K and D oscillations in the Standard Model and its extensions from $N_f = 2 + 1 + 1$ Twisted Mass LQCD

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in collaboration with

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Outline

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2. Simulation details
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   - Bare matrix elements: plateaux
   - Extraction of RCs
   - Extrapolation to the physical point
4. Preliminary results, error budget and comparison
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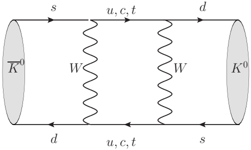
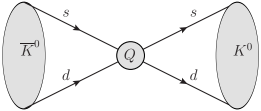
Meson oscillations



Integrating out the heavy degrees of freedom

OPE

$$Q_1 = [\bar{s}\gamma^{\mu}(1-\gamma_5)d]\,[\bar{s}\gamma_{\mu}(1-\gamma_5)d]$$



$$\langle \overline{K}^0 | H_{\rm eff}^{\Delta S=2} | K^0 \rangle = \underbrace{\frac{G_F^2 M_W^2}{16\pi^2}\left[\sum_{q,q'=u,c,t} \mathcal{C}_{qq'} V^*_{qs} V_{qd} V^*_{q's} V_{q'd}\right]}_{\equiv C_1 \text{ perturbative}} \underbrace{\langle \overline{K}^0 | \hat{Q}_1 | K^0 \rangle}_{\text{non perturbative}}$$

Beyond SM: Most General $\Delta F = 2$ hamiltonian

$$\langle \overline{K}^0 | H_{\rm eff}^{\Delta F=2} | K^0 \rangle = \frac{G_F^2 M_W^2}{16\pi^2}\left\{ \sum_{i=1}^{5} \overbrace{C_i(\mu)}^{\text{model dependent}} \underbrace{\overbrace{\langle \overline{K}^0 | \hat{Q}_i(\mu) | K^0 \rangle}^{\text{from lattice}}}_{\equiv \hat{B}_i \langle \overline{K}^0 | Q_i | K^0 \rangle_{VIA}} \right\}$$

UTfit Collaboration. M.Bona et al. JHEP 0803 (2008) 049

$$Q_1 = [\bar{h}^a \gamma^{\mu}(1-\gamma_5) l^a]\,[\bar{h}^b \gamma_{\mu}(1-\gamma_5) l^b]$$
$$Q_2 = [\bar{h}^a (1-\gamma_5) l^a]\,[\bar{h}^b (1-\gamma_5) l^b]$$
$$Q_3 = [\bar{h}^a (1-\gamma_5) l^b]\,[\bar{h}^b (1-\gamma_5) l^a]$$
$$Q_4 = [\bar{h}^a (1-\gamma_5) l^a]\,[\bar{h}^b (1+\gamma_5) l^b]$$
$$Q_5 = [\bar{h}^a (1-\gamma_5) l^b]\,[\bar{h}^b (1+\gamma_5) l^a]$$

Model independent constraints and NP from the UT analysis

The effective hamiltonian is parametrized by a Wilson coefficient of the form:

$$C_i(\Lambda) = \frac{F_i L_i}{\Lambda^2} \to \Lambda = \sqrt{\frac{F_i L_i}{C_i(\Lambda)}}$$

→Phenomenological allowed range for each C_i→lower bound of Λ

- F_i= NP coupling, depends on the flavor struture of the NP model
  - MFV: $F_1 = F_{SM} \sim (V_{tq} V^*_{tq'})^2$ and $F_{i\neq 1} = 0$
  - Generic flavor structure: $|F_i| \sim 1$
- L_i= loop factor:
  - L=1: tree-level FCNC
  - L=α_W^2: SM
  - L=α_{NP}^2: NP FCNC at loop level
- Λ= scale of NP, typical mass of the NP particles involved in the mixing

UTfit provides lower bounds for the scale of NP

$F_i \sim L_i \sim 1$	C_1	C_2	C_3	C_4	C_5
$\Lambda(\times 10^4\,\text{TeV})$ from $K^0 - \overline{K}^0$	~ 1.7	~ 22	~ 11	~ 46	~ 27
$\Lambda(\times 10^4\,\text{TeV})$ from $D^0 - \overline{D}^0$	~ 0.2	~ 0.7	~ 0.1	~ 1.2	~ 0.3

Simulation details

- Iwasaki gluon action
- Wilson Twisted Mass Action at maximal twist with $N_f = 2 + 1 + 1$ sea quarks

(details in *R. Baron et al. JHEP, 06:111, 2010, 1004.5284*)

- Osterwalder-Seiler valence quark action $\rightarrow \mathcal{O}(a)$ improvement & continuum like renormalization pattern

(*R. Frezzotti and G. C. Rossi. JHEP, 10:070, 2004, hep-lat/0407002*)

- RCs: $N_f = 4$ Twisted Mass action out of maximal twist.

(*B.Blossier et al.PoS, LATTICE2011:233, 2011, 1112.1540*)

- Three lattice spacings β =1.90, 1.95, 2.10 which correspond to a =0.09, 0.08, 0.06 fm (a^{-1}=2.2, 2.5, 3.2 GeV)
- Three volumes: $24^3 \times 48$, $32^3 \times 64$, $48^3 \times 96 \rightarrow L \approx 2, 3$fm
- Unitary setup: $\mu_{\rm sea} = \mu_\ell$. 1 light valence quark mass for each sea quark mass.
- $M_\pi = [220 : 500]$ MeV
- Three valence "strange" quark masses around the physical strange. Local propagators.
- Three valence "charmed" quark masses around the physical charm. Smeared propagators.
- $T/2$ separation between sources for $\overline{K}^0 - K^0$ while $T_{\rm sep} < T/2$ for $\overline{D}^0 - D^0$
- Stochastic propagators

RI-MOM out of maximal twist

- RCs are computed non perturbatively in the RI-MOM scheme.
- Dedicated $N_f = 4$ degenerate sea quark gauge configurations
- Instabilities in tuning to maximal twist $\rightarrow$ work out of maximal twist.

$\mathcal{O}(a)$ improvement out of maximal twist

We average over RC estimators computed at equal value of μ but opposite θ ($\tan\theta = Z_A m_{\mathrm{PCAC}}/\mu$)

$$\frac{Z[\theta] + Z[-\theta]}{2} \text{ is free of } \mathcal{O}(a) \text{ lattice artifacts}$$

R. Frezzotti and G. C. Rossi. JHEP, 10:070, 2004, hep-lat/0407002.

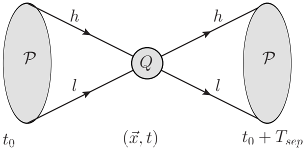
p/m ensembles: for each β, and for each M^{sea} we produce two ensembles with (nearly) opposite values of θ - **p**ositive and **n**egative values of θ (or m_{PCAC}) - with the same μ

Extracting B-parameters from lattice correlators

$$\langle \overline{P}^0 | \hat{Q}_i | P^0 \rangle = \hat{B}_i \langle \overline{P^0} | \hat{Q}_i | P^0 \rangle_{VIA} = \begin{cases} \hat{B}_1 \xi_1 f_P^2 M_P^2 & \\ \hat{B}_i \xi_i \left[\frac{M_P}{\hat{m}_h + \hat{m}_\ell} \right]^2 f_P^2 M_P^2 & i \geq 2 \end{cases}$$

$$\xi_i = \{8/3, -5/3, 1/3, 2, 2/3\}$$

$$\rightarrow \begin{cases} \frac{1}{\xi_1} \frac{\hat{C}_1^{(3)}(t)}{\hat{C}^{(2)}_{P5A0}(-t)\hat{C}^{(2)}_{P5A0}(t - T_{\rm sep})} \xrightarrow{0 \ll t \ll T_{sep}} \frac{\langle \overline{P}^0 | \hat{Q}_i | P^0 \rangle}{\xi_1 \langle \overline{P}^0 | \hat{A}_0 | 0 \rangle \langle 0 | \hat{A}_0 | P^0 \rangle} = \hat{B}_1 \\ \frac{1}{\xi_i} \frac{\hat{C}_i^{(3)}(t)}{\hat{C}^{(2)}_{P5P5}(-t)\hat{C}^{(2)}_{P5P5}(t - T_{\rm sep})} \xrightarrow{0 \ll t \ll T_{sep}} \frac{\langle \overline{P}^0 | \hat{Q}_i | P^0 \rangle}{\xi_i \langle \overline{P}^0 | \hat{P}_5 | 0 \rangle \langle 0 | \hat{P}_5 | P^0 \rangle} = \hat{B}_i \quad i \geq 2 \end{cases}$$



Meson walls

$$\mathcal{P}_5(t_0) = \sum_{\vec{y}} \bar{h}(\vec{y}, t_0)\gamma_5 l(\vec{y}, t_0) = \sum_{\vec{y}} P_5(\vec{y}) \quad ; \mathcal{P}_5(t_0)|0\rangle = P_0$$

Three- and two-point correlators

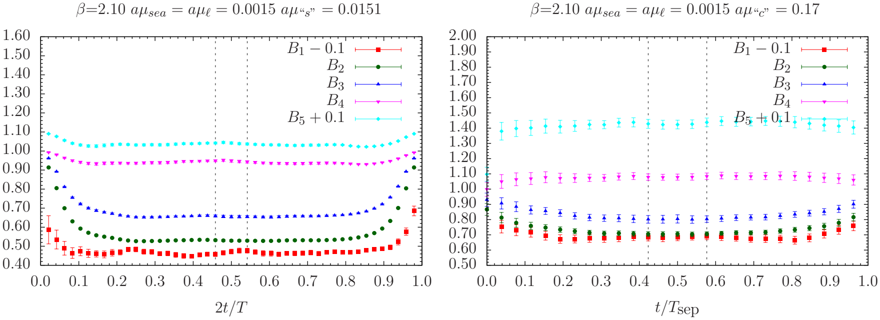
$$C^{(2)}_{P5A0}(t) = \sum_{\vec{x}} \langle 0 | A_0(\vec{x}, t)) \mathcal{P}_5(t_0) | 0 \rangle$$

$$C^{(2)}_{P5P5}(t) = \sum_{\vec{x}} \langle 0 | P_5(\vec{x}, t)) \mathcal{P}_5(t_0) | 0 \rangle$$

$$C_i^{(3)}(t) = \sum_{\vec{x}} \langle 0 | \mathcal{P}_5'(T_{sep}) Q_i(\vec{x}, t) \mathcal{P}_5(0) | 0 \rangle$$

Plateaux of bare correlators

$$\frac{C_i^{(3)}}{\xi_i C^{(2)} C'^{(2)}} \to B_i\left(a; \mu_\ell, \mu_h\right)$$



Example for the smallest lattice spacing ($a \sim 0.06$fm) with the lightest sea mass (M_π =220MeV)

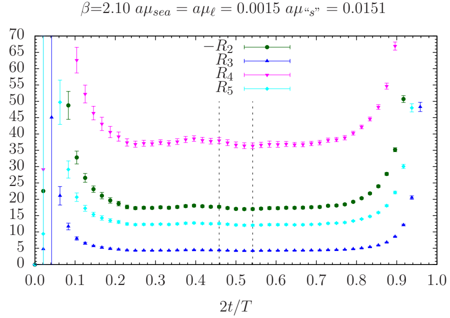
Extracting R-parameters from lattice correlators

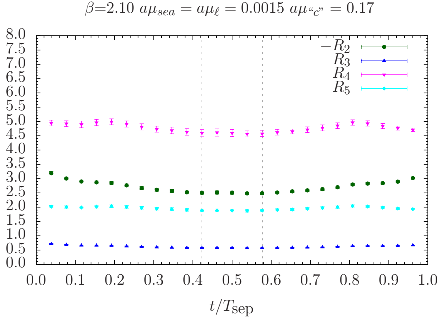
$$\hat{R}_i = \frac{\langle \overline{P}^0 | \hat{Q}_i | P^0 \rangle}{\langle \overline{P}^0 | \hat{Q}_1 | P^0 \rangle} \sim \frac{BSM}{SM} \quad i = 2,3,4,5 \quad \rightarrow \quad \frac{C_i^{(3)}(t)}{C_1^{(3)}(t)} \xrightarrow{0 \ll t \ll T_{sep}} R_i$$

To compensate the chiral vanishing of the $\langle \overline{P}^0 | \hat{Q}_1 | P^0 \rangle$ we define the rescaled quantity

(*R.Babich et al Phys. Rev. D74(2006)073009 hep-lat/0605016*)

$$\tilde{R}_i = \left(\frac{f_P}{M_P} \right)^2_{\text{exp}} \left[\left(\frac{M_{hl}}{F_{hl}} \right)^2 R_i \right]_{\text{lat}}$$





Example for the smallest lattice spacing ($a \sim 0.06$fm) with the lightest sea mass (M_π =220MeV)

Renormalization pattern

$$\langle \overline{P}^0|Q_i|P^0\rangle = c_w(\mu)\langle \overline{P}^0|\hat{Q}_i(\mu)|P^0\rangle = c_w(\mu)Z_{ij}(a\mu)\langle \overline{P}^0|Q_j(a)|P^0\rangle$$

On the lattice, the Wilson term induces explicit chiral symmetry breaking → mixing with operators of the same dimensionality but with the wrong naive chirality.

OS valence action

$$S^{\mathrm{val,OS}} = \sum_{f=h,h',l,l'} \sum_x \overline{q}_f(x) \left[\gamma\tilde{\nabla} - i\gamma_5 a\frac{r_f}{2}\sum_\mu \nabla^*_\mu\nabla_\mu - i\gamma_5 M_{cr}(r_f) + m_f\right] q_f(x)$$

→Continuum-like renormalization pattern for four fermion operator, with absence of wrong chirality mixing, if *(R.Frezzotti and G.C. Rossi. JHEP,2004)*

$$r_h = r_{h'} = r_l = -r_l'$$

$$\hat{Q} = Z_\chi[I+\Delta]Q$$

$$Z_\chi = \begin{pmatrix} Z_{11} & 0 & 0 & 0 & 0 \\ 0 & Z_{22} & Z_{23} & 0 & 0 \\ 0 & Z_{32} & Z_{33} & 0 & 0 \\ 0 & 0 & 0 & Z_{44} & Z_{45} \\ 0 & 0 & 0 & Z_{44} & Z_{55} \end{pmatrix} \quad \Delta = \begin{pmatrix} 0 & \Delta_{12} & \Delta_{13} & \Delta_{14} & \Delta_{15} \\ \Delta_{11} & 0 & 0 & \Delta_{24} & \Delta_{25} \\ \Delta_{31} & 0 & 0 & \Delta_{34} & \Delta_{35} \\ \Delta_{41} & \Delta 42 & \Delta_{43} & 0 & 0 \\ \Delta_{51} & \Delta_{52} & \Delta_{53} & 0 & 0 \end{pmatrix} \sim \mathcal{O}(a^2)$$

$$\hat{B}_1 = \frac{Z_{11}}{Z_V Z_A}B_1 \quad \hat{B}_i = \frac{Z_{ij}}{Z_P Z_S}B_j \quad \hat{R}_i = \frac{Z_{ij}}{Z_{11}}R_i$$

This setup has been successfully used in the ETMC $N_f = 2$ B-parameter analysis
ETMC, Phys Rev. D83 (2011) 014505 1009.5606 and *ETMC. JHEP, 1303:089, 2013, 1207.1287*

RC Analysis

Z_V from WI

M.Constantinou et al. JHEP1008(2010)068) hep-lat:1004.115

$$Z_V \partial_0 C^{(2)}_{P5V0}(t) = (\mu_1 + \mu_2) C^{(2)}_{P5P5}(t)$$

2F RCs

$$Z_\Gamma Z_q^{-1} \mathcal{V}_\Gamma = 1$$

$$\mathcal{V}_\Gamma = \mathrm{Tr}[P\Lambda_\Gamma]$$

tree-level projector: $P\Lambda^{(0)}_\Gamma = 1$

amputated 2F GF: $\Lambda_\Gamma = S^{-1} G_\Gamma S^{-1}$

4F RCs

$$Z = Z_q^2 \left[D^T\right]^{-1}$$

dynamical matrix: $D = P\Lambda$

tree-level projector: $P\Lambda^{(0)} = I$

amputated 4F GF: $\Lambda_{\Gamma_1\Gamma_2} = S^{-1} S^{-1} G_{\Gamma_1\Gamma_2} S^{-1} S^{-1}$

Two types of p^2-fit to subtract the residual $\mathcal{O}(a^2p^2)$ lattice artifacts:

- M1: linear combined fit in

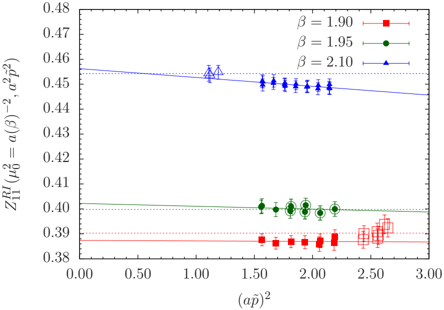
(full symbols)

$$1.5 \le a^2\tilde{p}^2 \le 2.2$$

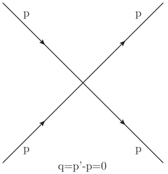
- M2: constant fit in

(open symbols)

$$p^2 \in [11 : 14]\,\mathrm{GeV}^2$$



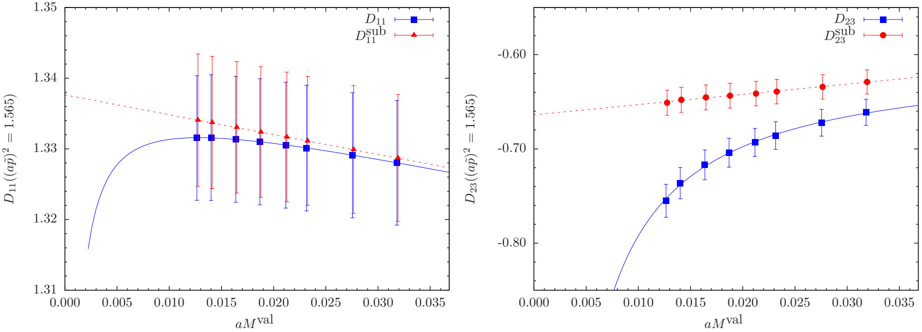
Chiral fits in valence: GBP subtraction



GB-pole

Specific care must be taken in the chiral extrapolation of $\mathcal{V}_{\mathcal{P}}$ and D_{ij} due to the coupling with the GB-pole $\propto M_{PS}^{-2}$

$$D_{ij}\left(M; a^2p^2\right) = D_{ij}^{(0)}(a^2p^2) + B_{ij}(a^2p^2)M + \frac{C_{ij}(a^2p^2)}{M_{PS}^2}$$



Example of GBP subtraction for D_{11} and D_{23} at the lightest sea quark mass of β=1.95 (a ∼0.08fm)

Extrapolation to the physical point

Quark masses extrapolation/interpolation

$$\mu_\ell \to \mu_{u/d}$$

$$\mu_h \to \mu_s \text{ or } \mu_h \to \mu_c$$

Preliminary ETMC values of the quark masses (*see talks by Paolo Lami and Lorenzo Riggio on Friday*)

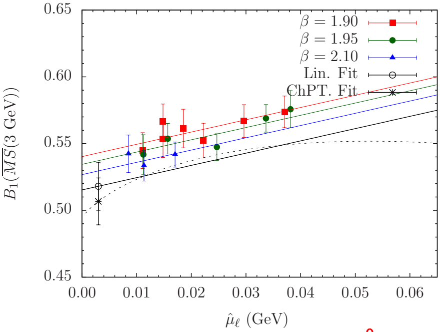
Combined chiral and continuum extrapolation

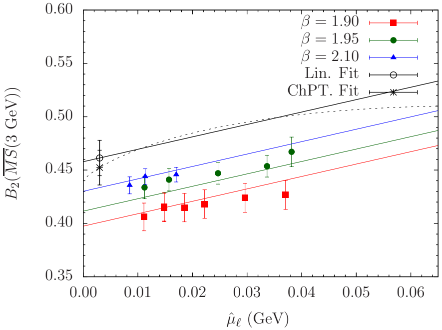
Linear fit: $\hat{B}_i = A + B\hat{\mu}_\ell + Da^2$

ChPT fit ansatz (*Phys.Rev.,D70:094036,2004,hep-lat/0408029*)

HMChPT fit ansatz

(*JHEP, 0706:003, 2007, hep-ph/0612224*)





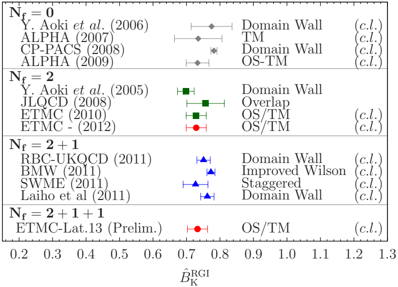
Example of B_1 and B_2 ($K^0 - \overline{K}^0$) chiral and continuum extrapolation

Preliminary results, error budget and comparison

Results in $\overline{MS}$ (*Buras et al., Nucl.Phys., B586:397–426, 2000, hep-ph/0005183*) at 3 GeV:

$K^0 - \overline{K}^0$					$D^0 - \overline{D}^0$				
B_1	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4	B_5
0.51(2)	0.46(2)	0.81(5)	0.76(3)	0.47(4)	0.76(4)	0.64(2)	1.02(7)	0.92(3)	0.95(5)

	$K^0 - \overline{K}^0$					$D^0 - \overline{D}^0$				
source of error (%)	B_1	B_2	B_3	B_4	B_5	B_1	B_2	B_3	B_4	B_5
stat. (corr+RC+$\mu_{s/c}$)	2.4	2.0	2.5	2.1	4.2	4.2	2.2	5.7	2.4	2.7
syst.										
chiral fit	1.2	1.0	1.0	1.2	1.2	0.1	0.1	0.1	0.7	0.7
RI-MOM	1.8	0.5	4.9	1.5	6.0	1.3	1.2	0.5	1.2	4.0
discr. effects	0.5	2.4	2.4	2.5	2.4	3.9	2.2	3.3	2.0	0.8
total	3.3	3.4	6.0	3.7	7.8	5.6	3.7	6.7	3.7	5.1



$\overline{K}^0 - K^0 \quad \overline{MS}(3\text{GeV})$			
	This work	ETMC [1]	*RBC-UKQCD* [2]
	$N_f = 2+1+1$	$N_f = 2$	$N_f = 2+1$
	c.l	c.l	$a \sim 0.086 fm$
B_1	0.51(2)	0.51(2)	*0.517(4)*
B_2	0.46(2)	0.47(2)	*0.43(5)*
B_3	0.81(5)	0.78(4)	*0.75(9)*
B_4	0.76(4)	0.76(3)	*0.69(7)*
B_5	0.47(4)	0.58(3)	*0.47(6)*

$\overline{D}^0 - D^0 \quad \overline{MS}(3\text{GeV})$		
	This work	ETMC [3]
	$N_f = 2+1+1$	$N_f = 2$
	c.l	c.l
B_1	0.76(4)	0.75(2)
B_2	0.64(2)	0.66(2)
B_3	1.02(7)	0.97(5)
B_4	0.92(3)	0.91(4)
B_5	0.95(5)	1.10(5)

[1] ETMC. JHEP, 1303:089, 2013, 1207.1287
[2] RBC and UKQCD Phys.Rev., D86:054028, 2012, 1206.5737
[3] ETMC. To be published

Preliminary results, error budget and comparison:

$K^0 - \overline{K}^0$ R-parameters

Results in $\overline{MS}$ (*Buras et al., Nucl.Phys., B586:397–426, 2000, hep-ph/0005183*) at 3 GeV:

R_2	R_3	R_4	R_5
-15.1(6)	5.4(3)	30.1(1.5)	6.0(3)

source of error (%)	R_2	R_3	R_4	R_5
stat. (corr+RC+$\mu_{s/c}$)	3.3	3.5	3.3	5.4
syst.				
chiral fit	0.1	0.3	2.7	2.9
RI-MOM	0.1	3.5	1.1	4.8
discr. effects	2.7	2.7	2.8	2.3
total	4.3	5.7	5.2	8.1

	This work	ETMC [1]	*RBC-UKQCD* [2]
	$N_f = 2+1+1$	$N_f = 2$	$N_f = 2+1$
	c.l	c.l	$a \sim 0.086 fm$
R_2	-15.1(6)	-15.6(5)	*-15.3(1.7)*
R_3	5.4(3)	5.3(3)	*5.4(0.6)*
R_4	30.1(1.5)	28.6(9)	*29.3(2.9)*
R_5	6.0(3)	7.8(4)	*6.6(0.9)*

[1] ETMC. JHEP, 1303:089, 2013, 1207.1287
[2] RBC and UKQCD Phys.Rev., D86:054028, 2012, 1206.5737

Conclusions

- Using data from $N_f = 2+1+1$ Twisted Mass simulations
  - with $a \sim [0.06 : 0.09]$ fm
  - and $M_\pi \sim [220 : 500]$ MeV

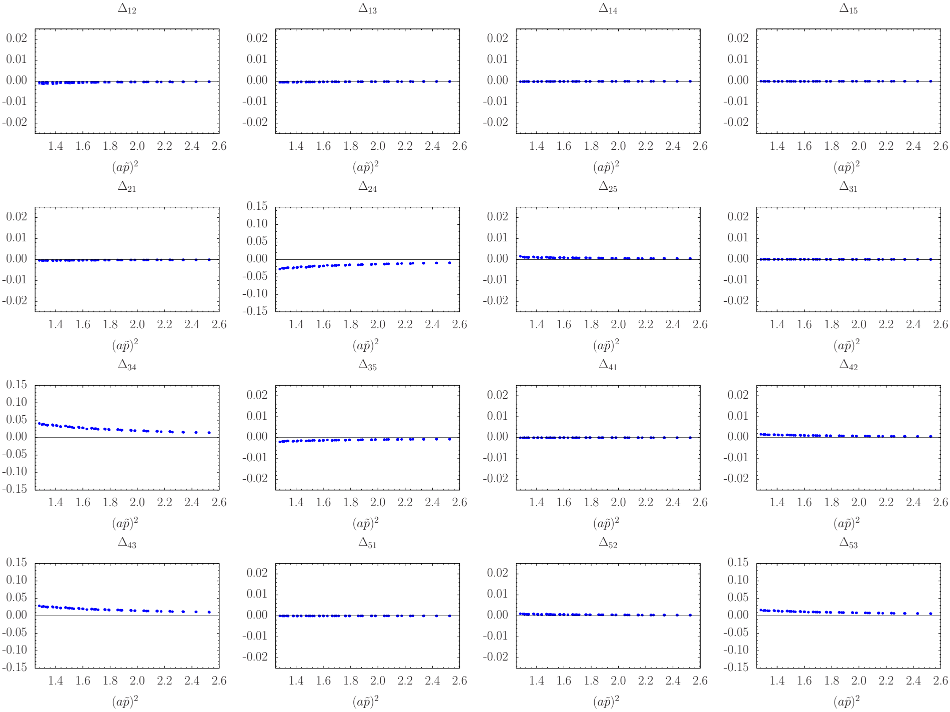
we have computed the SM and BSM contributions to $K^0 - \overline{K}^0$ and $D^0 - \overline{D}^0$ (complete basis of B_i parameters and BSM R_i parameters)

- $N_f = 2$ and $N_f = 2+1+1$ results are compatible $\rightarrow$ **There is not significant dependence in the number of dynamical sea quarks**.

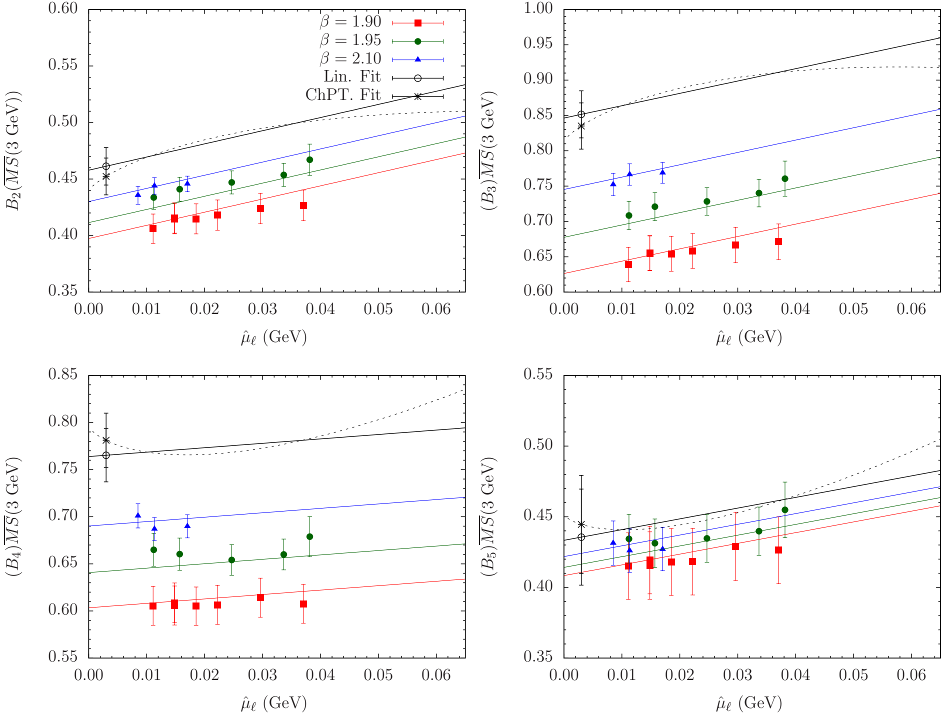
OUTLOOK

- Systematic effects due to the chiral extrapolations have to be considered with more accuracy $\rightarrow$ Results are still preliminary
- We plan to extend the analysis to $B^0 - \overline{B}^0$ and $B_s^0 - \overline{B}_s^0$ system with $N_f = 2+1+1$ dynamical sea quarks by applying the ratio method (*See talk by Petros Dimopoulos with* $N_f = 2$)

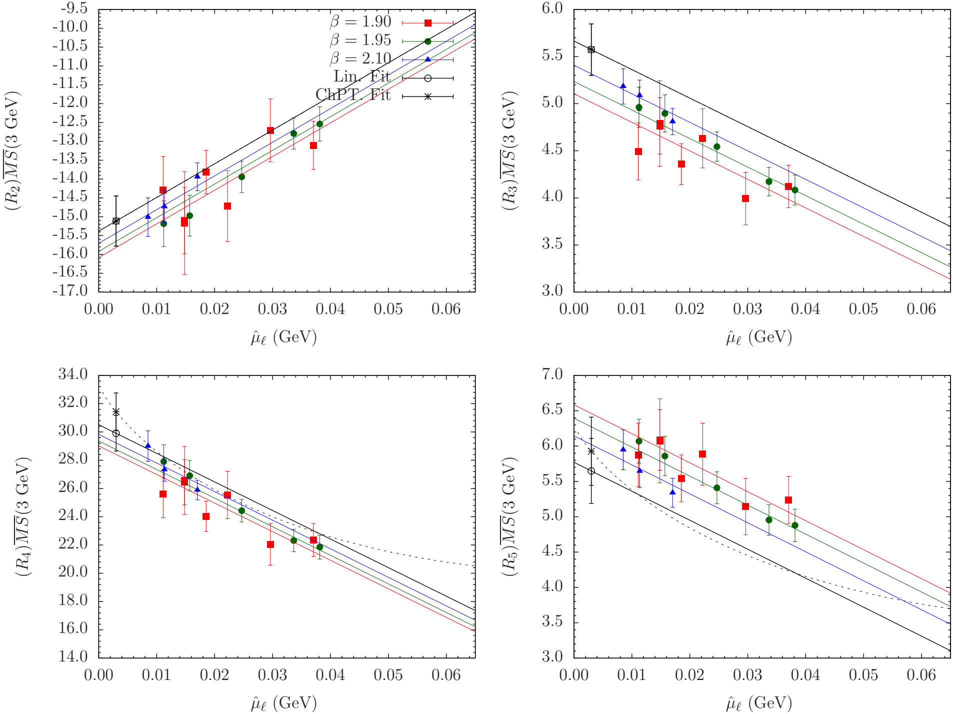
Backup slides



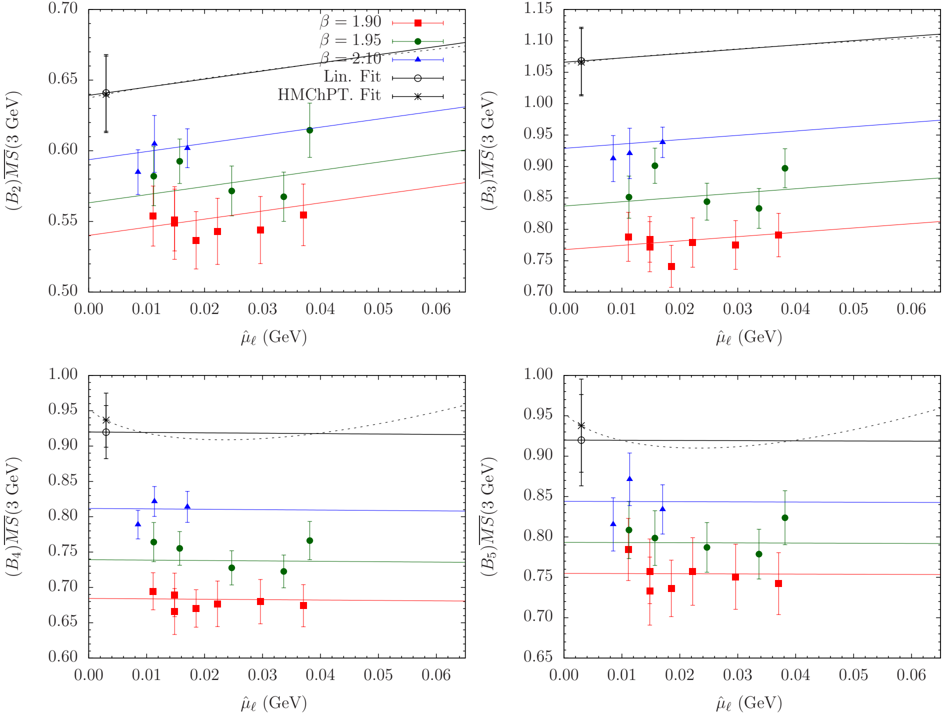
Δ12
Δ13
Δ14
Δ15
Δ21
Δ24
Δ25
Δ31
Δ34
Δ35
Δ41
Δ42
Δ43
Δ51
Δ52
Δ53
0.02
0.01
0.00
-0.01
-0.02
0.15
0.10
0.05
0.00
-0.05
-0.10
-0.15
1.4
1.6
1.8
2.0
2.2
2.4
2.6
$(a\tilde{p})^2$



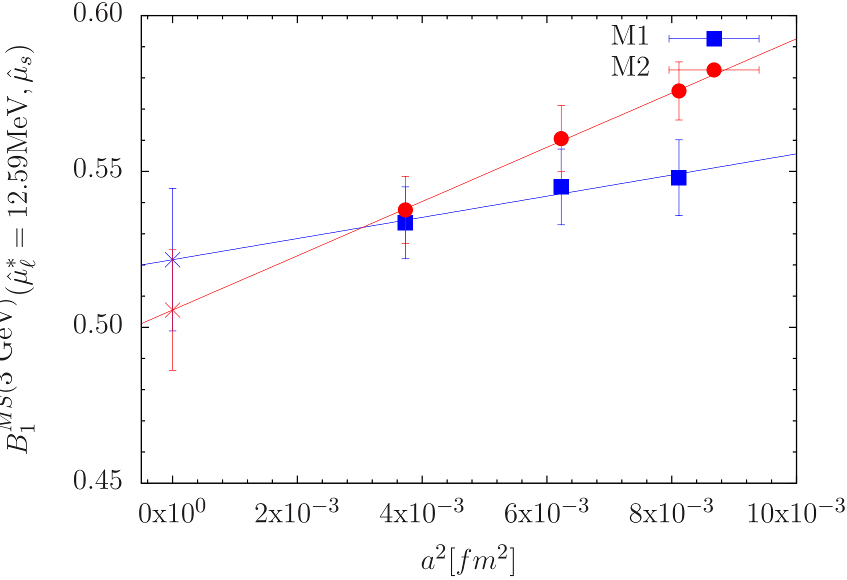
$\beta = 1.90$
$\beta = 1.95$
$\beta = 2.10$
Lin. Fit
ChPT. Fit
$B_2(\overline{MS}(3\text{ GeV}))$
$(B_3)\overline{MS}(3\text{ GeV})$
$(B_4)\overline{MS}(3\text{ GeV})$
$(B_5)\overline{MS}(3\text{ GeV})$
$\hat{\mu}_\ell$ (GeV)



$\beta = 1.90$
$\beta = 1.95$
$\beta = 2.10$
Lin. Fit
ChPT. Fit
$(R_2)\overline{MS}(3\text{ GeV})$
$(R_3)\overline{MS}(3\text{ GeV})$
$(R_4)\overline{MS}(3\text{ GeV})$
$(R_5)\overline{MS}(3\text{ GeV})$
$\hat{\mu}_\ell$ (GeV)



$\beta = 1.90$
$\beta = 1.95$
$\beta = 2.10$
Lin. Fit
HMChPT. Fit
$(B_2)\overline{MS}(3\text{ GeV})$
$(B_3)\overline{MS}(3\text{ GeV})$
$(B_4)\overline{MS}(3\text{ GeV})$
$(B_5)\overline{MS}(3\text{ GeV})$
$\hat{\mu}_\ell$ (GeV)



M1
M2
0.60
0.55
0.50
0.45
0x10^0
2x10^-3
4x10^-3
6x10^-3
8x10^-3
10x10^-3
$a^2[fm^2]$
$B_1^{\overline{MS}}(3\ \mathrm{GeV})_{(\hat{\mu}_\ell^* = 12.59\mathrm{MeV}, \hat{\mu}_s)}$

$$D_{ij}(p)^{\rm corr.} = D_{ij}(p) - \frac{g^2}{16\pi^2} a^2 \left[p^2 \left(d_{ij}^{(1)} + d_{ij}^{(2)} \log(a^2 p^2) \right) + d_{ij}^{(3)} \frac{\sum_\rho \tilde{p}_\rho^4}{\tilde{p}^2} \right]$$

$$\mathcal{V}_\Gamma(p)^{\rm corr.} = \mathcal{V}_\Gamma(p) - \frac{g^2}{12\pi^2} a^2 \left[\tilde{p}^2 \left(c_\Gamma^{(1)} + c_\Gamma^{(2)} \ln(a^2\, \tilde{p}^2) \right) + c_\Gamma^{(3)} \frac{\sum_\rho \tilde{p}_\rho^4}{\tilde{p}^2} \right]$$

$$\Sigma_1(p)^{\rm corr.} = \Sigma_1(p) - \frac{g^2}{12\pi^2} a^2 \left[\tilde{p}^2 \left(c_q^{(1)} + c_q^{(2)} \ln(a^2\, \tilde{p}^2) \right) + c_q^{(3)} \frac{\sum_\rho \tilde{p}_\rho^4}{\tilde{p}^2} \right]$$