SUNSET INTEGRALS AT FINITE VOLUME

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Overview

1. Why do this?
2. One-loop tadpoles
3. One-loop integrals
4. Why sunset?
5. Dividing up
6. One loop quantized
7. Two loops quantized
8. Numerical

Work done with E. Boström (Lund) and T. Lähde (Jülich)
Partially published in E. Boström, master thesis, $\boxed{\text{LU-TP 13-22}}$

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Motivation

- Lattice QCD calculates at different quark masses and volumes
- Chiral Perturbation Theory is useful for this
- Masses and decay constants known at infinite volume to two loops
- Finite volume: one-loop and two-loop-two-flavour known
  Colangelo,Haefeli, 2006
- Want to know also different mass case to two loops
- $1/m_\pi = 1.4$ fm
  may need to go beyond leading $e^{-m_\pi L}$ terms
- Convergence of ChPT is given by $1/m_\rho \approx 0.25$ fm

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The underlying formulas

- Underlying formula in one dimension
  periodic boundary condition $F(x=0)=F(x=L)$

$$\int \frac{dp}{2\pi} F(p) \longrightarrow \frac{1}{L}\sum_{p_n=2\pi n/L} F(p_n) \equiv \int_L \frac{dp}{2\pi} F(p)$$

- Poisson summation formula

$$\frac{1}{L}\sum_{p_n=2\pi n/L} F(p_n) = \sum_{\ell=nL}\int \frac{dp}{2\pi} e^{i\ell p} F(p)$$

- If twist angle θ, $\phi(L)=e^{-i\theta}\phi(0)$: $p_n = \frac{2\pi}{L}n + \frac{\theta}{L}$
- Poisson summation formula

$$\frac{1}{L}\sum_{p_n=2\pi n/L+\theta/L} F(p_n) = \sum_{\ell=nL}\int \frac{dp}{2\pi} e^{i(\ell p-\ell(\theta/L))} F(p)$$

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One-loop tadpole

$$\lfloor X \rfloor = \int_V \frac{d^d r}{(2\pi)^d} \frac{X}{(r^2+m^2)^n},$$

Poisson trick for three spatial dimensions:

$$\lfloor X \rfloor = \sum_{l_r} \int \frac{d^d r}{(2\pi)^d} \frac{X\, e^{il_r \cdot r - il_r \cdot \Theta}}{(r^2+m^2)^n},$$

$l_r = (0, n_1 L, n_2 L, n_3 L)$, $\Theta = (0, \vec{\theta}/L)$

Split in infinite volume $l_r = 0$ term and rest

$$\lfloor X \rfloor = \lfloor X \rfloor^\infty + \lfloor X \rfloor^V$$

Bring up denominator using 'α' parameters: $1/a = \int_0^\infty d\lambda e^{-\lambda a}$

$$\lfloor 1 \rfloor^V = \frac{1}{\Gamma(n)} \sum_{l_r}{}' \int \frac{d^d r}{(2\pi)^d} \int_0^\infty d\lambda \lambda^{n-1}\, e^{il_r \cdot r - il_r \cdot \Theta} e^{-\lambda(r^2+m^2)}$$

$\sum_{l_r}'$ means sum without $l_r = 0$ (all components zero)

One-loop tadpole

Shift $r = \bar{r} + i l_r/(2\lambda)$ to $\lfloor 1 \rfloor^V = \frac{1}{\Gamma(n)} \sum_{l_r}' \int_0^\infty d\lambda \lambda^{n-1} \, e^{-\lambda m^2 - \frac{l_r^2}{4\lambda} - i l_r \cdot \Theta} \int \frac{d^d \bar{r}}{(2\pi)^d} e^{-\lambda \bar{r}^2}$

Master formula for tadpoles:

$$\boxed{\lfloor 1 \rfloor^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \sum_{l_r}' \int_0^\infty d\lambda \lambda^{n-\frac{d}{2}-1} \, e^{-\lambda m^2 - \frac{l_r^2}{4\lambda} - i l_r \cdot \Theta}}$$

- Do the λ integral — Gasser, Leutwyler, 1988
  leads to sums over Bessel functions
- Do the $\sum_{l_r}$ — Becirevic, Villadoro, 2003
  leads to an integral over Jacobi theta functions

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One-loop tadpole: Bessel functions

We use

$$\mathcal{K}_\nu(Y,Z) = \int_0^\infty d\lambda\, \lambda^{\nu-1} e^{-Z\lambda - Y/\lambda} = 2\left(\frac{Y}{Z}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{YZ}\right)$$

to obtain

$$\boxed{\lfloor 1 \rfloor^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \sum_{l_r}{}' e^{-il_r\cdot\Theta} \mathcal{K}_{n-\frac{d}{2}-1}\left(m^2, \frac{l_r^2}{4}\right)}$$

- $d = 4 - 2\epsilon$ and can expand in ϵ
- Triple sum can be simplified: $\sum'_{l_r} f(l_r^2) = \sum_{k>0} x(k) f(k)$
  $k = l_r^2$ and $x(k)$ number of times $l_r^2 = kL^2$
- $K_i(mL\sqrt{k}) \approx \sqrt{\frac{\pi}{2mL\sqrt{k}}} e^{-mL\sqrt{k}}$

One-loop tadpole: (Jacobi) theta functions

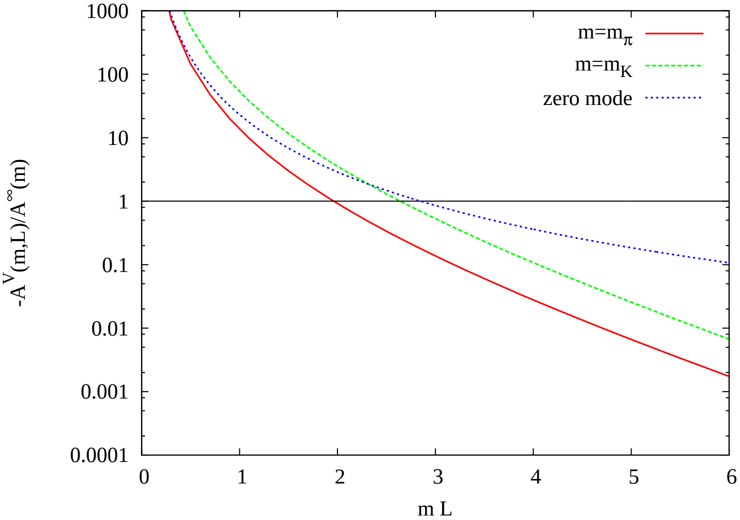
Define $\theta_3(u|\tau) = \sum_n e^{i\pi\tau n^2 + 2\pi i u n}$ which satisfies

$\theta_3(u+n|\tau) = \theta_3(u|\tau)$ and $\theta_3(u|\tau) = \frac{1}{\sqrt{-i\tau}} e^{-\pi i \frac{u^2}{\tau}} \theta_3\left(\frac{u}{\tau}\middle| \frac{-1}{\tau}\right)$

$$\boxed{\begin{aligned}\lfloor 1 \rfloor^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \int_0^\infty d\lambda \lambda^{n-\frac{d}{2}-1}\, e^{-\lambda m^2} \times \\ \left[\Pi_{j=x,y,z}\theta_3\left(-\theta_j/(2\pi)|iL^2/(4\pi\lambda)\right) - 1\right]\end{aligned}}$$

- No twisting, goes to a cubed theta function
- rescale λ to get L out of the argument
- Small λ, use the identity above, large λ sum converges fast

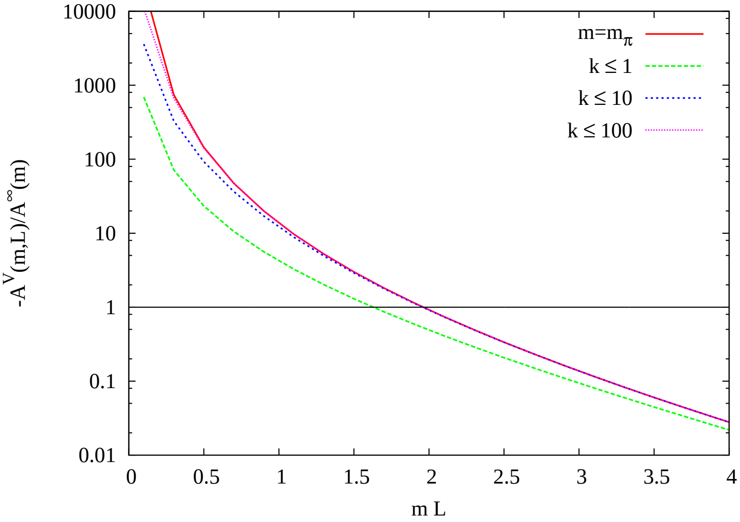
A numerical example



Relative correction to the infinite volume integral

$$A = \int_V \frac{d^d p}{(2\pi)^d} \frac{1}{p^2+m^2}$$

A numerical example



Convergence of the Bessel sum as a function of k

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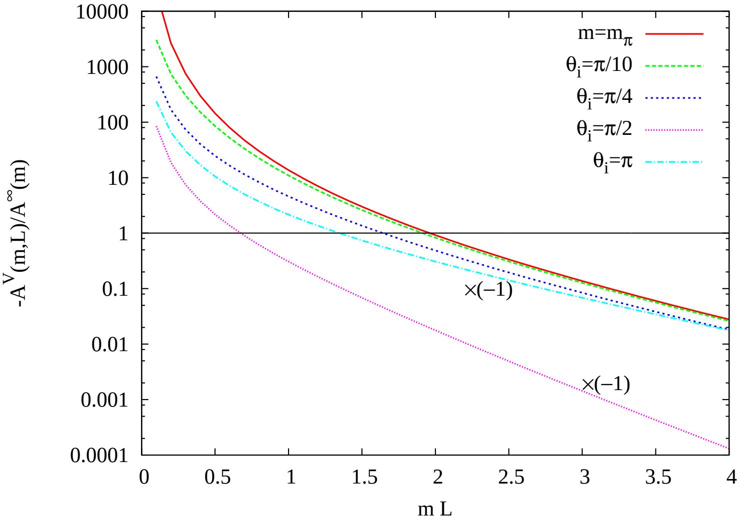
A numerical example



For various valus of $\theta_x = \theta_y = \theta_z = \theta_i$

For $\theta = \pi/2$ all terms with an $(l_r)_i$ odd cancel

One-loop integrals

Integrals with numerators $r_\mu r_\nu$

- Shift with $r = \bar{r} + i l_r/(2\lambda)$
- Integrals then done with $\bar{r}_\mu \bar{r}_\nu \to \bar{r}^2 \delta_{\mu\nu}/d$
- Similar for more complicated numerators
- But extra term show up:
  box and twisting break Lorentz invariance

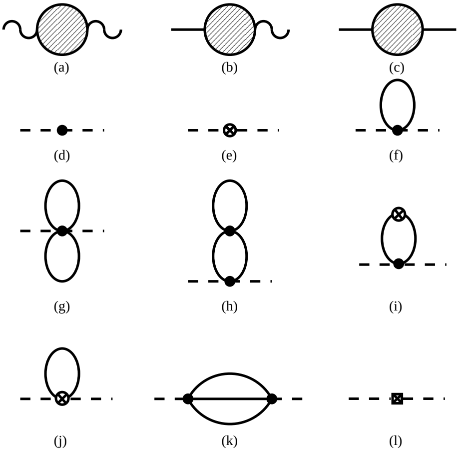
Integrals with more denominators and external momentum

- Combine denominators with Feynman parameters
- Shift with $r = \bar{r} + i l_r/(2\lambda) + (1-x)p$
- Gives extra factors like $e^{-i(1-x)l_r\cdot p}$
- Center of mass system $p = (p, 0, 0, 0) \Longrightarrow l_r \cdot p = 0$
- Moving frame: deal with as for twist angle

All the previous goes through (stay below threshold)

One-loop integrals

Corrections to masses and decay constants involve



Diagrams (f)-(j) can be done as earlier

(k) or the sunset needs new methods

Sunset overall

$$\langle\langle X \rangle\rangle = \int_V \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{X}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}}$$

Poisson summation twice

$$\langle\langle X \rangle\rangle = \sum_{l_r,l_s} \int \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{X\, e^{il_r\cdot r} e^{il_s\cdot s}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}}$$

Stick to simplest case $X = 1$, $n_1 = n_2 = n_3 = 1$ and $p = (p, 0, 0, 0)$

l_r and l_s are of the form $l_i = (0, n_1 L, n_2 L, n_3 L)$

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Divide and conquer

Split off part with $l_r = l_s = 0$

$$\langle\langle X\rangle\rangle = \langle\langle X\rangle\rangle^\infty + \langle\langle X\rangle\rangle^V,$$

Split of where only one loop momentum feels the boundary

$$\langle\langle 1\rangle\rangle^V = \langle\langle 1\rangle\rangle_r + \langle\langle 1\rangle\rangle_s + \langle\langle 1\rangle\rangle_t + \langle\langle 1\rangle\rangle_{rs},$$

Defined as $\langle\langle 1\rangle\rangle_{\{r,s,t,rs\}} = \left\{ \sum_{l_r}{}', \sum_{l_s}{}', \sum_{l_t}{}', \sum_{l_r,l_s}{}'' \right\} \times$

$$\int \frac{d^d r}{(2\pi)^d}\frac{d^d s}{(2\pi)^d} \frac{\{e^{il_r\cdot r}, e^{il_s\cdot s}, e^{il_t\cdot(r+s)}, e^{il_r\cdot r}e^{il_s\cdot s}\}}{(r^2+m_1^2)(s^2+m_2^2)((r+s-p)^2+m_3^2)}$$

Sums are over

$\{l_r \neq 0, l_s = 0, l_r = 0, l_s \neq 0, l_t \equiv l_r = l_s \neq 0, l_r \neq 0, l_s \neq 0, l_r \neq l_s\}$

$\langle\langle 1 \rangle\rangle_r, \langle\langle 1 \rangle\rangle_s, \langle\langle 1 \rangle\rangle_t$

- We have sums over full momentum integrals
- $(r \leftrightarrow s, l_r \leftrightarrow l_s) \Longrightarrow$
  $$\langle\langle 1 \rangle\rangle_s(m_1, m_2, m_3) = \langle\langle 1 \rangle\rangle_r(m_2, m_1, m_3)$$
- $(r \leftrightarrow t = r - s - p, l_t \leftrightarrow -l_r, e^{il_t \cdot p} = 1) \Longrightarrow$
  $$\langle\langle 1 \rangle\rangle_t(m_1, m_2, m_3) = \langle\langle 1 \rangle\rangle_r(m_3, m_2, m_1)$$
- So we only need $\langle\langle 1 \rangle\rangle_r$

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$\langle\langle 1 \rangle\rangle_r$

- The s integral is standard infinite volume
- $\langle\langle 1 \rangle\rangle_r = \sum_{l_r}' \int \frac{d^d r}{(2\pi)^d} \frac{e^{i l_r \cdot r}}{(r^2 + m_1^2)} \int_0^1 dx \frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \left(\overline{m}^2\right)^{\frac{d}{2}-2}$
- $\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \left(\overline{m}^2\right)^{\frac{d}{2}-2} = \frac{1}{16\pi^2}\left[\lambda_0 - 1 - \log(\overline{m}^2)\right] + \mathcal{O}(\epsilon)$

  $\lambda_0 = 1/\epsilon + \log(4\pi) + 1 - \gamma.$
  $\overline{m}^2 = (1-x)m_2^2 + x m_3^2 + x(1-x)(r-p)^2$

- The part containing λ_0 we call $\langle\langle 1 \rangle\rangle_{r,A}$
  should cancel in the final result (of a physical quantity), so ignore for now
- Do partial integration in x for the $\log(\overline{m}^2)$ term
- $\langle\langle 1 \rangle\rangle_r = \langle\langle 1 \rangle\rangle_{r,A} + \langle\langle 1 \rangle\rangle_{r,G} + \langle\langle 1 \rangle\rangle_{r,H}$

  $\langle\langle 1 \rangle\rangle_{r,G} = -\frac{1+\log(m_3^2)}{16\pi^2}\overline{A}^V(m_1^2)$

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$\langle\langle 1\rangle\rangle_r$

$$\langle\langle 1\rangle\rangle_{r,H} =$$
$$\frac{1}{16\pi^2}\sum_{l_r}{}' \int \frac{d^4 r}{(2\pi)^4}\frac{e^{il_r\cdot r}}{(r^2+m_1^2)}\int_0^1 dx x\frac{m_3^2-m_2^2+(1-2x)(r-p)^2}{\overline{m}^2}$$

Bring up denominators with 'α' parameters, shift r, do $\tilde{r}$ integral and finally symmetrize expression

$$\boxed{\begin{aligned}\langle\langle 1\rangle\rangle_{r,H} &= \frac{-1}{(16\pi^2)^2}\sum_{l_r}{}'\int_0^\infty d\lambda_1 d\lambda_2 d\lambda_3 \frac{\lambda_3}{\tilde{\lambda}^2}e^{-M^2}\times\\ &\left(m_3^2-m_2^2+\frac{\lambda_3-\lambda_2}{\tilde{\lambda}}\left(2+\frac{\lambda_3+\lambda_2}{\tilde{\lambda}}\tilde{p}^2\right)\right)\end{aligned}}$$

with $M^2 = \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 + \frac{\lambda_1\lambda_2\lambda_3}{\tilde{\lambda}}p^2 + \frac{\lambda_2+\lambda_3}{\tilde{\lambda}}\frac{l_r^2}{4}$

$\tilde{p} = \frac{il_r}{2} - \lambda_1 p, \qquad \tilde{\lambda} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$

Triple integral triple sum

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Two momenta quantized: $\langle\langle 1 \rangle\rangle_{rs}$

- Bring up denominators
- Shift integration momenta, do the momentum integrals
- Finite, set $d = 4$

$$\boxed{\langle\langle 1 \rangle\rangle_{rs} = \frac{1}{(16\pi^2)^2} \sum_{l_r, l_s}'' \int_0^\infty d\lambda_1 d\lambda_2 d\lambda_3 \, \tilde{\lambda}^{-2} e^{-\tilde{M}^2}}$$

$$\tilde{M}^2 = \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 + \frac{\lambda_1\lambda_2\lambda_3}{\tilde{\lambda}} p^2 + \frac{\lambda_2}{\tilde{\lambda}} \frac{l_r^2}{4} + \frac{\lambda_1}{\tilde{\lambda}} \frac{l_s^2}{4} + \frac{\lambda_3}{\tilde{\lambda}} \frac{(l_r - l_s)^2}{4}$$

$$\tilde{\lambda} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

Triple integral, sextuple sum (very similar to previous page)

Numerical sunset

- $\lambda_1 = \lambda x$, $\lambda_2 = \lambda y$, $\lambda_3 = \lambda(1 - x - y)$
- Do λ integral: sums over bessel functions
  sextuple sum to triple sum over $k_1 = l_r^2$, $k_2 = l_s^2$,
  $k_3 = (l_r - l_s)^2$ and $x(k_1, k_2.k_3)$
- Do sum in terms of Riemann or Siegel theta function:

$$\theta^{(g)}(z|\tau) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i \left(\frac{1}{2} n^T \tau n + n^T z\right)}$$

Some properties:

$$\theta^{(g)}(z|\tau) = \theta^{(g)}(az|a\tau a^T) \text{ (a and a^{-1} integer)}$$

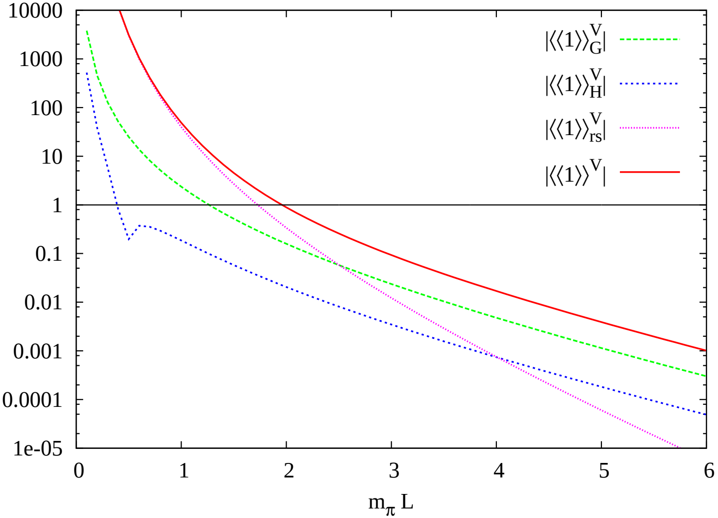
$$\theta^{(g)}(\tau^{-1}z| - \tau^{-1}) = \sqrt{\det(-i\tau)} e^{\pi i z^T \tau^{-1} z} \theta^{(g)}(z|\tau)$$

Can be used to speed up computation

Numerics

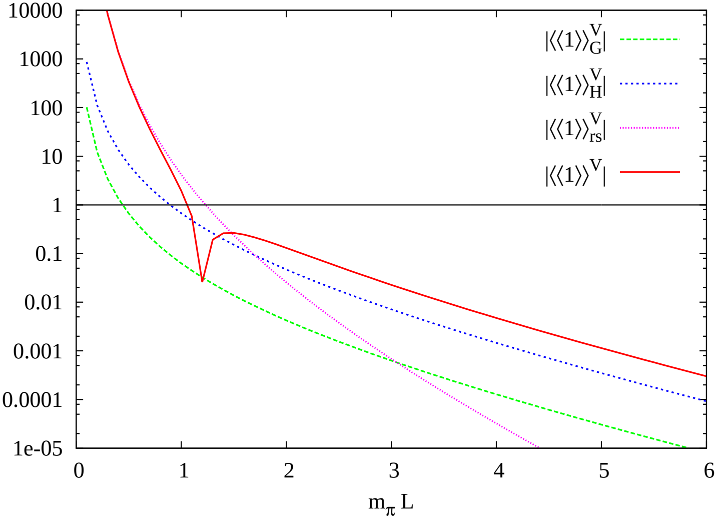
- Getting 5-6 digits of precision for $\langle\langle 1\rangle\rangle_{rs}$ goes fine (takes a while but not too bad)
- Only works below threshold
- large mL Bessel fastest
- small or medium mL theta function fastest
- $\langle\langle 1\rangle\rangle_{rs}$ Bessel only implemented to $l_i^2 < 40$
- The two methods always agree

Numerics



$m_1 = m_2 = m_3 = m_\pi$, $p^2 = -m_\pi^2$ (pion on shell)
Relative to $\langle\langle 1\rangle\rangle^\infty = 3.7384\ 10^{-5}$ GeV^2
Funny bumps: result went through zero

Numerics



$m_1 = m_2 = m_\pi$, $m_3 = m_K$, $p^2 = -m_K^2$ (kaon on shell)
Relative to $\langle\langle 1\rangle\rangle^\infty = 6.7407\ 10^{-5}\ \mathrm{GeV}^2$
Funny bumps: result went through zero

Conclusions and future

Conclusions

- An important hurdle in two-loop ChPT at finite volume taken
- The various pieces in the sunset are all needed dependent on the inputs used
- Riemann theta function in all its varieties not present in mathematica (need its derivatives)

Future

- The cases with numerators are in progress
- Moving frame and/or twisting not studied yet, but I see no obvious new problems appearing
- Then need to redo the two-loop 3-flavour ChPT from scratch since not all integral relations at infinite volume remain valid

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