The magnetic susceptibility in QCD

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Outline

1. QCD at finite temperature: diamagnetic or paramagnetic?
2. Why it is not trivial to answer
3. How to answer
4. Numerical results
5. Conclusions & developments

QCD in external magnetic field

External magnetic fields can be relevant for the phenomenology of

- primordial universe
- heavy-ion collisions

From a theoretical point of view they are also interesting as a different way to test the non-perturbative dynamics of QCD.

External E.M. fields couple to QCD through the Dirac matrix. In the unimproved staggered formulation

$$D_{i,j} = am\delta_{i,j} + \frac{1}{2}\sum_{\nu=1}^{4}\eta_\nu(i)\Big[u_\nu(i)U_\nu(i)\delta_{i,j-\hat\nu} - u^*_\nu(i-\hat\nu)U^\dagger_\nu(i-\hat\nu)\delta_{i,j+\hat\nu}\Big]$$

$U_\nu(i)$ = non abelian link variables
$u_\nu(i)$ = abelian link variables

The QCD medium

We are interested in the magnetic properties of QCD at finite temperature.

For "small" external magnetic B we can write the free energy as

$$F(B,T) = F(B=0,T) + F_1(T)B + \frac{1}{2}F_2(T)B^2 + \mathcal{O}(B^3)$$

$F_1 \equiv 0$ if no ferromagnetism
F_2 is proportional to the magnetic susceptibility (see later)
$F_2 < 0$ paramagnetic medium
$F_2 > 0$ diamagnetic medium

The basic point we want to settle:
is QCD paramagnetic or diamagnetic?

The standard way and a no-go

The determination of magnetic susceptibilities is a standard problem in statistical physics. An estimator of F_2 is obtained by using the relation

$$F_2(T) = \left. \frac{\partial^2 F(B,T)}{\partial B^2} \right|_{B=0}$$

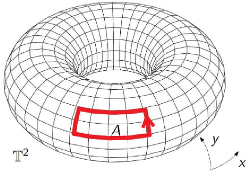
which amounts to compute the mean value of some lattice observable at $B = 0$.

In QCD this is not possible: to reduce finite size effects simulations are performed on compact manifold without boundary and as a consequence the possible values of the homogeneous magnetic field are quantized.

$\frac{\partial}{\partial B}$ on the lattice is not well defined!

The magnetic field on the lattice

On a compact manifold with no boundary there is an ambiguity in the application of the Stokes theorem: which part of the manifold is the interior and which is the exterior of a path?



For an homogeneous magnetic field $B\hat{z}$ we have

$$\oint A_\mu \mathrm{d}x_\mu = \mathcal{A}B \qquad \oint A_\mu \mathrm{d}x_\mu = -(\ell_x\ell_y - \mathcal{A})B$$

This does not affect the motion of a charged particle if we impose

$$\exp(iqB\mathcal{A}) = \exp(iqB(\mathcal{A} - \ell_x\ell_y)) \quad \Rightarrow \quad \boxed{qB = \frac{2\pi b}{\ell_x\ell_y} \quad b \in \mathbb{Z}}$$

(the ℓ_μ's are the lengths in physical units)

The magnetic field on the lattice (2)

A simple choice of the lattice discretization is

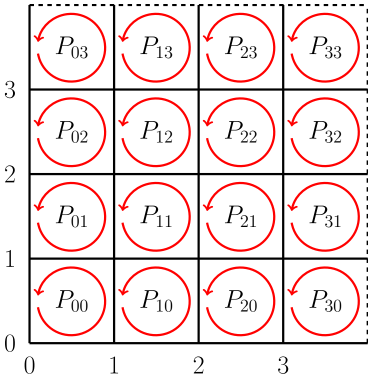
$$u_y(n) = e^{ia^2 qBn_x} \quad u_x(L_x - 1) = e^{-ia^2 qBL_x n_y} \quad \text{otherwise} \quad u_j(n) = 1$$

An example for $L_x = L_y = 4$.

The E.M. plaquettes are given by

- $P_{ij} = e^{ia^2 qB}$ for $(i,j) \neq (3,3)$
- $P_{33} = \exp(ia^2 qB + ia^2 qBL_x L_y)$

Everything is ok if $a^2 qBL_x L_y = 2\pi b$ with $b \in \mathbb{Z}$. The idea is the same as the Dirac quantization condition for monopoles (i.e. "invisible" string).



A way to go

We are interested in studying the B dependence of F, i.e.

$$\Delta F(B_k, T) = F(B_k, T) - F(0, T) \qquad a^2 q B_k = \frac{2\pi k}{L_x L_y} \quad k \in \mathbb{Z}$$

$M(B, T) = \frac{\partial F(B,T)}{\partial b}$ *is not* the magnetization, but we can evaluate it at non quantized values of B in order to get

$$\Delta F(B_k, T) = \int_0^k M(b, T) \mathrm{d}b$$

All the "periodicity" artefacts that affect M simplify in the integral to give us the correct answer!

We work on finite lattices, so everything is analytic and we adopt the previous expression for the $u_i(n)$ also for non quantized B values. These values of B are non physical but are needed only for the purpose of reconstructing ΔF for integer b.

Renormalization prescription

The free energy renormalizes additively so it is not enough to fix the sign of $\Delta F(B_k, T)$ to determine the nature of the magnetic QCD medium.

We can remove the additive renormalization by subtracting the zero temperature value:

$$(\Delta F)_R(B_k, T) = \Delta F(B_k, T) - \Delta F(B_k, T = 0)$$

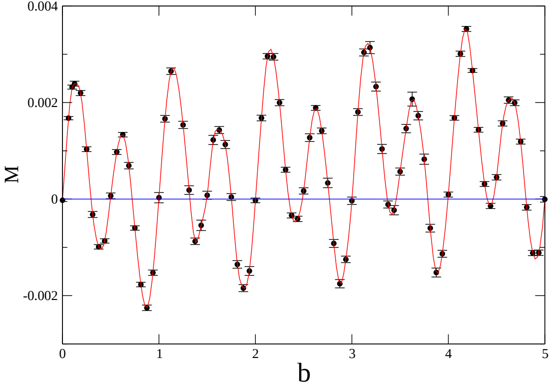
This is motivated by the idea that we want to study the properties of the thermal medium so the zero temperature value has to be subtracted as a normalization.

Our procedure is thus the following:

1. compute the “magnetization” M for different temperatures and for non quantized B values
2. integrate M to get $\Delta F(B_k, T)$ for the quantized B_k values
3. compute the renormalized magnetic free energy $(\Delta F)_R(B_k, T)$

How M looks like

M computed on a 16^4 lattice, $N_f = 1+1$, $m_\pi \approx 480$MeV, $a \approx 0.188$fm. The continuous line is a 3^{rd} order spline interpolation.



The numerical integration of M to get ΔF is performed by means of spline interpolations together with a bootstrap analysis for the error estimation.

Extracting the quadratic term

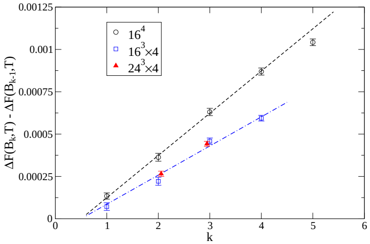
We now need to estimate f_2 defined by $\Delta F(B_k, T) \approx \frac{1}{2} f_2(T) k^2$ ($B_k \propto k$).
In order to minimize the error propagation in the integration we fit

$$\Delta F(B_k, T) - \Delta F(B_{k-1}, T) = \int_{k-1}^{k} M(b, T) \mathrm{d}b$$

with the function

$$\frac{1}{2} f_2(T) \left[k^2 - (k-1)^2\right] = \frac{1}{2} f_2(T)(2k-1)$$

Results for $4 \times 16^3, 4 \times 24^3$ and 16^4 lattices with $m_\pi \approx 480$MeV and $a \approx 0.188$fm ($T \approx 175$MeV).



A note on the susceptibility

In an usual linear medium we have (in SI units)

$$\mathbf{M} = \chi \mathbf{H} \quad \mathbf{B} = \mu \mathbf{H} \quad \mu = \mu_0(1+\chi)$$

and the expression for the total free energy (for volume unit) is

$$F/V = \int \mathbf{H} \cdot \mathrm{d}\mathbf{B} = \frac{1}{2\mu} B^2 = \frac{1}{2\mu_0(1+\chi)} B^2$$

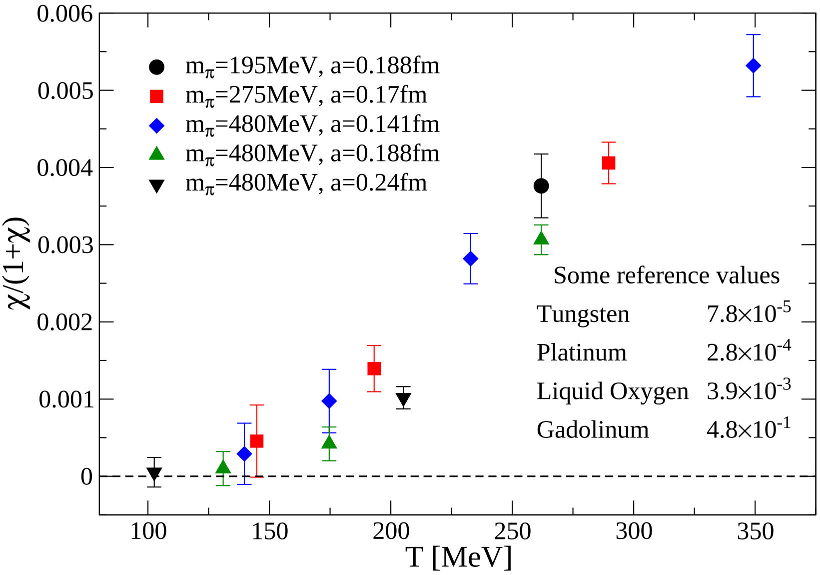
In our QCD simulations the magnetic field is quenched and the energy of the electromagnetic field "without QCD" must be subtracted:

$$F/V = -\int \mathbf{M} \cdot \mathrm{d}\mathbf{B} = -\frac{\chi}{\mu_0(1+\chi)} \int \mathbf{B} \cdot \mathrm{d}\mathbf{B}$$

moreover there is no back reaction of the medium on the magnetic field, so that $\mathbf{B} = \mu_0 \mathbf{H}$ and

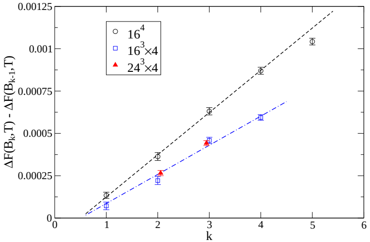
$$F/V = -\frac{\chi}{2\mu_0(1+\chi)} \mathbf{B}^2$$

The final result



Check for systematics

- dependence on the volume



- dependence on the spline interpolation and/or the number of points

s	16 points	32 points
1	0.001192(32)	0.001187(25)
2	0.001188(35)	0.001186(25)
3	0.001184(35)	0.001188(25)
4	0.001183(34)	0.001188(27)

- dependence on the B field extension out of integers

one string	0.00211(5)
two strings	0.00208(4)

Systematics are always less than statistical errors

Conclusions & developments

- We presented a simple new strategy to study the magnetic properties of QCD
- Our results show that the QCD medium near deconfinement behaves as a strong paramagnet
- A study with improved fermions at physical quark masses is ongoing. Preliminary results are in qualitative and reasonable quantitative agreement with the unimproved staggered result presented here.