$D$ and $D_s$ decay constants from a chiral analysis on HISQ ensembles

Speaker: Claude Bernard  
Washington University  
St. Louis, USA

Fermilab Lattice and MILC Collaborations


*Ph.D thesis research

Lattice 2013, Mainz, Germany
For higher precision than available with the asqtad action, we have moved to the HISQ action [Follana et al. [HPQCD], PRD 75 (2007) 054502].

- Reduced $O(\alpha_s a^2)$ and $O(\alpha_s^2 a^2)$ [taste-violation] errors with respect to asqtad.
- $\alpha_s (am_c)^2$, $(am_c)^4$ errors also reduced.
  - Non-relativistic expansion says that $m_c$ errors further reduced in heavy-light physics by powers of charm quark velocity.
  - $\Rightarrow$ treat charm with same relativistic action as light quarks.

- Ensembles include charm sea quarks:
  - Although error of quenching charm is probably quite small in most cases, at today’s level of precision it is safer to include charm in the sea; doesn’t cost much.
D meson decay constants with HISQ

🔹 Advantage of HISQ is that charm may be treated with same action as light quarks.
  • avoid renormalization errors and many tuning issues.
  • share to some degree the small statistical errors of staggered light pseudoscalars.
  • how large a value of $m_c a$ may reasonably be used is not obvious a priori.
    • HPQCD has included HISQ valence on asqtad lattices as coarse as $a=0.15$ fm, for which $m_c a = 0.85$.
    • We consider HISQ on HISQ up to $a=0.15$ fm also.
      – Power counting estimate is that errors are ~5%, with further reduction by dimensionless factors possible.
      – We currently are keeping $a=0.15$ fm data in central values, but compare with fits dropping it in systematic error estimate.
### Ensembles Used

Ensembles with physical strange quark mass:

<table>
<thead>
<tr>
<th>$\approx a$ (fm)</th>
<th>$m_l/m_s$</th>
<th>$N_s^3 \times N_t$</th>
<th>$M_\pi L$</th>
<th>$M_\pi$ (MeV)</th>
<th>$N_{\text{lat}}$</th>
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</thead>
<tbody>
<tr>
<td>0.15</td>
<td>1/5</td>
<td>$16^3 \times 48$</td>
<td>3.78</td>
<td>306.9(5)</td>
<td>1020</td>
</tr>
<tr>
<td>0.15</td>
<td>1/10</td>
<td>$24^3 \times 48$</td>
<td>3.99</td>
<td>214.5(2)</td>
<td>1000</td>
</tr>
<tr>
<td>0.15</td>
<td>1/27</td>
<td>$32^3 \times 48$</td>
<td>3.30</td>
<td>131.0(1)</td>
<td>1000</td>
</tr>
<tr>
<td>0.12</td>
<td>1/5</td>
<td>$24^3 \times 64$</td>
<td>4.54</td>
<td>305.3(4)</td>
<td>1040</td>
</tr>
<tr>
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<tr>
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<td>$32^3 \times 64$</td>
<td>4.29</td>
<td>216.9(2)</td>
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<tr>
<td>0.12</td>
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<td>217.0(2)</td>
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<tr>
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<td>$96^3 \times 192$</td>
<td>3.95</td>
<td>135.5(2)</td>
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</tbody>
</table>

*Red = ensemble generation still in progress*
Ensembles Used

Ensembles with strange quark mass lighter than physical:

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<th>$m'_s/m_s$</th>
<th>$N_s^3 \times N_t$</th>
<th>$N_{lats}$</th>
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<td>$32^3 \times 64$</td>
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<td>0.10</td>
<td>0.45</td>
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<tr>
<td>0.12</td>
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<td>0.60</td>
<td>$32^3 \times 64$</td>
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<td>0.25</td>
<td>0.25</td>
<td>$24^3 \times 64$</td>
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<tr>
<td>0.12</td>
<td>0.175</td>
<td>0.45</td>
<td>$32^3 \times 64$</td>
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</table>

(These ensembles are not crucial to D decay project, but are useful for adjusting for mistunings in strange (and light) masses.)
MILC HISQ Ensembles

$N_f = 2+1+1$ Hisq MILC ensembles

$M_{\pi}$ [MeV] vs $a$ [fm]

- Red circles: completed
- Green circles: in progress
- Red crosses: planned
- Green up arrow: unphysical $m_s$

3 volumes
MILC HISQ Ensembles

$\mathcal{N}_f = 2+1+1$ Hisq MILC ensembles

![Graph showing $M_\pi$ vs $a$ for different sea masses and ensemble completions.](image)

Sea masses: $m_{\pi}^{RMS}$ [MeV]

- completed
- in progress
- planned
- unphysical $m_s$

Ensembles:
- 3 volumes

C. Bernard, Lattice 2013, 7/31/13
## Valence Masses Used

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<thead>
<tr>
<th>$\beta$</th>
<th>$am_l$</th>
<th>$am_s$</th>
<th>$am_C$</th>
<th>light masses $m_x$</th>
<th>heavy mass $m_y$</th>
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<td></td>
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<td>$(m_x/m_y)$</td>
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<td>5.80</td>
<td>0.013</td>
<td>0.065</td>
<td>0.838</td>
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<td>0,9,1.0</td>
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<td>6.00</td>
<td>0.0102</td>
<td>0.0509</td>
<td>0.635</td>
<td>0.1,0.15,0.2,0.3,0.4,0.6,0.8,1.0</td>
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<td>0.0507</td>
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Physical sea mass ensembles have physical valence masses. (Volumes chosen appropriately.)
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<td>0.0008</td>
<td>0.022</td>
<td>0.260</td>
<td>0.036, 0.068, 0.1, 0.15, 0.2, 0.3, 0.4, 0.6, 0.8, 1.0</td>
<td>0.9, 1.0</td>
</tr>
</tbody>
</table>

Physical sea mass ensembles have physical valence masses. (Volumes chosen appropriately.)

C. Bernard, Lattice 2013, 7/31/13
Chiral Perturbation Theory

✧ J. Komijani and CB have worked out appropriate heavy-meson chiral perturbation theory when both light and heavy quarks are staggered: “Heavy-meson, rooted, all-staggered chiral perturbation theory” (HM\textit{r\textit{AS}}\textit{χPT}) [arXiv:1211.0785, and to appear].
  
  • Used here to fit all heavy-light data and interpolate/extrapolate to physical quark masses and to continuum.
  
  • Useful also for understanding the pattern of taste-symmetry breaking in the heavy-light meson masses [\textit{MILC: A. Bazavov et al.}, PRD 87 054505 (2013)].
Chiral Perturbation Theory

An alternative, simpler analysis that avoids $\chi$PT and focuses on the physical-mass ensembles will be presented by Doug Toussaint tomorrow (session 8C).

- That analysis also includes light-light mesons and provides the absolute scale setting (through $f_\pi$) and physical quark masses needed here.

- At the cost of significant complications, the $\chi$PT analysis allows us to use all our data (not just physical-mass ensembles), thereby reducing statistical errors, and to help control the continuum extrapolation.

- Ultimately will extend the $\chi$PT analysis to the light-light sector to have a completely self-contained version of the analysis.
Chiral Perturbation Theory

✦ NLO form in HM\textsubscript{rAS}χPT, including hyperfine and flavor splittings in heavy-light masses:

\[
\frac{f_{D_{x\Xi}}}{\kappa} \sqrt{M_{D_{x\Xi}}} = 1 + \frac{1}{16\pi^2 f^2} \frac{1}{2} \left\{ -\frac{1}{16} \sum_{\delta,\Xi'} \ell(m^2_{S_{x,\Xi'}}) \right. \\
- \frac{1}{3} \sum_{j \in M^{(3,x)}_I} \frac{\partial}{\partial m^2_{X,I}} \left[ R_j^{[3,3]} (M^{(3,x)}_I; \mu^{(3)}_I) \ell(m^2_j) \right] \\
- \left( a^2 \delta' \sum_{j \in M^{(4,x)}_V} \frac{\partial}{\partial m^2_{X,V}} \left[ R_j^{[4,3]} (M^{(4,x)}_V; \mu^{(3)}_V) \ell(m^2_j) \right] + [V \rightarrow A] \right) \\
- 3g^2_{\pi} \frac{1}{16} \sum_{\delta,\Xi'} J(m_{S_{x,\Xi'}}, \Delta^*_x + \delta_{S_{x}}) \\
- g^2_{\pi} \sum_{j \in M^{(3,x)}_I} \frac{\partial}{\partial m^2_{X,I}} \left[ R_j^{[3,3]} (M^{(3,x)}_I; \mu^{(3)}_I) J(m_j, \Delta^*_x) \right] \\
- 3g^2_{\pi} \left( a^2 \delta' \sum_{j \in M^{(4,x)}_V} \frac{\partial}{\partial m^2_{X,V}} \left[ R_j^{[4,3]} (M^{(4,x)}_V; \mu^{(3)}_V) J(m_j, \Delta^*_x) \right] + [V \rightarrow A] \right) \left\} \right.
\]

\[+ c_s (m_u + m_d + m_s) + c_v m_x + c_{a,\Xi} a^2.\]
Chiral Perturbation Theory

✧ NLO form in HMrASχPT, including hyperfine and flavor splittings in heavy-light masses:

\[
\frac{f_{D_x\Xi}}{\kappa} = 1 + \frac{1}{16\pi^2 f^2} \frac{1}{2} \left\{ \frac{1}{16} \sum_{s,\Xi'} \ell(m_{Sx,\Xi'}^2) \right\} - \frac{1}{3} \sum_{j \in \mathcal{M}_I^{(3,x)}} \frac{\partial}{\partial m_{X,I}^2} \left[ R_j^{[3,3]}(\mathcal{M}_I^{(3,x)}; \mu_I^{(3)}) \ell(m_j^2) \right]

- \left( a^2 \delta_V' \sum_{j \in \mathcal{M}_V^{(4,x)}} \frac{\partial}{\partial m_{X,V}^2} \left[ R_j^{[4,3]}(\mathcal{M}_V^{(4,x)}; \mu_V^{(3)}) \ell(m_j^2) \right] + [V \rightarrow A] \right)

- 3g_\pi^2 \frac{1}{16} \sum_{s,\Xi'} J(m_{Sx,\Xi'}, \Delta^* + \delta_{Sx})

- g_\pi^2 \sum_{j \in \mathcal{M}_I^{(3,x)}} \frac{\partial}{\partial m_{X,I}^2} \left[ R_j^{[3,3]}(\mathcal{M}_I^{(3,x)}; \mu_I^{(3)}) J(m_j, \Delta^*) \right]

- 3g_\pi^2 \left( a^2 \delta_V' \sum_{j \in \mathcal{M}_V^{(4,x)}} \frac{\partial}{\partial m_{X,V}^2} \left[ R_j^{[4,3]}(\mathcal{M}_V^{(4,x)}; \mu_V^{(3)}) J(m_j, \Delta^*) \right] + [V \rightarrow A] \right) \right\}

+ c_s (m_u + m_d + m_s) + c_v m_x + c_{a,\Xi} a^2.

\]
Chiral Perturbation Theory

**NLO form in HMrASχPT, including hyperfine and flavor splittings in heavy-light masses:**

\[
\frac{f_{Dx} \sqrt{M_{Dx}}}{\kappa} = 1 + \frac{1}{16\pi^2 f^2} \frac{1}{2} \left\{ -\frac{1}{16} \sum_{s,\Xi'} \ell(m_{sx,\Xi'}) \\
- \frac{1}{3} \sum_{j \in \mathcal{M}_I^{(3,x)}} \frac{\partial}{\partial m_{X,I}^2} \left[ R_j^{[3,3]} \left( \mathcal{M}_I^{(3,x)} ; \mu_I^{(3)} \right) \ell(m_j^2) \right] \\
- \left( a^2 \delta'_V \sum_{j \in \mathcal{M}_V^{(4,x)}} \frac{\partial}{\partial m_{X,V}^2} \left[ R_j^{[4,3]} \left( \mathcal{M}_V^{(4,x)} ; \mu_V^{(3)} \right) \ell(m_j^2) \right] + [V \to A] \right) \\
- \frac{3g_\pi^2}{16} \sum_{s,\Xi'} J(m_{sx,\Xi'}, \Delta^* + \delta_{sx}) \\
- g_\pi^2 \sum_{j \in \mathcal{M}_I^{(3,x)}} \frac{\partial}{\partial m_{X,I}^2} \left[ R_j^{[3,3]} \left( \mathcal{M}_I^{(3,x)} ; \mu_I^{(3)} \right) J(m_j, \Delta^*) \right] \\
- \frac{3g_\pi^2}{16} \left( a^2 \delta'_V \sum_{j \in \mathcal{M}_V^{(4,x)}} \frac{\partial}{\partial m_{X,V}^2} \left[ R_j^{[4,3]} \left( \mathcal{M}_V^{(4,x)} ; \mu_V^{(3)} \right) J(m_j, \Delta^*) \right] + [V \to A] \right) \right\} \\
+ c_s(m_u + m_d + m_s) + c_v m_x + c_{a,\Xi} a^2.
\]
Chiral Perturbation Theory

✦ NLO form in HM\(\chi\)PT, including hyperfine and flavor splittings in heavy-light masses:

\[
\frac{f_{D_x \Xi}}{\kappa} = 1 + \frac{1}{16\pi^2 f^2} \frac{1}{2} \left\{ - \frac{1}{16} \sum_{\bar{s}, \Xi'} \ell(m^2_{\bar{s}x, \Xi'}) 
\right.
\]

\[
- \frac{1}{3} \sum_{j \in \mathcal{M}_{I}^{(3,x)}} \frac{\partial}{\partial m^2_{X,I}} \left[ R_j^{[3,3]} (\mathcal{M}^{(3,x)}_I; \mu_I^{(3)}) \ell(m^2_j) \right] 
\]

\[
- \left( a^2 \delta' \sum_{j \in \mathcal{M}_V^{(4,x)}} \frac{\partial}{\partial m^2_{X,V}} \left[ R_j^{[4,3]} (\mathcal{M}^{(4,x)}_V; \mu_V^{(3)}) \ell(m^2_j) \right] + [V \to A] \right) 
\]

\[
- \frac{3g_\pi^2}{16} \sum_{\bar{s}, \Xi'} J(m_{\bar{s}x, \Xi'}, \Delta^* + \delta_{\bar{s}x}) 
\]

\[
- g^2_\pi \sum_{j \in \mathcal{M}_{I}^{(3,x)}} \frac{\partial}{\partial m^2_{X,I}} \left[ R_j^{[3,3]} (\mathcal{M}^{(3,x)}_I; \mu_I^{(3)}) J(m_j, \Delta^*) \right] 
\]

\[
- 3g^2_\pi \left( a^2 \delta' \sum_{j \in \mathcal{M}_V^{(4,x)}} \frac{\partial}{\partial m^2_{X,V}} \left[ R_j^{[4,3]} (\mathcal{M}^{(4,x)}_V; \mu_V^{(3)}) J(m_j, \Delta^*) \right] + [V \to A] \right) \}
\]

\[
+ c_s (m_u + m_d + m_s) + c_v m_x + c_{a,\Xi} a^2 .
\]
Chiral Perturbation Theory

**✦ NLO form in HMrASχPT, including hyperfine and flavor splittings in heavy-light masses:**

\[
\frac{f_{D_x} \pm \sqrt{M_{D_x}}}{\kappa} = 1 + \frac{1}{16\pi^2 f^2} \frac{1}{2} \left\{ -\frac{1}{16} \sum_{S,\Xi'} \ell(m_{Sx,\Xi}') - \frac{1}{3} \sum_{j \in \mathcal{M}_{I}^{(3,x)}} \frac{\partial}{\partial m_{X,I}^2} \left[ R_{j}^{[3,3]}(\mathcal{M}_{I}^{(3,x)}; \mu_{I}^{(3)}) \ell(m_{j}^2) \right] \right. \\
\left. - \left( a^2 \delta' V \sum_{j \in \mathcal{M}_{V}^{(4,x)}} \frac{\partial}{\partial m_{X,V}^2} \left[ R_{j}^{[4,3]}(\mathcal{M}_{V}^{(4,x)}; \mu_{V}^{(3)}) \ell(m_{j}^2) \right] + [V \to A] \right) \\
- 3g_\pi^2 \frac{1}{16} \sum_{S,\Xi'} J(m_{Sx,\Xi'}, \Delta^* + \delta_{Sx}) \\
- g_\pi^2 \sum_{j \in \mathcal{M}_{I}^{(3,x)}} \frac{\partial}{\partial m_{X,I}^2} \left[ R_{j}^{[3,3]}(\mathcal{M}_{I}^{(3,x)}; \mu_{I}^{(3)}) I(m_{j}, \Delta^*) \right] \\
- 3g_\pi^2 \left( a^2 \delta' V \sum_{j \in \mathcal{M}_{V}^{(4,x)}} \frac{\partial}{\partial m_{X,V}^2} \left[ R_{j}^{[4,3]}(\mathcal{M}_{V}^{(4,x)}; \mu_{V}^{(3)}) I(m_{j}, \Delta^*) \right] + [V \to A] \right) \right\} \\
+ c_s (m_u + m_d + m_s) + c_v m_x + c_{a,\Xi} a^2 .
\]
Chiral Perturbation Theory

* NLO form in HMṛASχPT, including hyperfine and flavor splittings in heavy-light masses:

\[
\frac{f_{D_x \xi}}{\kappa} = 1 + \frac{1}{16\pi^2 f^2} \frac{1}{2} \left\{ -\frac{1}{16} \sum_{\bar{s}, \Xi} \ell(m_{\bar{s}x, \Xi}) 
\right.
\]

\[
- \frac{1}{3} \sum_{j \in \mathcal{M}_I^{(3, x)}} \frac{\partial}{\partial m_{X, I}^2} \left[ R_j^{[3, 3]}(\mathcal{M}_I^{(3, x)}; \mu_I^{(3)})\ell(m_j^2) \right] 
\]

\[
- \left( a^2 \delta'_V \sum_{j \in \mathcal{M}_V^{(4, x)}} \frac{\partial}{\partial m_{X, V}^2} \left[ R_j^{[4, 3]}(\mathcal{M}_V^{(4, x)}; \mu_V^{(3)})\ell(m_j^2) \right] + [V \to A] \right) 
\]

\[
- 3g_\pi^2 \frac{1}{16} \sum_{\bar{s}, \Xi} J(m_{\bar{s}x, \Xi}, \Delta^* + \delta_{\bar{s}x}) 
\]

\[
- g_\pi^2 \sum_{j \in \mathcal{M}_I^{(3, x)}} \frac{\partial}{\partial m_{X, I}^2} \left[ R_j^{[3, 3]}(\mathcal{M}_I^{(3, x)}; \mu_I^{(3)})J(m_j, \Delta^*) \right] 
\]

\[
- 3g_\pi^2 \left( a^2 \delta'_V \sum_{j \in \mathcal{M}_V^{(4, x)}} \frac{\partial}{\partial m_{X, V}^2} \left[ R_j^{[4, 3]}(\mathcal{M}_V^{(4, x)}; \mu_V^{(3)})J(m_j, \Delta^*) \right] + [V \to A] \right) \right\} 
\]

\[
+ c_s (m_u + m_d + m_s) + c_v m_x + c_{a, \Xi} a^2 .
\]

only taste-dependence at this order
Convenient to redefine LECs in terms of natural dimensionless factors of $\chi$PT:
\[
c_{s}(m_u + m_d + m_s) + c_vm_x + c_a a^2
\rightarrow L_s (x_u + x_d + x_s) + L_v (x_x) + L_a \frac{x_\bar{\Delta}}{2}
\]
\[
x_{u,d,s,x} \equiv \frac{4B}{16\pi^2 f_\pi^2} m_{u,d,s,x}, \quad x_\bar{\Delta} \equiv \frac{2}{16\pi^2 f_\pi^2} \bar{\Delta}
\]

where $B$ is the LEC that gives pion mass: $m_\pi^2 = B(m_u + m_d)$, and $\bar{\Delta}$ is the mean-squared pion taste splitting.

With these definitions, LECs are expected to be $O(1)$. 
Have very precise data (~0.2% stat errors), with ~200--366 points, depending on cuts.

Need to add higher-order analytic terms to the fit function:

• “Generic” $a$-dependence of NLO LECs (a NNLO effect).
  
  • e.g., $L_s (x_u + x_d + x_s) \rightarrow (L_s + L_s \delta \alpha S a^2) (x_u + x_d + x_s)$.
  
  • So add parameters $\kappa_\delta$, $L_s \delta$, $L_v \delta$, $\delta_{A \delta}'$, $\delta_{V \delta}'$, $g_{\pi \delta}^2$, $L a \delta$.

• NNLO and NNNLO terms in quark masses needed to fit masses $\sim m_s$:

<table>
<thead>
<tr>
<th>$q_1(x_x^2)$</th>
<th>$c_1(x_x)^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_2(2x_l + x_s)^2$</td>
<td>$c_2(x_x)(2x_l + x_s)^2$</td>
</tr>
<tr>
<td>$q_3(2x_l + x_s)(x_v)$</td>
<td>$c_3(x_x)(2x_l^2 + x_s^2)$</td>
</tr>
<tr>
<td>$q_4(2x_l^2 + x_s^2)$</td>
<td>$c_4(x_x)^2(2x_l + x_s)$</td>
</tr>
<tr>
<td>$c_5(2x_l + x_s)^3$</td>
<td>$c_6(2x_l + x_s)(2x_l^2 + x_s^2)$</td>
</tr>
<tr>
<td>$c_7(2x_l^3 + x_s^3)$</td>
<td></td>
</tr>
</tbody>
</table>
Because of mistunings in $m_c$, and especially if $m_c = 0.9 m_c^{\text{phys}}$ valence masses are included, need higher order HQET terms.

- multiply by $1 + c_{m_1} (\Lambda_{QCD} / m_c) + c_{m_2} (\Lambda_{QCD} / m_c)^2$, where we take $\Lambda_{QCD} \sim 350 \text{ MeV}$.

- If $\beta = 5.8$ ($a = 0.15$ fm) ensembles are included, additional discretization correction for large $am_c$ needed, although difficult to distinguish $\alpha_s (am_c)^2$ from $(am_c)^4$.  

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Lattice Scale

✧ Relative lattice scales are determined by $f_{p4s}$, the decay constant when valence masses are 0.4 $m_s^{\text{phys}}$.
  • (sea masses are physical).

✧ Has very small statistical errors (comparable to that of Symanzik or Wilson flow $w_0$ when computed on the same numbers of configurations).

✧ Absolute scale is set by computing $f_{p4s}/f_{\pi}$, extrapolated to the continuum.
Lattice Data

For each color, higher points have \( m_c = m_{c,\text{phys}} \); lower points have \( m_c = 0.9 m_{c,\text{phys}} \).

Data for unphysical \( m_s \) ensembles (as well as multiple \( a=0.12 \) fm volumes) not shown, but included in fits.

366 data points, total.
Central Chiral Fit

- Fit is to all 366 data points; 31 parameters.
- correlated $\chi^2$/dof=325/335; $p=0.64$.
- Black burst shows continuum-extrapolated physical result.
- Statistical errors from jackknife (including all inputs) are tiny.
Central Chiral Fit

- Fit is to all 366 data points; 31 parameters.
- correlated $\chi^2$/dof=325/335; $p=0.64$.
- Black burst shows continuum-extrapolated physical result.
- Statistical errors from jackknife (including all inputs) are tiny.
• Fit does not include $a=0.15$ fm data.
• Fit is to 314 data points; 29 parameters.
• Does not include $\alpha_s (am_c)^2, (am_c)^4$ terms.
• Correlated $\chi^2$/dof=298/285; $p=0.28$. 

An Alternative Chiral Fit
Dependence on $a^2$

- clear competition of $a^2$ and $a^4$ fit terms, needed for parabolic shape of data.
- total variation with $a$, as well as individual contributions to $a$-dependence, are ~ 2--3%.
- Scale dependence is different in physical-point analysis (Doug Toussaint’s talk), which uses $f_\pi$ for relative scale setting, but rough shape is similar.

- data (adjusted for mistunings to physical masses)
- staggered chiral log contribution
- contribution from fit $a$-dependence ($\alpha a^2$, $\alpha^2 a^2$, $a^4$)
Discussion of Systematic Errors

For continuum extrapolation/chiral interpolation errors, use two methods:

• By straightforward comparison with various continuum extrapolations of physical-mass ensemble results (Doug Toussaint’s talk).

• “Self-contained” error analysis:
  • Have 10 acceptable chiral fits (p>0.05), which:
    – keep or drop \( a=0.15 \text{ fm} \) ensembles.
    – keep or drop \( (a m_c)^4 \) and \( \alpha_s (a m_c)^2 \) terms.
    – constrain higher order chiral terms and/or discretization terms with priors, or leave them unconstrained.
  • Have 6 versions of inputs (quark masses, \( f_{p4s} \) in physical units from \( f_{hn} \)) from physical-mass ensemble results.
  • Histogram results of 60 composite analyses.
Continuum Extrapolation Error

\[ \Phi_D \equiv f_D + \sqrt{M_{D^+}} \]

\[ \Phi_{D_s} \equiv f_{D_s} \sqrt{M_{D_s}} \]
Continuum Extrapolation Error

\[ \Phi_D \equiv f_{D^+} \sqrt{M_{D^+}} \]

\[ \Phi_{D_s} \equiv f_{D_s} \sqrt{M_{D_s}} \]

“Central” fit
→ 9187 MeV\(^{3/2}\)
Continuum Extrapolation Error

\[ \Phi_D \equiv f_D + \sqrt{M_{D^+}} \]

\[ \Phi_{D_s} \equiv f_{D_s} \sqrt{M_{D_s}} \]

"Central" fit
\[ \rightarrow 9187 \text{ MeV}^{3/2} \]

Systematic error:
\[ +14 \text{ MeV}^{3/2} \]
\[ -47 \text{ MeV}^{3/2} \]
Continuum Extrapolation Error

\[ \Phi_D \equiv f_{D+} \sqrt{M_{D+}} \]

\[ \Phi_{D_s} \equiv f_{D_s} \sqrt{M_{D_s}} \]

“Central” fit
→ 9187 MeV^{3/2}

“Central” fit
→ 11045 MeV^{3/2}

Systematic error:
-47 MeV^{3/2}
+14 MeV^{3/2}
Continuum Extrapolation Error

\[ \Phi_D \equiv f_D + \sqrt{M_{D^+}} \]

\[ \Phi_{D_s} \equiv f_{D_s} \sqrt{M_{D_s}} \]

"Central" fit
\[ \rightarrow 9187 \text{ MeV}^{3/2} \]

"Central" fit
\[ \rightarrow 11045 \text{ MeV}^{3/2} \]

Systematic error:

-47 MeV\(^{3/2}\) → +14 MeV\(^{3/2}\)

-55 MeV\(^{3/2}\) → +13 MeV\(^{3/2}\)
Continuum Extrapolation Error
Continuum Extrapolation Error

\[ \Phi_{D_s}/\Phi_D \]

“Central” fit
\[ \rightarrow 1.2023 \]
Continuum Extrapolation Error

\[ \Phi_{D_s}/\Phi_D \]

"Central" fit
\[ \rightarrow 1.2023 \]

Systematic error:

-0.0025 to +0.0012
Continuum Extrapolation Error

- Roughly speaking, the chiral fits to account for ~2/3 of the variance, while the inputs of scale and quark masses account for ~1/3.

Systematic error:

-0.0025

+0.0012
Finite size effects

- “Direct” finite size effects on heavy-light decay constants and masses are negligible.
- But note that results here are in lattice units.
- Small but non-negligible finite-size effects enter from scale setting through $f_{\pi}$. 

![Graphs showing finite size effects](image_url)
EM effects included in error are only the ones coming from light quark masses.

Do not include effects on $m_c$ estimate (i.e. haven’t taken out EM effects from the $D_s$ mass, used to set $m_c$).

- This appears to be a not-insignificant source of error (especially for $f_{D_s}/f_D$).
- Can be relatively easily improved with an expansion of ongoing MILC EM project.

In addition, errors associated with matching complete theory to pure QCD are not included (would be relevant for comparing to experiment).

- Gläßle and Bali, arXiv:1111.3958 and Davies, et al., PRD 82 (2010) 114504 expect such errors are $< 0.5\%$
Results

- Preliminary results, from “self-contained” chiral fit analysis only:

\[
\begin{align*}
  f_D &= 212.5 \pm 0.5_{\text{stat}}^{+0.3}_{-1.1} |a^2\text{extrap} \pm 0.2_{\text{FV}} \pm 0.0_{\text{EM}} \pm 0.3_{f_\pi\text{expt}} \text{MeV} \\
  f_{D_s} &= 248.9 \pm 0.2_{\text{stat}}^{+0.3}_{-1.2} |a^2\text{extrap} \pm 0.2_{\text{FV}} \pm 0.1_{\text{EM}} \pm 0.4_{f_\pi\text{expt}} \text{MeV} \\
  f_{D_s}/f_D &= 1.1717(20)_{\text{stat}}^{+12}_{-24} |a^2\text{extrap}(3)_{\text{FV}}(3)_{\text{EM}} 
\end{align*}
\]

- Preliminary results, including comparison with continuum extrapolation of physical ensembles results only (Doug Toussaint talk):

\[
\begin{align*}
  f_D &= 212.5 \pm 0.5_{\text{stat}}^{+0.3}_{-1.4} |a^2\text{extrap} \pm 0.2_{\text{FV}} \pm 0.3_{\text{EM}} \pm 0.3_{f_\pi\text{expt}} \text{MeV} \\
  f_{D_s} &= 248.9 \pm 0.2_{\text{stat}}^{+0.3}_{-1.5} |a^2\text{extrap} \pm 0.2_{\text{FV}} \pm 0.1_{\text{EM}} \pm 0.4_{f_\pi\text{expt}} \text{MeV} \\
  f_{D_s}/f_D &= 1.1717(20)_{\text{stat}}^{+52}_{-24} |a^2\text{extrap}(4)_{\text{FV}}(5)_{\text{EM}} 
\end{align*}
\]

- For now, we’ve taken larger error values in each case to be conservative.
Results

✦ Summary of our best current results (still preliminary):

\[
\begin{align*}
 f_D &= 212.5 \pm 0.5_{\text{stat}}^{+0.6}_{-1.5}_{\text{sys}} \text{ MeV} \\
 f_{D_s} &= 248.9 \pm 0.2_{\text{stat}}^{+0.5}_{-1.6}_{\text{sys}} \text{ MeV} \\
 f_{D_s}/f_D &= 1.1717(20)_{\text{stat}}(+52)_{-25}^{+52}_{-25}_{\text{sys}} 
\end{align*}
\]

✦ In progress:

• finish 3 partial ensembles.
• chiral/continuum fits still need work:
  • Can we do with fewer parameters? Improve stability?
  • Understanding \(a\)-dependence better (esp. \(\alpha_s (am_c)^2\), \((am_c)^4\) terms).
  • Choosing a more “central” central fit would be preferable, if one can be found that has comparable p value to current one and reasonable consistency with expectations of PT.
• Extending chiral-fit approach to light-light sector.
Compare to Previous Work

- Red points have statistical error only; blue include systematic errors.