The QCD phase diagram at strong coupling including auxiliary field fluctuations

Research talk about severity of the sign problem at strong coupling to investigate the QCD phase diagram on a large size lattice.

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Finite chemical potential region

- The sign problem
  - Caused by chemical potential
  - Complexity of the weight
  - Weight cancellation
  - Difficulty in studying finite chemical potential region

Avoiding or weakening the sign problem

- Ways to study finite chemical potential region
  - Reweighting Z. Fodor, S. D. Katz, (2002)…
  - Taylor expansion C. R. Allton *et al* (2002,2005) R. V. Gavai,S. Gupta (2008), S. Ejiri et al., (2010) …
  - Imaginary chemical potential M. G. Alford et al,. (1999). P. de Forcrand and O. Philipsen, (2002)…
  - Complex Langevin Matsui and Nakamura (1987) G. Aarts et al. (2010) …
  - Canonical approach Miller and Redlich (1987) Engels et al. (1999) A.Li, Meng et al.(2010) …
  - Strong coupling

  ………….etc.

Sign problem & Strong coupling lattice QCD

$$S_{\mathrm{LQCD}} = S_{\mathrm{F}} + S_{\mathrm{G}}$$

$$\frac{1}{g^2} \square U \sim F_{\mu\nu}^2$$

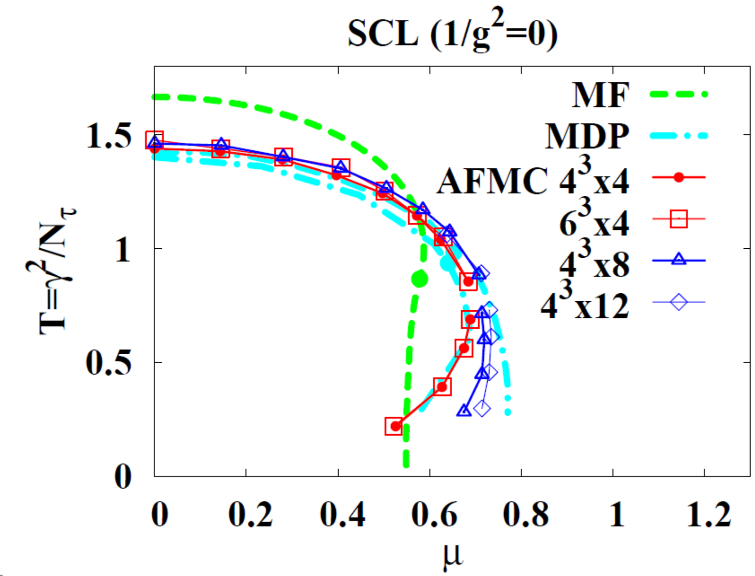
- Characteristics
  - Starting from lattice QCD
  - $1/g^2$ expansion
    - Expansion by inverse coupling
- No sign problem in the mean field approximations
  - Chiral transition

    N. Kawamoto and J. Smit (1981), P. H. Damgaard, N. Kawamoto and K. Shigemoto(1984) etc.
  - The QCD phase diagram

    Bilic, Karsch, Redlich ('92), Fukushima ('04), Nishida ('04) etc.
- "The sign problem" with fluctuations
  - Monomer-Dimer-Polymer simulations

    W. Unger, Ph. de Forcrand,
    J. Phys. G: Nucl. Part. Phys. **38** 124190 (2011)
  - Auxiliary field Monte-Carlo method

    A. Ohnishi, T. I. and T. Z. Nakano : arXiv:1211.2282



Severity of the sign problem

- $\Delta f (= f^{\text{full}} - f^{\text{p.q.}})$, the difference of the free energy density in full and phase quenched MC simulations

$$e^{-\Omega \Delta f} = \frac{Z_{\text{full}}}{Z_{\text{p.q.}}} = \frac{\int \mathcal{D}\sigma \mathcal{D}\pi e^{-S_{\text{eff}}^{\text{p.q.}}} e^{i\theta}}{\int \mathcal{D}\sigma \mathcal{D}\pi e^{-S_{\text{eff}}^{\text{p.q.}}}} = \left\langle e^{i\theta} \right\rangle_{\text{p.q.}}$$

$Z = \exp[-F/T] = \exp[-\Omega f]$

Ω : space-time volume

p. q. : phase quenched

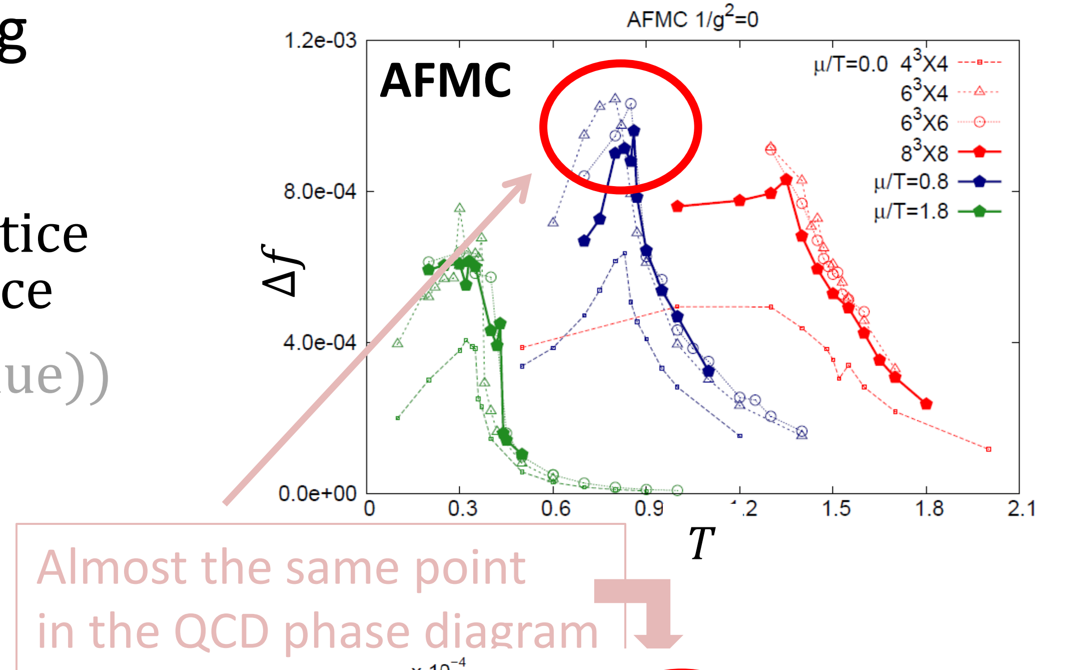
  - Case : lower reliability in numerical simulations
    1. Large $\Delta f (= f^{\text{full}} - f^{\text{p.q.}})$
    2. Large lattice size

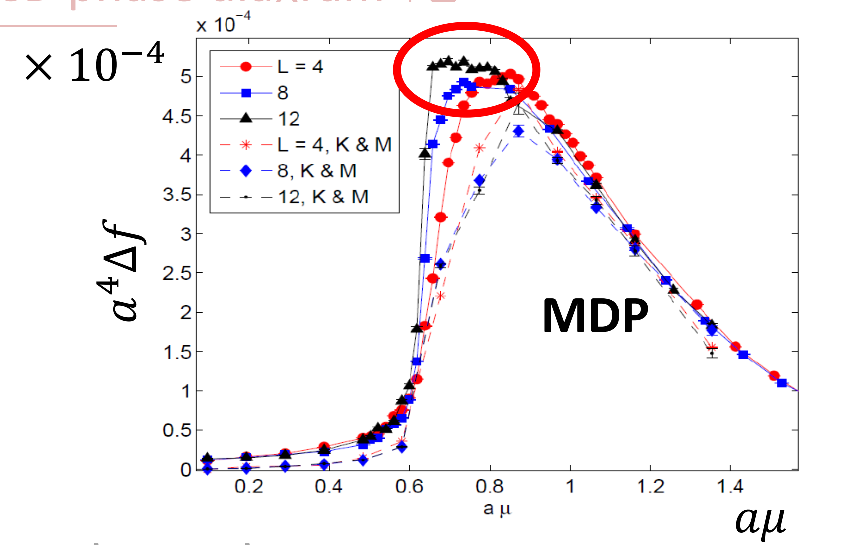
⇨ small average phase factor

⇨ severe weight cancellation

"The sign problem" at strong coupling

- Two ways at strong coupling
  - Auxiliary field Monte-Carlo (AFMC) method
    - Saturated value for a lattice larger than $6^3 \times N_\tau$ lattice
    - $\Delta f\,(\text{AFMC(Saturated value)}) \cong 1.0 \times 10^{-3}$
  - Monomer-Dimer-Polymer (MDP) simulation
    - $\Delta f\,(\text{MDP}) \cong 0.5 \times 10^{-3}$





- AFMC has more sever cancellation
  - $\Delta f\,(\text{AFMC}) \cong 2 \times \Delta f\,(\text{MDP})$

⇨ we need to improve AFMC method for a larger lattice

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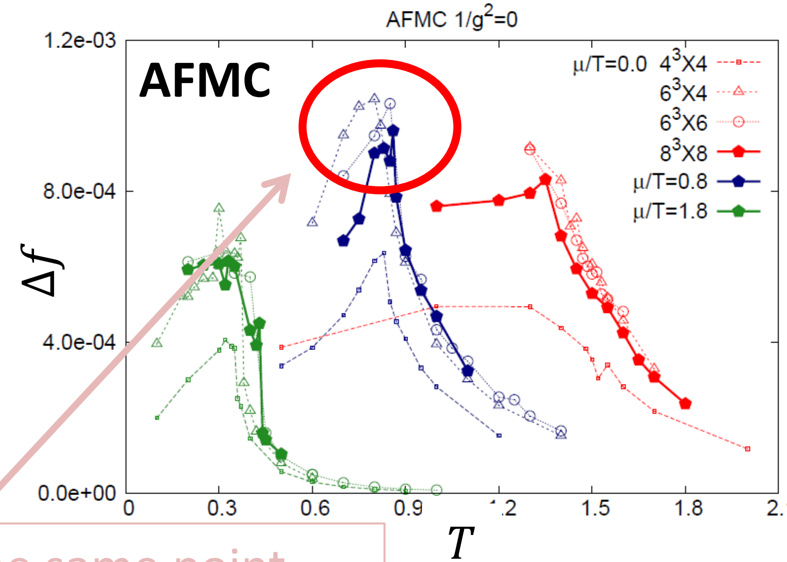
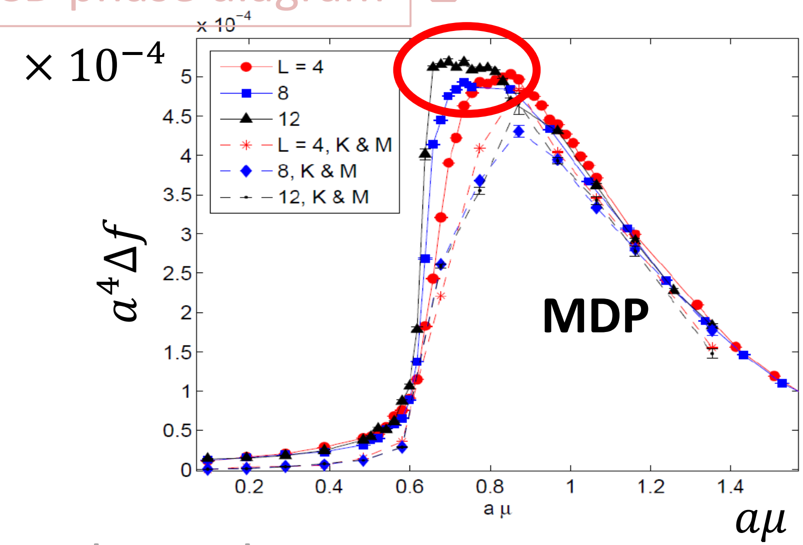


Almost the same point in the QCD phase diagram



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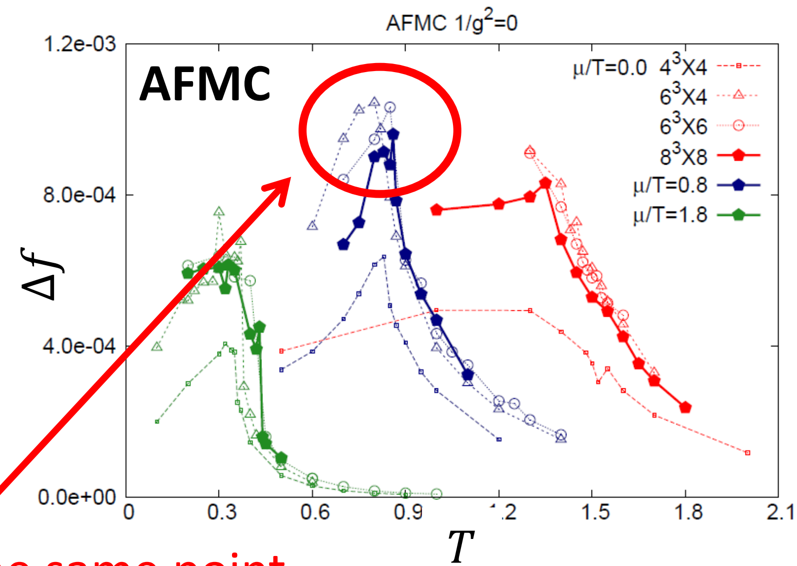

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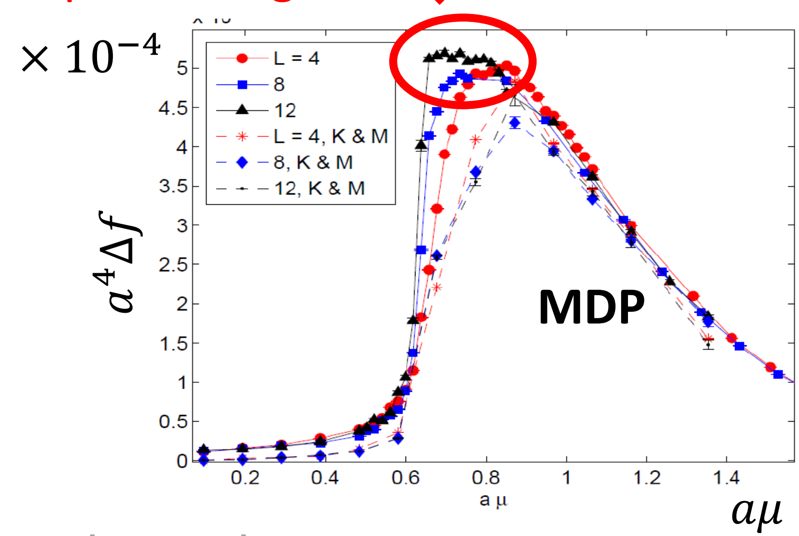

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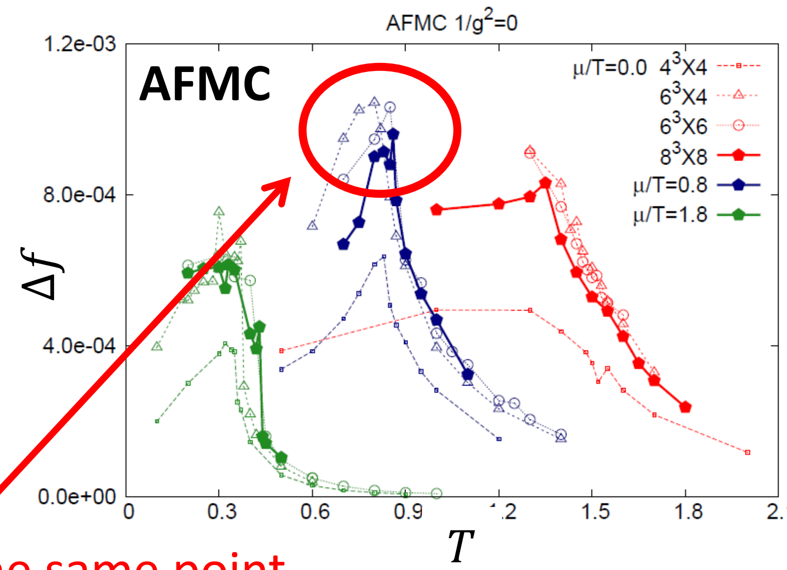

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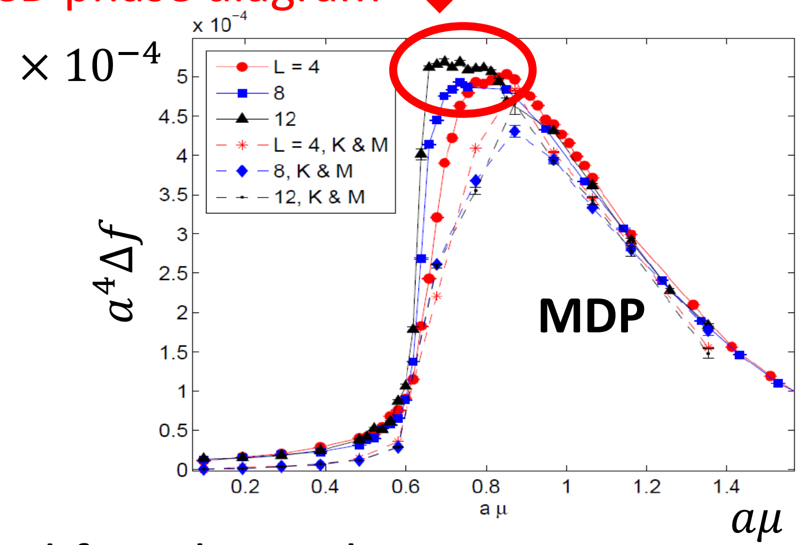


- AFMC has more sever cancellation
  - $\Delta f\,(\mathrm{AFMC}) \cong 2 \times \Delta f\,(\mathrm{MDP})$

⇨ we need to reduce Δf in AFMC method for a larger lattice

Purpose

- To discuss the source of “the sign problem” in Auxiliary field Monte-Carlo (AFMC) method
- To explore the possibility of applying AFMC method on a large lattice

The effective action & AFMC method

- Effective action for Auxiliary fields   Faldt and Petersson (1986)

$$S_{\rm eff}^{\rm AF} = \frac{L^3}{4N_c} \sum_{\boldsymbol{k},\tau,f(\boldsymbol{k})>0} f(\boldsymbol{k}) \left[|\sigma_{\boldsymbol{k},\tau}|^2 + |\pi_{\boldsymbol{k},\tau}|^2 \right] - \sum_{\boldsymbol{x}} \log \left[X_{N_\tau}(\boldsymbol{x})^3 - 2X_{N_\tau}(\boldsymbol{x}) + 2\cosh(N_c\mu/T) \right]$$

  - $X_{N_\tau} = X_{N_\tau}[m_x], \quad m_x = m + \frac{1}{4N_c}\sum_j (\sigma + i\epsilon\pi)_{x\pm\hat{j}}$ complex
  - Smaller phase at larger μ

$$f(\boldsymbol{k}) = \sum_{j=1}^{d} \cos k_j$$

$$\epsilon = (-1)^{x_0+\cdots+x_d}$$

- The Auxiliary field Monte-Carlo (AFMC) method

$$Z = \int \mathcal{D}\left[\sigma_{\boldsymbol{k},\tau}, \pi_{\boldsymbol{k},\tau}\right] e^{-S_{\rm eff}^{\rm AF}}$$

Auxiliary fields

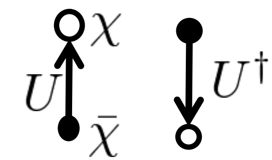
Most of previous works
 : Mean Field approximations
This go-round
 : Monte-Carlo simulations

Strong coupling lattice QCD action

- Procedure in the strong coupling limit
  - Unrooted staggered fermion, anisotropic lattice, strong coupling limit

$$S_{\mathrm{SCL-LQCD}} = \frac{1}{2}\sum_{\nu=0}^{d}\sum_{x}\left[\eta_{\nu,x}^{+}\bar{\chi}_x U_{\nu,x}\chi_{x+\hat{\nu}} - \eta_{\nu,x}^{-}(\mathrm{H.C.})\right]$$

Kinetic



$$+m\sum_{x}\bar{\chi}_x\chi_x$$

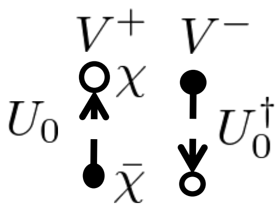
Mass

$$\eta_{\nu,x}^{\pm} = \left(\gamma e^{\pm\mu a_\tau}, (-1)^{x_1+\cdots+x_{\nu-1}}\right)$$

  - 1/*d* expansion, U_j integration N.Bilic et al. (1992), G. Faldt et al.(1986)

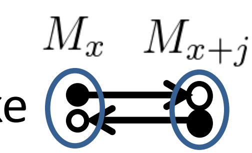
$$S_{\mathrm{eff}} = \frac{1}{2}\sum_{x}\gamma\left[e^{\mu a_\tau}\bar{\chi}_x U_0\chi_{x+\hat{0}} - e^{-\mu a_\tau}\bar{\chi}_{x+\hat{0}}U_0^{\dagger}\chi_x\right]$$

Temporal Kinetic



$$-\frac{1}{4N_c}\sum_{x,j}M_x M_{x+\hat{j}}$$

4 fermi-like



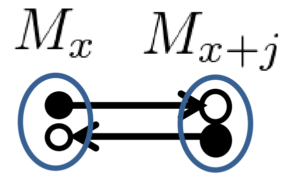
$$+m\sum_{x}M_x$$

Mass

$$M_x = \bar{\chi}_x\chi_x$$

The AFMC method at strong coupling

- Extended HS transformation
  - Taking different value at each site
  - Necessity to introduce complex term



4 fermi-like

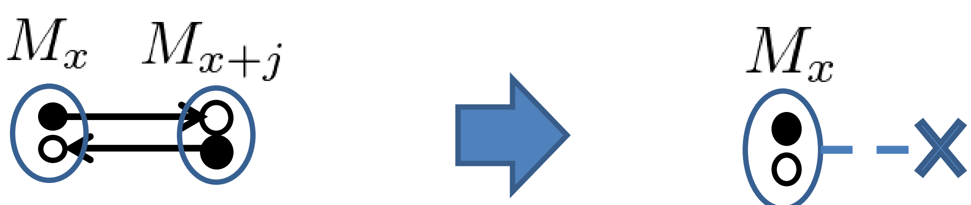
$M_x = \bar{\chi}_x \chi_x$

$$\exp\left[\alpha AB\right]$$

Notice !

$$= \int \mathcal{D}\left[\phi, \varphi\right] \exp\left[-\alpha\left[\phi^2 + \varphi^2 + (A+B)\varphi - i(A-B)\phi\right]\right]$$

- Bosonization



4 fermi-like

$M_x = \bar{\chi}_x \chi_x$

AFMC method

Auxiliary fields

$\sigma_x = -\left\langle \bar{\chi}_x \chi_x \right\rangle$

$\pi_x = \left\langle i\epsilon \bar{\chi}_x \chi_x \right\rangle$

$\epsilon = (-1)^{x_0 + \cdots + x_d}$

$$Z = \int \mathcal{D}\left[\sigma, \pi\right] e^{-S_{\rm eff}}$$

- Effective mass

$$m_x = m + \frac{1}{4N_c} \sum_j (\sigma + i\epsilon\pi)_{x \pm \hat{j}}$$

The source of the "sign problem"

- Integrating out numerically?
  - Complex effective mass causes complex weight

$$m_x = m + \frac{1}{4N_c}\sum_j (\sigma + i\epsilon\pi)_{x\pm\hat{j}}$$

$$\epsilon = (-1)^{x_0+\cdots+x_d}$$

$$Z = \int \mathcal{D}[\sigma,\pi]\, e^{-S_{\text{eff}}}$$

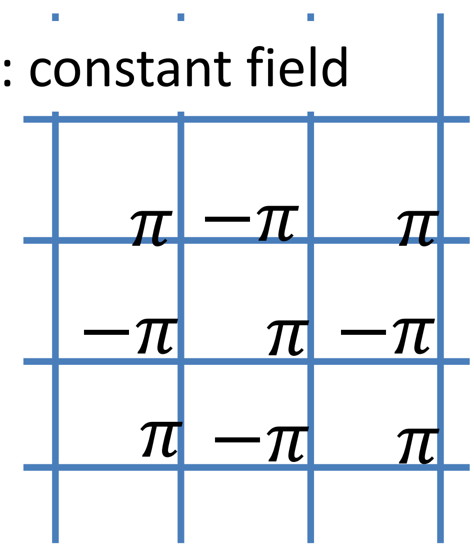
$$\sigma_x = -\langle \bar{\chi}_x \chi_x \rangle$$

$$\pi_x = \langle i\epsilon \bar{\chi}_x \chi_x \rangle$$

- High momentum auxiliary fields
  - Low momentum
    - Cancellation mechanism
    - Small phase
  - High momentum
    - No cancellation mechanism
    - Severe weight cancellation (?)

⇨ We confirm qualitatively

Case : constant field



Results

Reservations

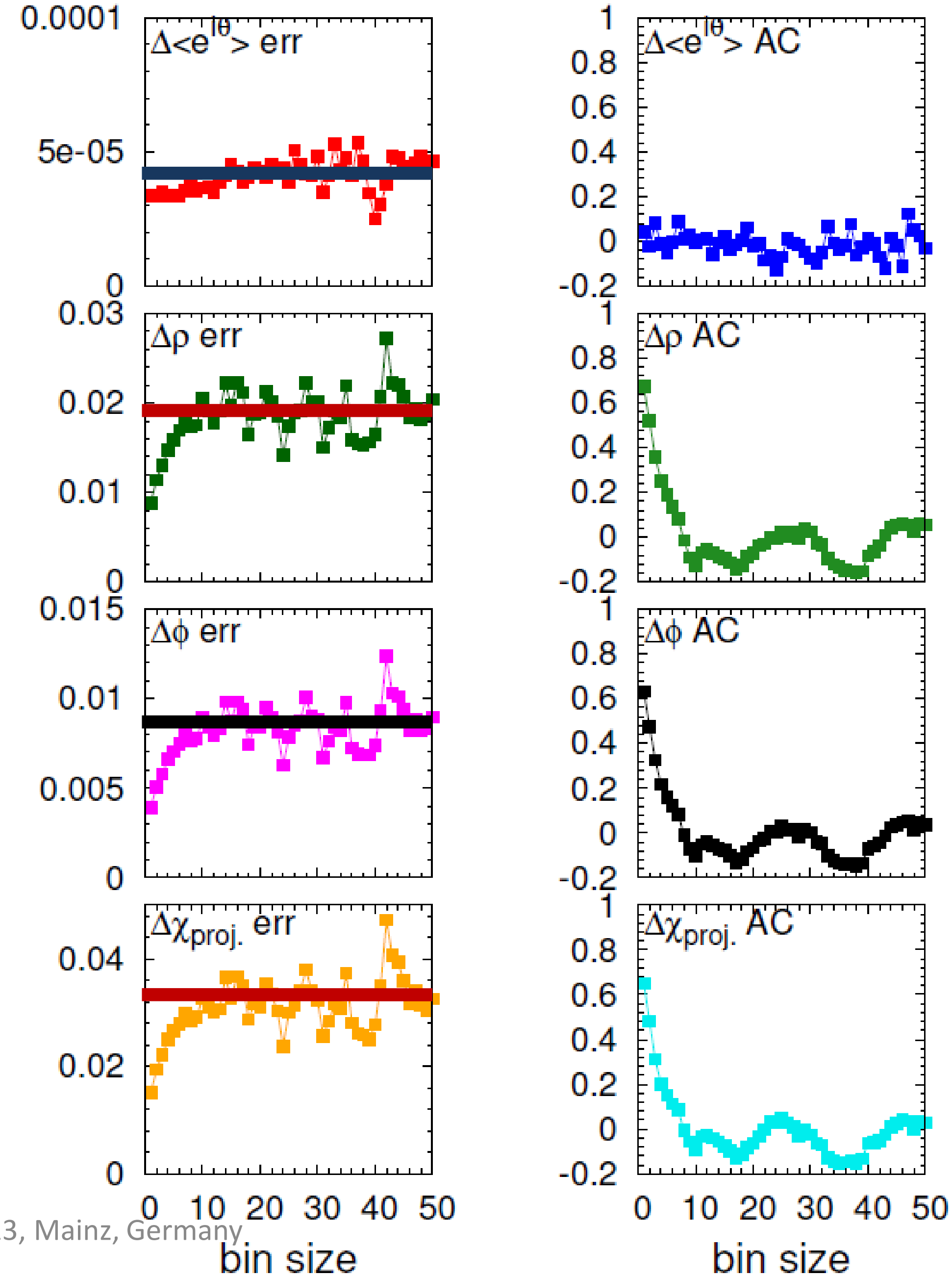
- Unrooted staggered fermion
- anisotropic lattice
- chiral limit
- All results are in lattice unit
- Jack knife methods

Evaluation of error bars

- Jack knife method
  - Error bars
    - taking plateau value after auto correlation disappear

$8^3 \times 8, \mu/T = 0.8,$
$T = 0.85,\ \Lambda = 0.00$ data

Jack knife error　　Auto Correlation

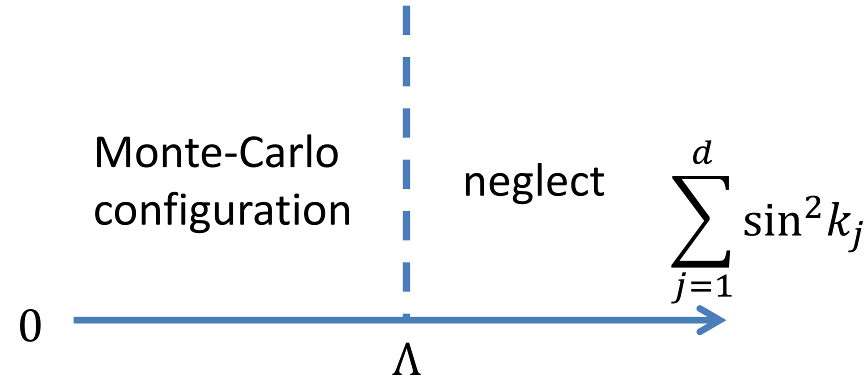


Auxiliary field momentum cut off

$$\sum_{j=1}^{d} \sin^2 k_j$$

- High momentum
  ≡High momentum modes of spatial kinetic momentum

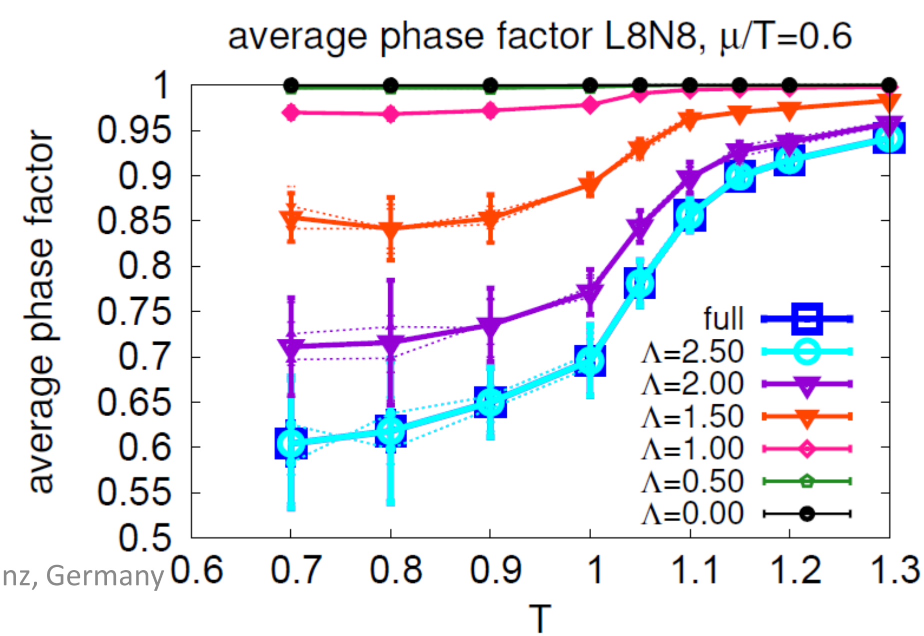
- cutting off high momentum auxiliary field components



Reductions of weight cancellations?

- Qualitative confirmations
  - Average phase factor goes to 1
  - Weight cancellations weaken

e.g. size $8^3 \times 8, \mu/T = 0.6$



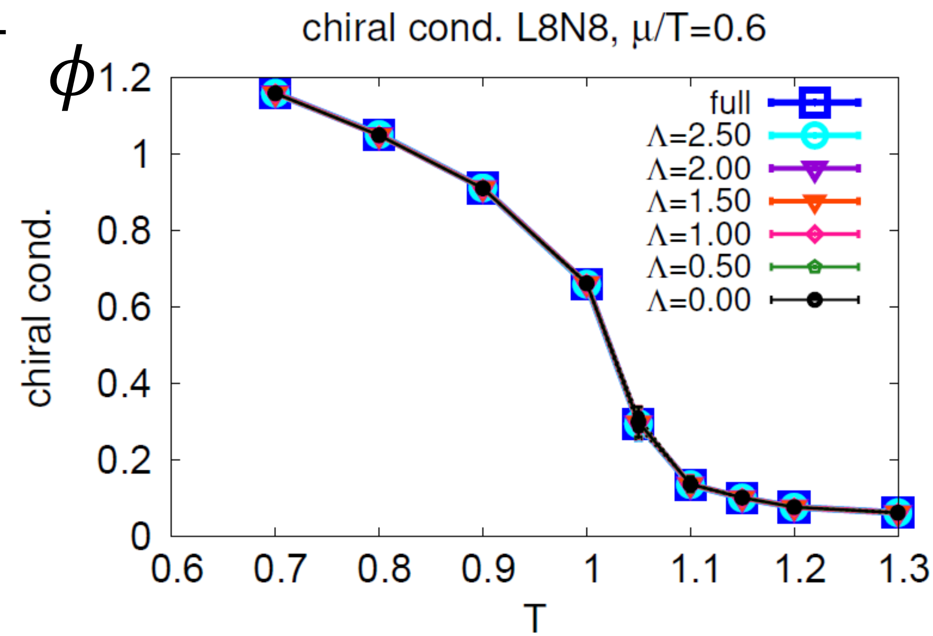
Cut-off dependence of order parameters

- Results of size $8^3 \times 8, \mu/T = 0.6$

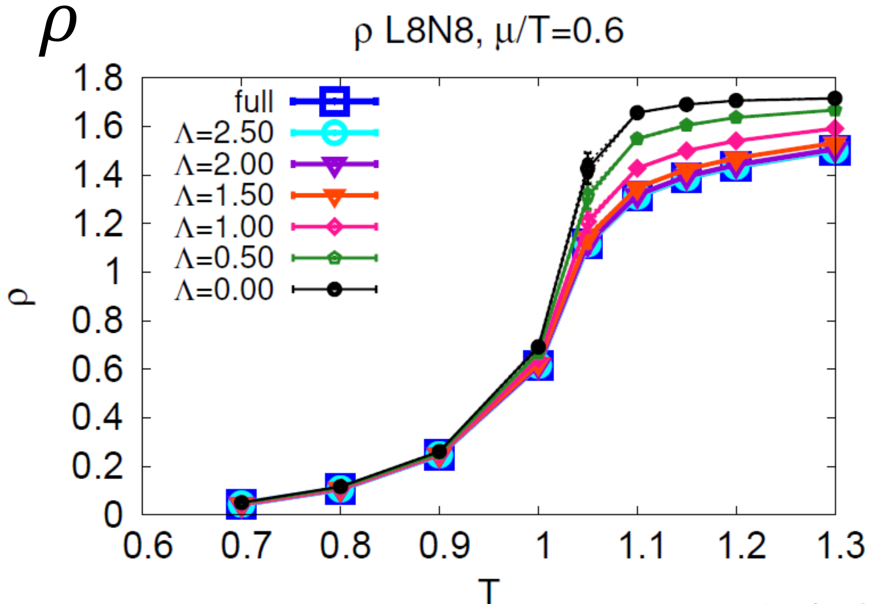
■ Chiral condensate : $\phi = \sqrt{\sigma^2 + \pi^2}$ because of chiral limit

- □ increasing around phase boundary
- □ almost no cut-off dependence $\phi(\Lambda = \text{full}) \simeq \phi(\Lambda \geq 0.0)$

ϕ



ρ



■ Quark number density

- □ increasing around phase boundary
- □ same value as long as $\Lambda \gtrsim 2.0$ $\rho(\Lambda = \text{full}) \simeq \rho(\Lambda \geq 2.0)$

$$\rho = -\frac{1}{N_\tau L^3} \frac{\partial \log Z}{\partial \mu}$$

Susceptibilities

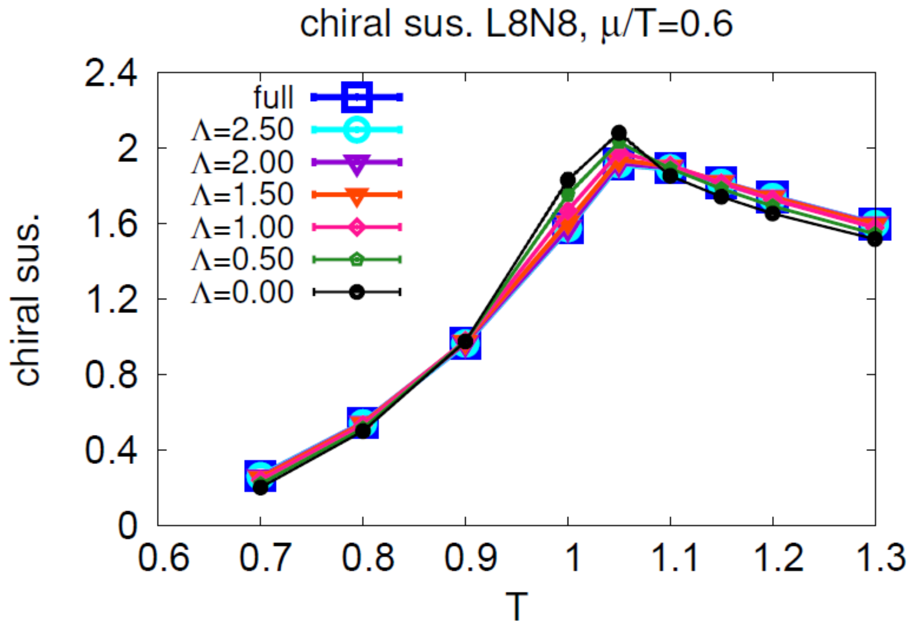
- Results of size $8^3 \times 8, \mu/T = 0.6$

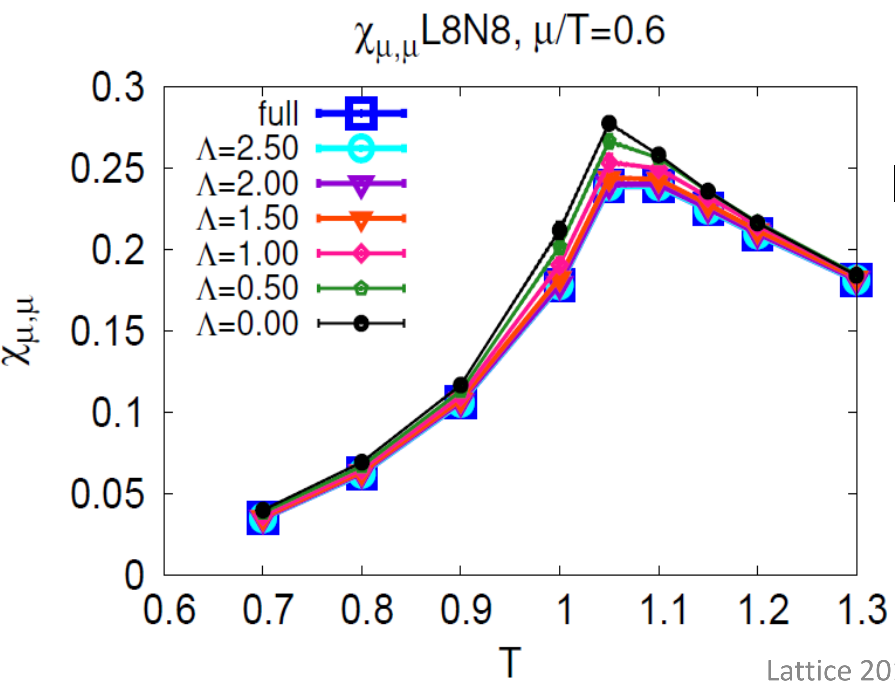
■ Chiral susceptibility

□ peak around phase boundary

□ same value as long as $\Lambda \gtrsim 2.0$

$\chi(\Lambda = \text{full}) \simeq \chi(\Lambda \geq 2.0)$





■ Quark number susceptibility

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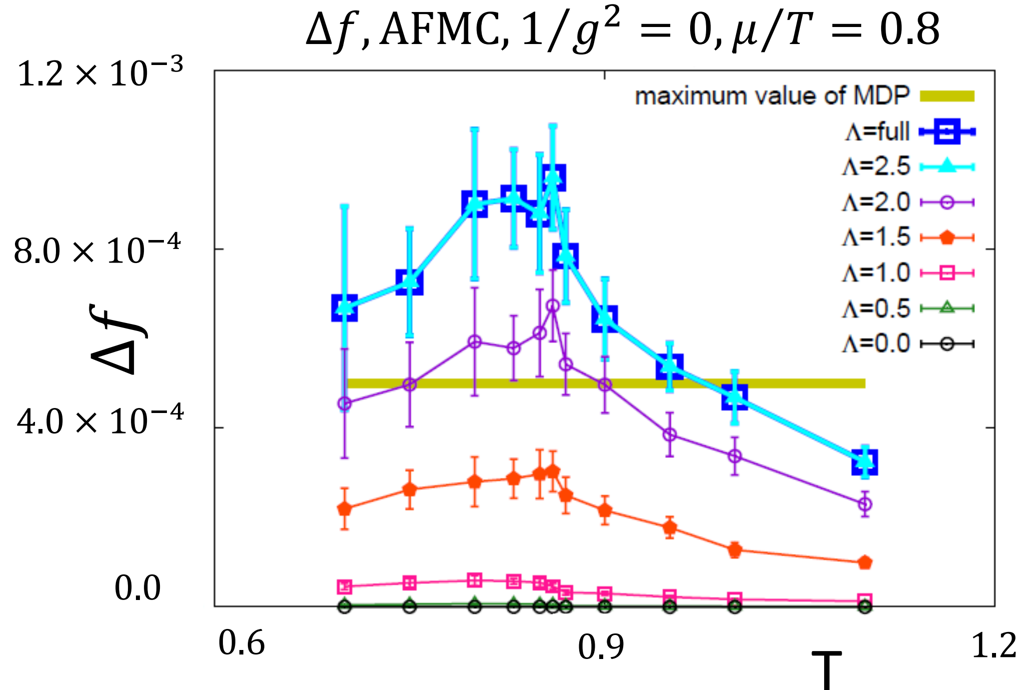
$\chi_{\mu,\mu}(\Lambda = \text{full}) \simeq \chi_{\mu,\mu}(\Lambda \geq 2.0)$

$$\chi_{\mu,\mu} = -\frac{1}{N_\tau L^3} \frac{\partial^2 \log Z}{\partial \mu^2}$$

Cut-off dependence of each quantities

- Critical cut-off where average phase factor becomes large



& order parameters do not depend on cut-off

$\Phi(\Lambda = \mathrm{full}) \simeq \Phi(\Lambda > \Lambda_c)$

These results indicate

we could investigate phase transition phenomena on a large lattice by cutting off or by approximately integrating out the high momentum auxiliary fields.

Summary

- The source of the “sign problem” in AFMC method
  - High momentum auxiliary field components
- Cutting off high momentum auxiliary fields
  - Weight cancellations weaken
  - The region where quantities of phase transition phenomena do not depend on cut-off
- We try for a much larger lattice