The gradient flow in a twisted box

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Acknowledgments

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Introduction: gradient flow [Lüscher '10]

- Add "extra" (flow) time coordinate t ($\neq x_4$). Define gauge field $B_\mu(x,t)$

$$
\begin{aligned}
G_{\nu\mu}(x,t) &= \partial_\nu B_\mu(x,t) - \partial_\nu B_\mu(x,t) + [B_\nu(x,t), B_\mu(x,t)] \\
\frac{dB_\mu(x,t)}{dt} &= D_\nu G_{\nu\mu}(x,t) \quad \left(\sim -\frac{\delta S_{\rm YM}[B]}{\delta B_\mu}\right)
\end{aligned}
$$

  with initial condition $B_\mu(x, t=0) = A_\mu(x)$.
- Flow tends to a classical solution. UV fluctuations are suppressed.
- PT gives some intuition[M. Lüscher].

$$B_\mu(x,t) = \sum_{n=1}^{\infty} B_{\mu,n}(x,t) g_0^n$$

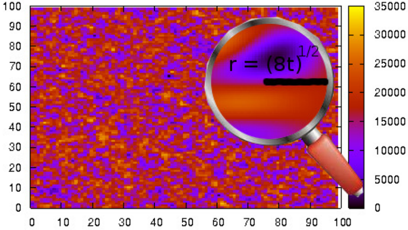
and to Leading order GF=Heat Flow.

$$\dot{B}_{\mu,1}(x,t) = \partial_\nu^2 B_{\mu,1}(x,t) + \text{gauge terms}$$

With solution

$$B_{\mu,1}(x,t) = \frac{1}{\sqrt{4\pi t}} \int d^4y\, e^{-\frac{(x-y)^2}{4t}} A_\mu(y)$$

We "average" with a resolution $\sim \sqrt{8t}$.



Introduction: GF as a renormalized coupling [Lüscher '10]

Energy density finite after usual renormalization of bare parameters

$$\langle E(t) \rangle = \frac{1}{4\mathcal{Z}} \int \mathcal{D}A_\mu \, G^a_{\mu\nu}(x,t) G^a_{\mu\nu}(x,t) e^{-S[A]}$$

In perturbation theory we have:

$$\langle E(t) \rangle = \frac{3 g^2_{\overline{MS}}}{16\pi^2 t^2} (1 + c_1 g^2_{\overline{MS}} + \mathcal{O}(g^4_{\overline{MS}}))$$

and we can use $\langle E(t) \rangle$ for a NP definition of the coupling at scale $\mu = 1/\sqrt{8t}$.

$$\alpha(\mu) = \frac{4\pi}{3} t^2 \langle E(t) \rangle = \alpha_{\overline{MS}}(\mu) + \dots$$

Applications

- Scale setting (t_0) [Lüscher, '10; BWM '12; Bruno lat13; Sommer lat13].
- Running coupling by identifying μ with L (finite size scaling) [Fodor et al. '12; Fritzsch, Ramos '13; Fritzsch lat13].

Running coupling schemes

Finite size scaling [Lüscher, Weisz, Wolff '91]

Define a coupling that depends only on one scale given by the finite volume

$$g^2(L) = \mathcal{N}^{-1}(t,\dots)t^2\langle E(t)\rangle|_{\sqrt{8t}=cL} = g^2_{\rm MS} + \dots$$

Periodic box ($\mathbb{T}^4$) [Fodor et al. '12]

- Leading order contribution from zero momentum modes is not quadratic [A. Gonzalez-Arroyo et al. 1983].

$$\langle A_\mu A_\nu \rangle \sim \int \mathcal{D}A\; A_\mu A_\nu e^{-A_\mu^2 - \tilde{A}_\mu^4(0)}$$

- Difficult (but has been solved).
- Coupling is not analytic in $g^2_{\rm MS}$

SU(2): $g^2(L) = g^2_{\rm MS} + \mathcal{O}(g^2_{\rm MS}\log g^2_{\rm MS})$

and for $N > 2$

$$\text{SU(N)}: g^2(L) = g^2_{\rm MS} + \mathcal{O}(g^3_{\rm MS})$$

Schrödinger Functional [Fritzsch, Ramos '13]

- SF: Dirichlet b.c. in x_0 eliminates zero momentum mode contribution [Lüscher, Narayanan, Weisz, Wolff '92].

- Reaction of the system to the change in boundary values: $g^2_{\rm SF}(L)$.
- Manifold with boundaries: $\mathcal{O}(a)$ cutoff effects.
- Removed by boundary counterterms.
  - Computed in PT: Hard work.
  - Changes with action/matter/...

Twisted boundary conditions for $SU(N)$ YM.

Key idea

In a periodic world only gauge invariant quantities need to be periodic.

$$A_\mu(x + L\hat{\nu}) = \Omega_\nu(x) A_\mu(x) \Omega_\nu^+(x) + \imath \Omega_\nu(x) \partial_\mu \Omega_\nu^+(x).$$

- We choose to twist the plane $x_1 - x_2$.

$$\begin{aligned} \Omega_{3,4}(x) &= 1 \\ \Omega_1 \Omega_2 &= e^{2\pi \imath m/N} \Omega_2 \Omega_1 \end{aligned}$$

- No "zero mode" problem [A. Gonzalez-Arroyo et al. 1983, Lüscher, Weisz 1986, TPL scheme, . . .].
- Invariance under translations (no boundaries).
- Gauge connections can be expanded

$$A_\mu^a(x) T^a = \frac{1}{L^4} \sum_{p,\tilde{p} \neq 0} \tilde{A}_\mu(P) e^{\imath P x} \hat{\Gamma}(P).$$

with $P_\mu = p_\mu + \tilde{p}_\mu$ and $(\mu = 1, 2, 3, 4; i = 1, 2)$.

$$\begin{aligned} p_\mu &= \frac{2\pi n_\mu}{L} \, (n_\mu \in \mathbb{Z}) \\ \tilde{p}_i &= \frac{2\pi \tilde{n}_i}{NL} \, (\tilde{n}_i = 0, \ldots, N-1) \\ \hat{\Gamma}(P) &= e^{\imath \alpha(P)} \Omega_1^{-k\tilde{n}_2} \Omega_2^{k\tilde{n}_1} \end{aligned}$$

How is $\langle E(t)\rangle$ to LO in a twisted box?

$$\begin{aligned} \dot{B}_\mu(x,t) &= D_\nu G_{\nu\mu}(x,t)\,, \qquad B_\mu(x,0) = A_\mu(x)\,, \\ G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \end{aligned}$$

$B_\mu(x,t)$ has an asymptotic expansion in g_0

$$B_\mu(x,t) = \sum_n B_{\mu,n}(x,t) g_0^n$$

After gauge fixing and to leading order

$$\dot{B}_{\mu,1}(x,t) = \partial_\nu^2 B_{\mu,1}(x,t) \quad (B_{\mu,1}(x,0) = A_\mu(x))$$

with solution

$$B_{\mu,1}(x,t) = \frac{1}{L^4} \sum_{p,\tilde{p}\neq 0} e^{-P^2 t} \tilde{A}_\mu(P) e^{\imath P x} \hat{\Gamma}(P).$$

And finally $\langle E(t)\rangle = \frac{1}{4}\langle G_{\mu\nu}(t) G_{\mu\nu}(t)\rangle = \mathcal{E}(t) + \mathcal{O}(g_0^4)$

$$\mathcal{E}(t) = \frac{g_0^2(d-1)}{2L^4} \sum_{p,\tilde{p}\neq 0} e^{-P^2 t}$$

in the lattice $\hat{P}_\mu = \frac{2}{a}\sin\left(a\frac{P_\mu}{2}\right)$; $\mathring{P}_\mu = \frac{1}{a}\sin\left(aP_\mu\right)$; $C_\mu = \cos\left(a\frac{P_\mu}{2}\right)$

$$\hat{\mathcal{E}}_{\rm clover}(t, a/L) = \frac{g_0^2}{2L^4} \sum_{p,\tilde{p}\neq 0} e^{-\hat{P}^2 t} \frac{\mathring{P}^2 C^2 - \sum_\mu (\mathring{P}_\mu C_\mu)^2}{\hat{P}^2},$$

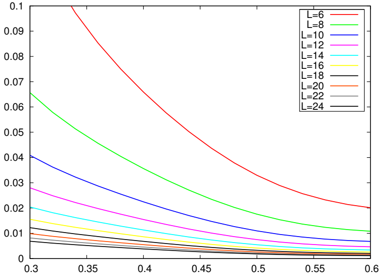
Coupling definition

Reading the value of $\langle E(t)\rangle$ at $\sqrt{8t} = cL$, we define

$$\mathcal{N}_T(c) = \frac{(d-1)c^4}{128} \sum_P{}' e^{-\frac{c^2L^2}{4}P^2} = \frac{g_0^2(d-1)c^4}{128} \sum_{n_\mu=-\infty}^{\infty} \sum_{\tilde{n}_i=0}^{N-1}{}' e^{-\pi^2 c^2 (n^2 + \tilde{n}^2/N^2 + 2\tilde{n}_i n_i/N)}$$

Twisted gradient flow coupling

$$g^2_{TGF}(L) = \mathcal{N}_T^{-1}(c) t^2 \langle E(t)\rangle \Big|_{\sqrt{8t}=cL} = g^2_{\overline{\rm MS}} + \mathcal{O}(g^4_{\overline{\rm MS}})$$

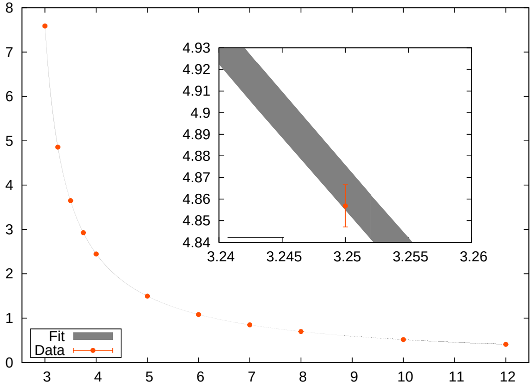


- Different c's: different schemes.
- Larger c smaller cutoff effects (more smooth).
- Larger c larger autocorrelations.
- Larger c smaller signal to noise.
- $c \in [0.3, 0.5]$ reasonable range.

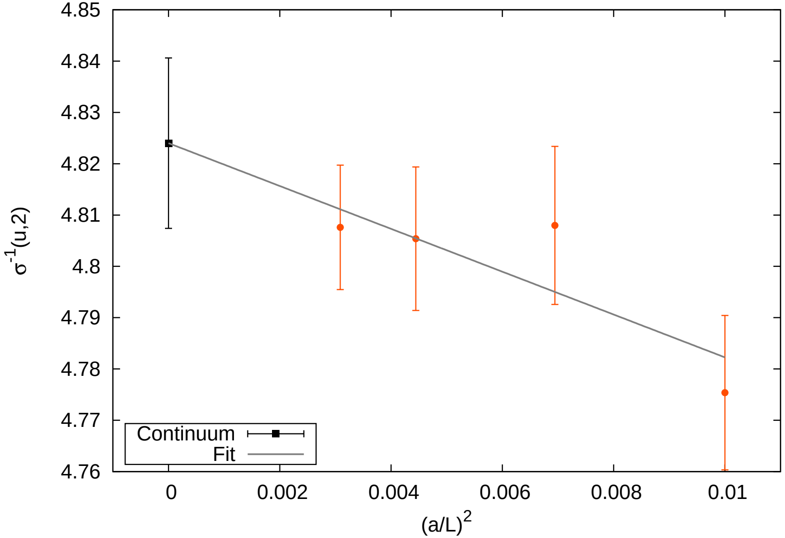
$SU(2)$ YM running coupling

- Simulations for $L/a = 10, 12, 15, 18, 20, 24, 30, 36$ at $\beta \in [2.75, 12]$.
- Modest statistics: 2048 independent measurements of g^2_{TGF}.
- Between 0.15-0.25% precision in g^2_{TGF} *for all* L/a.
- Padè fit (constrain to PT), 4 parameters, $\chi^2/\mathrm{ndof} = 5.9/7$.
- Example: $L/a = 36$

β	$g^2_{TGF}(L)$
12.0	0.41078(64)
10.0	0.51809(83)
8.0	0.6987(11)
7.0	0.8497(13)
6.0	1.0819(18)
5.0	1.4968(25)
4.0	2.4465(44)
3.75	2.9277(54)
3.5	3.6494(69)
3.25	4.8568(99)
3.0	7.587(20)
2.9	10.610(32)
2.8	16.752(47)
2.75	22.168(59)



- Modest cutoff effects. Starting recursion with $u = 7.5$.



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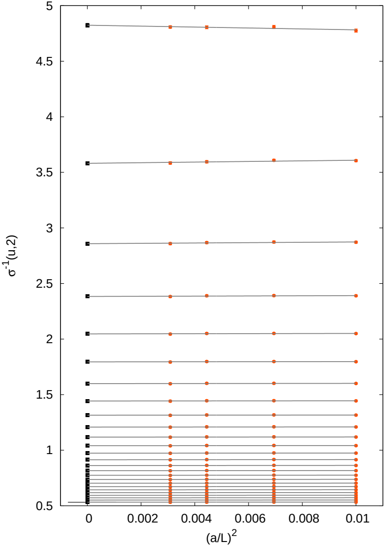
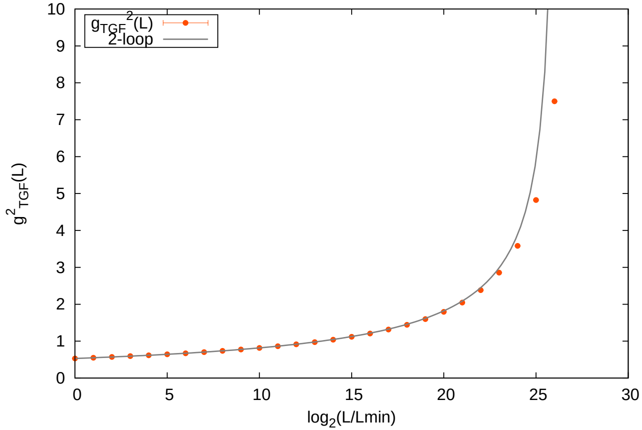


$g^2_{TGF}(L)$ for pure gauge $SU(2)$



Since $\Lambda = \mu(b_0 g^2(\mu))^{-b1/2b_0^2} e^{-1/2b_0 g^2(\mu)} e^{-\int_0^{g^2(\mu)}\left\{\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right\}}$ and $\mu = 1/cL$.

$$\Lambda L_{\rm max} = 1.509(44) \quad (@g^2_{TGF}(L) = 1.7948(93))$$

$$\Lambda L_{\rm max} = 1.532(69) \quad (@g^2_{TGF}(L) = 1.4430(92))$$

Fermions

"We love the gluons, we love the quarks as well"
– Simon D. (German Composer) –

Fundamental fermions and twisted boundary conditions

$$\psi(x + L\hat{\mu}) = \Omega_\mu \psi(x)$$

And therefore

$$\begin{aligned} \psi(x + L\hat{1} + L\hat{2}) &= \Omega_1 \Omega_2 \psi(x) \\ \psi(x + L\hat{2} + L\hat{1}) &= \Omega_2 \Omega_1 \psi(x) = e^{2\pi \imath m/N} \Omega_1 \Omega_2 \psi(x) \end{aligned}$$

- $SU(N)$ with N_f fundamental fermions only possible if $N_f/N \in \mathbb{Z}$ [Parisi '83].
- No problem with multi-index representations (i.e. adjoint same as gauge field).

Automatic $\mathcal{O}(a)$ improvement

Massless ($m^{\mathrm{PCAC}} = 0$) quarks (Wilson) do not need c_{SW}.

Conclusions

- New running coupling scheme based on Wilson Flow and twisted bc.
- Compared with periodic bc:
  - "Easy" perturbation theory. Improved coupling definition.
  - $g^2_{TGF} = g^2_{\overline{\rm MS}} + \mathcal{O}(g^4_{\overline{\rm MS}})$ for all $SU(N)$ group and matter content.
  - Universal two loop β-function.
- Compared with Schrödinger functional
  - No boundaries $\rightarrow$ No need of boundary counterterms. No $c_t, c_{\tilde{t}}$.
  - "Automatic" $\mathcal{O}(a)$ improved, even with (massless) Wilson fermions. No need of c_{SW}.
- Signal to noise ratio (almost) independent on a/L.
- Only one drawback: $SU(N)$ with N_f fermions in the fundamental representation requires $N_f/N \in \mathbb{Z}$.
- Fermions in multi-index (adjoint) representation are ok.
- Nice scheme for $SU(2)$ with adjoint/sextet fermions, $SU(3)$ with 3, 6, 9, 12 fundamental fermions, etc...
- I have shown the viability:
  - $SU(2)$ YM running coupling over 8 orders of magnitude.
  - Precise even with modest statistics.
  - Mild cutoff effects.

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Thanks!