)$$

- But we know that:

$$\langle O \rangle_{\rho(t)} = \int dx \, \rho(x, t) O(x), \quad \rho(x) = e^{-S(x)}$$

- Therefore we want that: $\quad \langle O \rangle_{\rho(t)} = \langle O \rangle_{P(t)}$

- Introducing the Langevin Operator: $\tilde{L} = \left[\partial_z - (\partial_z S(z)) \right] \partial_z$, the criterion for correctness is given by:
$$C_O \equiv \langle \tilde{L} O(z) \rangle = 0$$
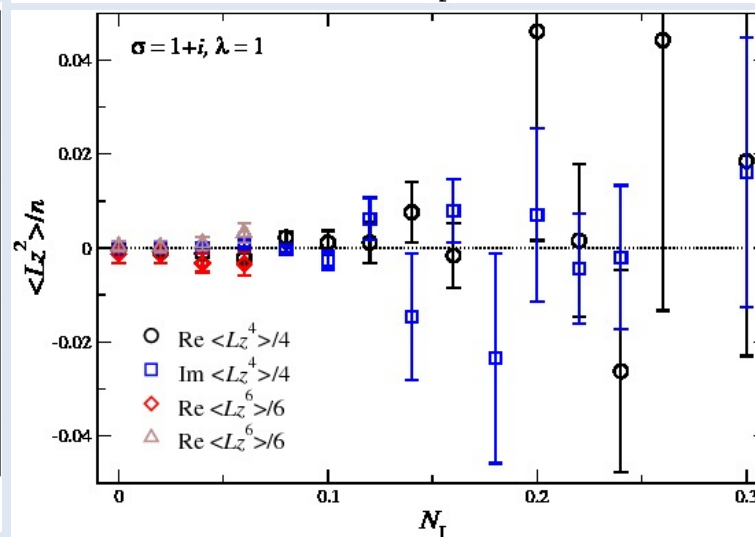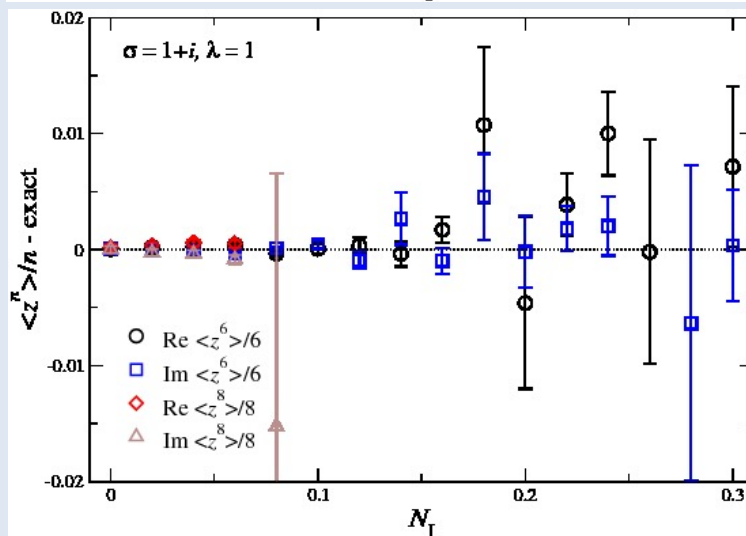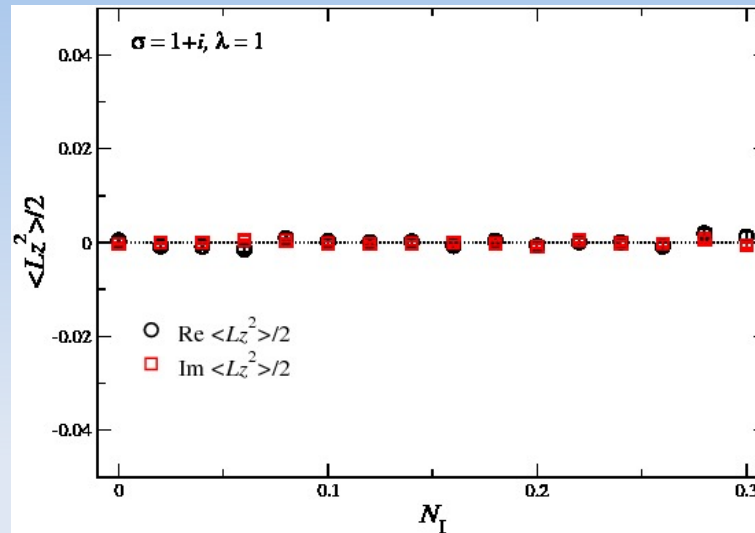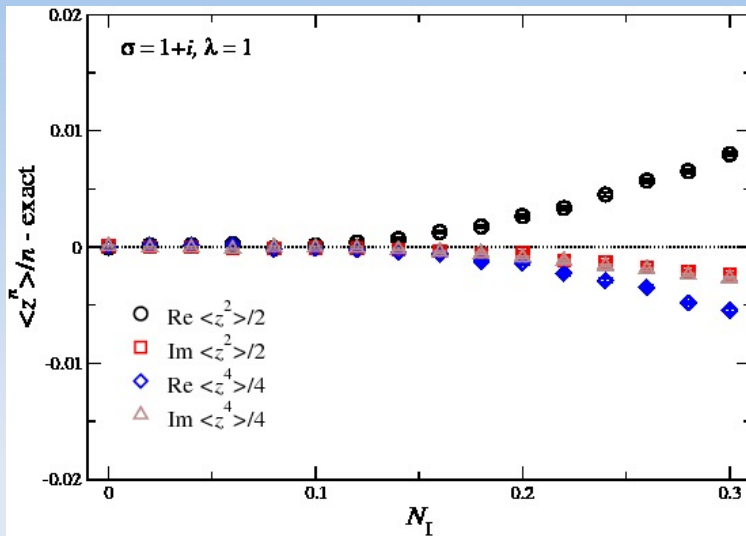(to be satisfied for **a complete set** of observables)

- We consider the observables: $O_n(z) = \dfrac{1}{n}z^n$ and the criterion: $C_n \equiv \dfrac{1}{n}\left\langle \tilde{L}z^n \right\rangle = 0$
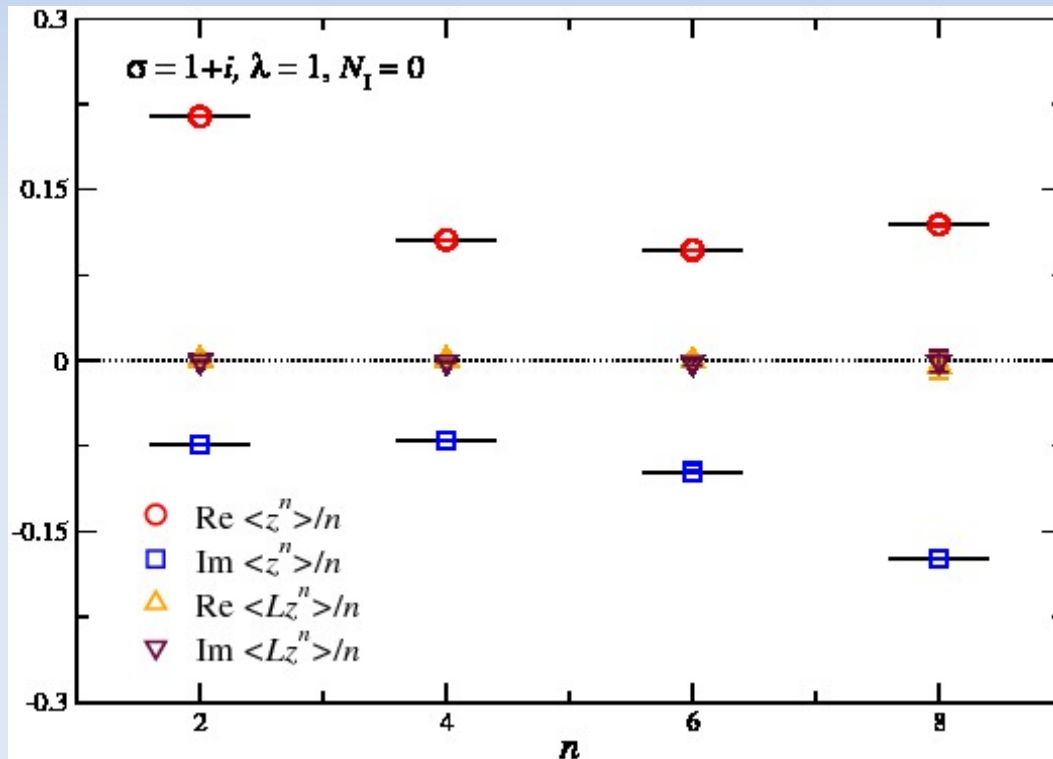
Perfect agreement and criterion for correctness satisfied!

Complex noise

Note: C_2 always consistent with zero; strong fluctuation for large N_I

Solving FPE

- We want to solve the FP equation: $\dot{P}(x,y;t) = L^T P(x,y;t)$

- To do that we solve the eigenvalue problem: $-L^T P_\kappa(x,y) = \kappa \; P_\kappa(x,y)$

- If we have a unique ground state P_0 with eigenvaue $\kappa = 0$, then the solution is: $P(x,y;t) = P_0(x,y) + \sum_{\kappa \neq 0} e^{-\kappa\,t} P_\kappa(x,y)$

- In [A.Duncan, M.Niedermaier, Annals Phys.329 (2013) 93] P(x,y) is expanded in a basis of Hermite functions:

$$P(x,y) = \sum_{k=0}^{N_H-1} \sum_{l=0}^{N_H-1} c_{kl} H_k\left(\sqrt{w}x\right) H_l\left(\sqrt{w}y\right)$$

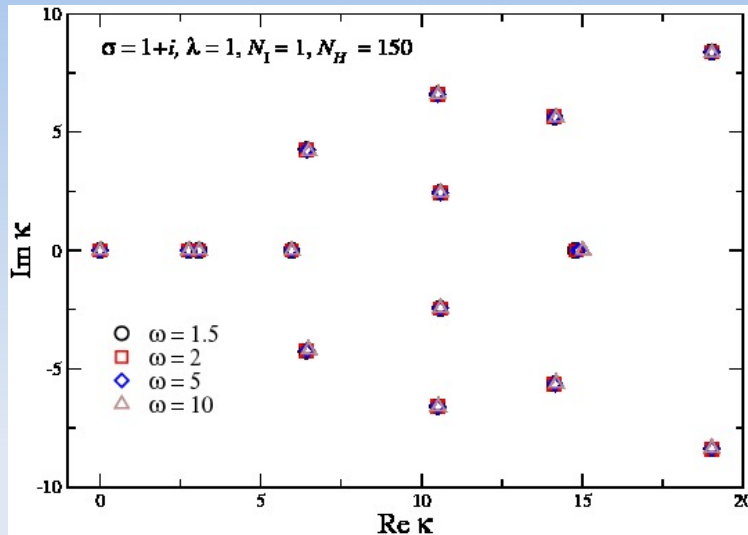- This was done introducing creation and annihilation operators, a and b:

$$x = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right), \quad p_x = -i\partial_x = i\sqrt{\frac{\omega}{2}}\left(a^\dagger - a\right),$$
$$y = \frac{1}{\sqrt{2\omega}}\left(b + b^\dagger\right), \quad p_y = -i\partial_y = i\sqrt{\frac{\omega}{2}}\left(b^\dagger - b\right)$$

- We determine the matrix elements: $\langle kl|L^T|mn\rangle$ where $|mn\rangle = \frac{1}{\sqrt{m!n!}} a^{\dagger m} b^{\dagger n}|0\rangle$ and therefore $H_m(\sqrt{\omega}x) = \langle x|m\rangle$
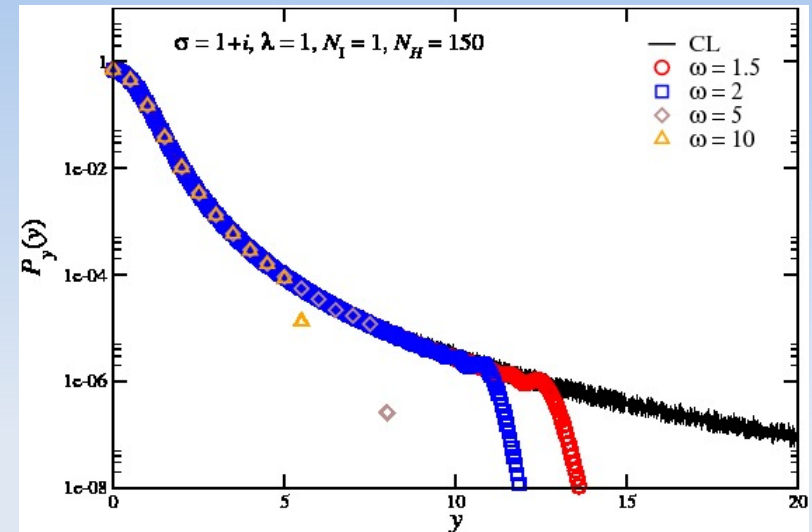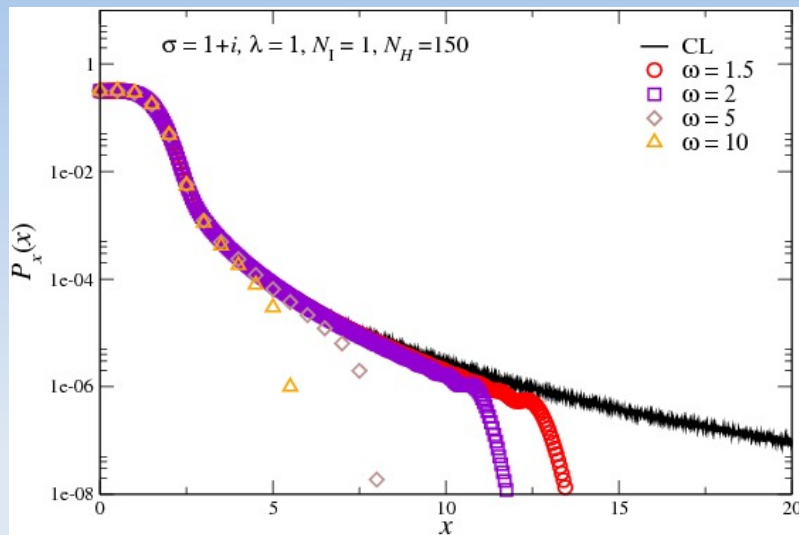
- Note: ω and N_H
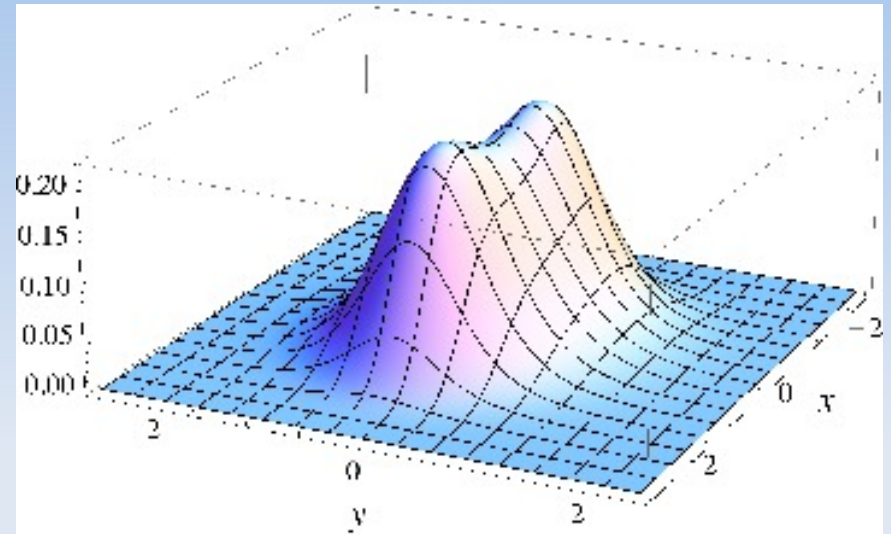
- The eigenvalues around the origin are independent of ω

- Ground state: $\omega = 1.5$ and $N_H = 150$

- We find that there is an interval for ω for which:

  - There is always an eigenvalue consistent with zero

  - The other eigenvalues are in the right half-plane

  - The ground state is stable under variation of ω and N_H

Complex noise (integrat. distr. & power decay)

- Partially integrated distr: $P_x(x) = \int_{-\infty}^{\infty} dy \, P(x, y), \quad P_y(y) = \int_{-\infty}^{\infty} dx \, P(x, y)$

- Manifestation of the truncation in N_H

- We observe a power decay with power 5: $\quad P_x(x) \sim \frac{1}{|x|^5}, \quad P_y(y) \sim \frac{1}{|y|^5}$

- This suggests: $P(x, y) \sim \frac{1}{(x^2 + y^2)^3}$
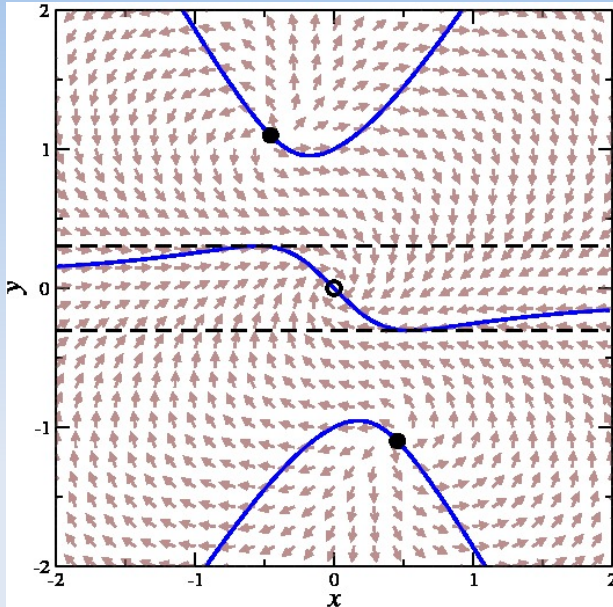
Real noise (eigenvalues & distr.)
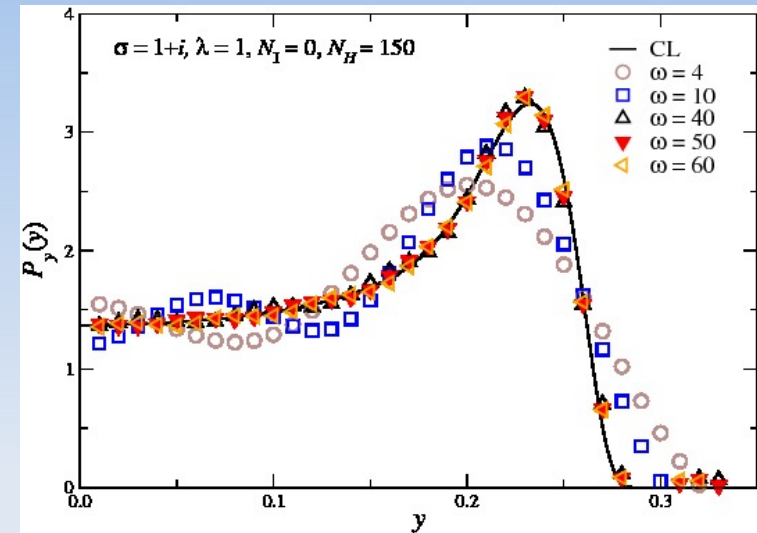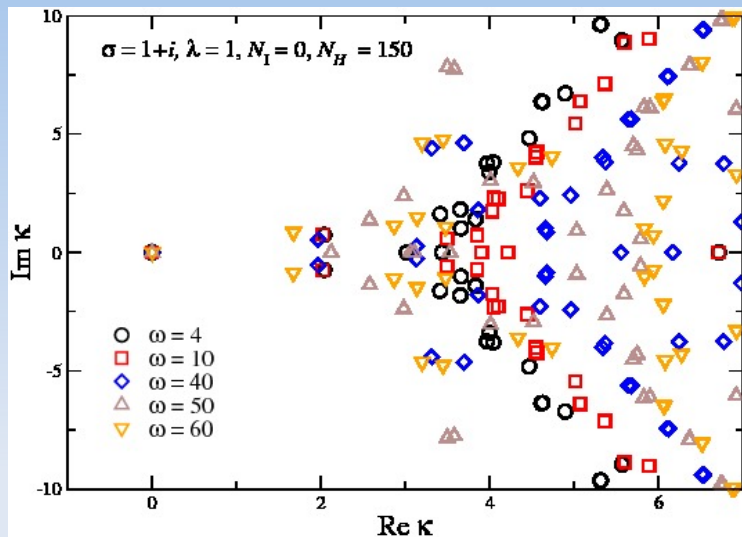
- There is an eigenvalue at the origin but in general they depend on ω

- From $P_y(y)$ we see convergence only for large values of ω

- Distribution very localised, drops to zero around $y \approx 0.28$
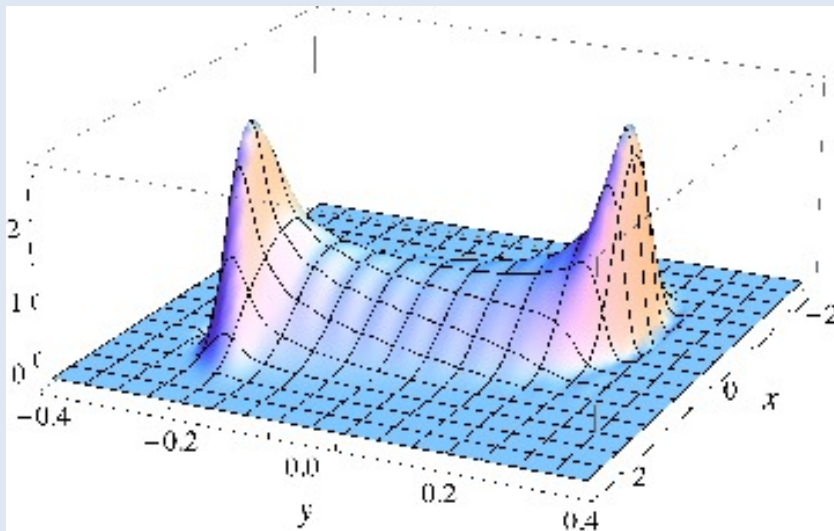
Real noise (truncation & 3d plot)
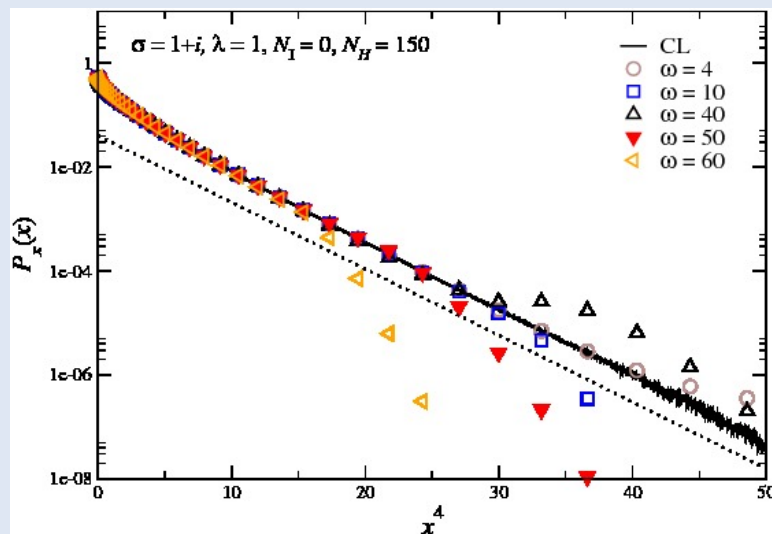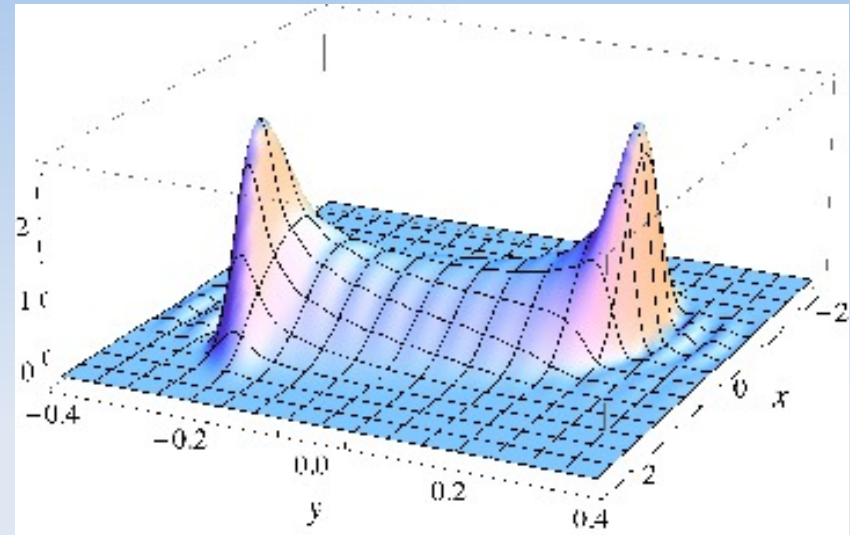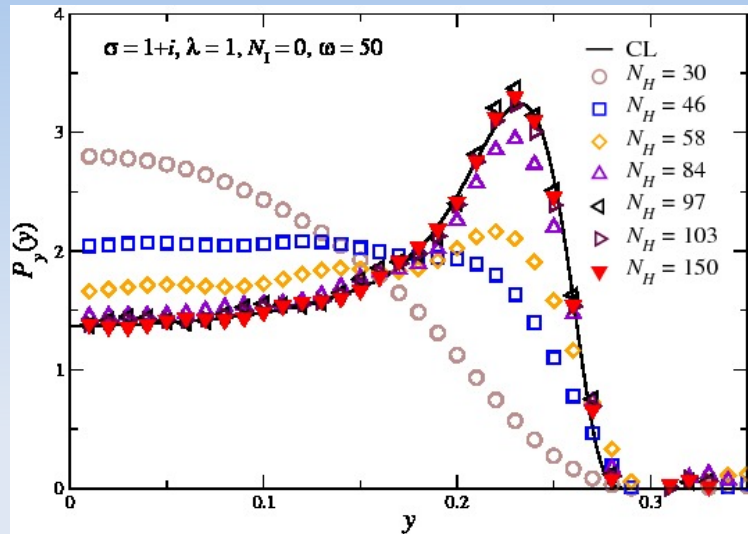
- $P_x(x)$ exponential decay!
- $P_x(x) \sim e^{-ax^4}, \quad a \sim 0.295.$

Classical flow

- $(K_x(x,y), K_y(x,y))$ for $\sigma = 1 + i$ and $\lambda = 1$
- 3 fixed points (where $K_x(x,y) = K_y(x,y) = 0$):
  - An attractive point at (x,y)=(0,0)
  - Two repulsive points at $(\pm 0.455, \mp 1.10)$
- Blue lines where $K_y(x,y)$ changes sign
- Dynamics confined between the dashed lines!!! (we have: $-0.3029 < y < 0.3029$)

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Conservation law

- The classical flow result can be made more rigorous

- We note that the FPE takes the form of a conservation law:

$$\dot{P}(x,y;t) = \partial_x J_x(x,y;t) + \partial_y J_y(x,y;t)$$

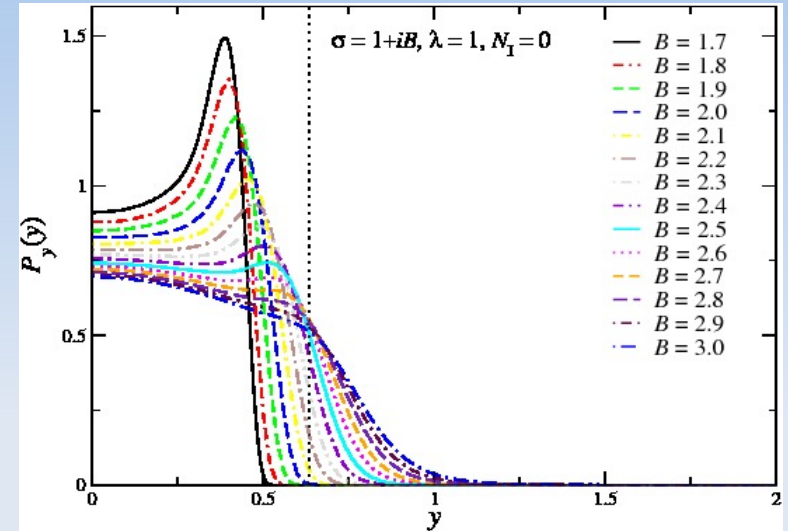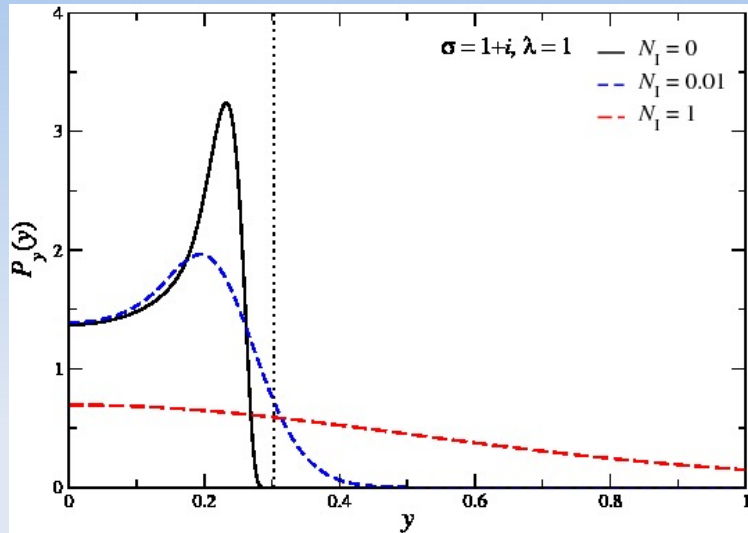$$J_x = (N_R \partial_x - K_x) P, \quad J_y = (N_I \partial_y - K_y) P$$

- We can now introduce the charge $Q(y,t) = \int_{-\infty}^{\infty} dx \, J_y(x,y;t)$

- Assuming sufficient decay, i.e. $K_{x,y}(x,y)P(x,y) \to 0$ and real noise we have: $Q(y) = \int_{-\infty}^{\infty} dx \, K_y(x,y)P(x,y) = 0$

- Since $P(x,y)$ is not negative, if $K_y(x,y)$ has a definite sign as a function of x for a given y, then $P(x,y)$ has to vanish for this y value
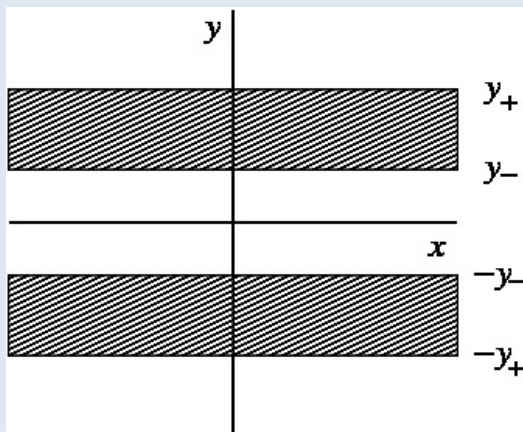
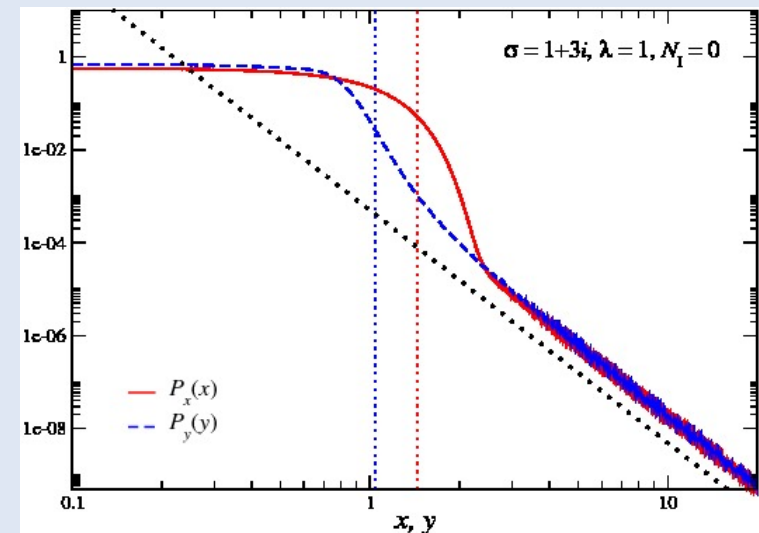- The distribution is strictly zero in the two strips provided that $3A^2 > B^2$ and $N_I = 0$

- Where:

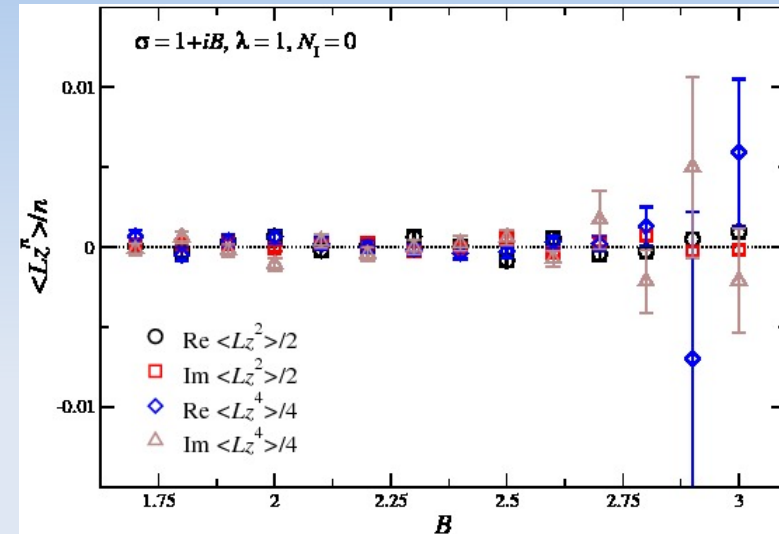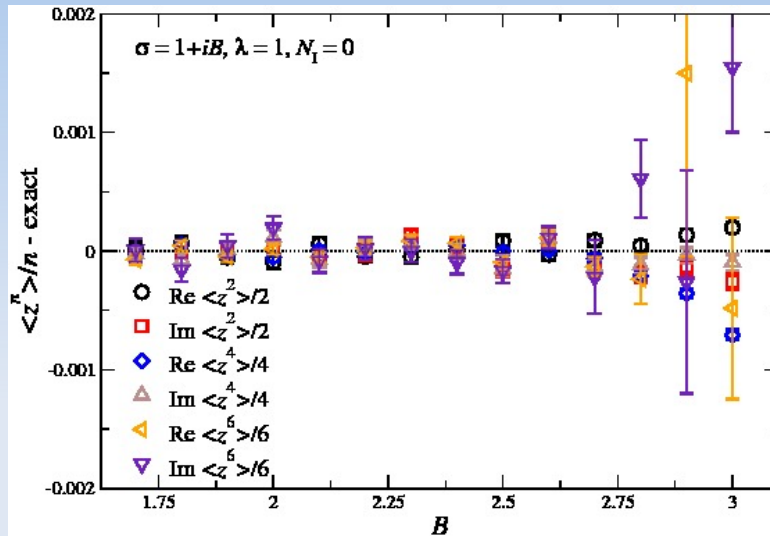$$y_{\pm}^2 = \frac{A}{2\lambda}\left(1 \pm \sqrt{1 - \frac{B^2}{3A^2}}\right)$$

- For complex noise there are no strips!

- Always power decay:
$$P_y(y) \sim 1/|y|^5$$

- For real noise no strip if $3A^2 < B^2$

- Increasing B similar to increasing N_I

Criterion for correctness vs B

- Also from here we see that the effect of increasing B is very similar to increasing the value of N_I

Universal decay behaviour

- It is possible to understand the universal power decay!

- Starting from FPE: $\dot{P}(x, y; t) = L^T P(x, y; t)$

- And substituting the Ansatz: $P(x, y) = \dfrac{c}{(x^2 + y^2)^\alpha}$

- We find that:

$$\alpha \frac{x^2 - y^2 + 2\alpha(N_R x^2 + N_I y^2)}{(x^2 + y^2)^2} + A(1 - \alpha) + \lambda(3 - \alpha)(x^2 - y^2) = 0$$

- At large x and y, only the last term dominates and we have: $\alpha = 3$

- And therefore: $P_x(x) \sim \dfrac{1}{|x|^5}, \quad P_y(y) \sim \dfrac{1}{|y|^5}$

Conclusions

- In order to justify the results obtained with CL the probability distribution has to be sufficiently localised

- Here we have studied the properties of the distribution via a number of methods: classical flow, histogram by brute force, explicit solution of FPE, criterion for correctness

- We have found:

  - For real noise as $3A^2 > B^2$, the distribution has support only in a strip and it has an exponential decay in the real direction; criterion for correctness satisfied and correct results obtained!

  - When $3A^2 < B^2$ or the noise is complex the distribution is NOT localised; the distribution has a power law: $P(x,y) \sim (x^2 + y^2)^{-3}$, because of this slow decay high moments are not well-defined; criterion for correctness suffer of large fluctuations: signal of failure!

- A consistent picture of the dynamics can be obtained already from a combination of partially integrated distribution and criterion for correctness

- These tools are readily available to study SU(N) gauge theories (plus gauge cooling...)