The Phase Diagram of Strong Coupling Lattice QCD including Gauge Corrections

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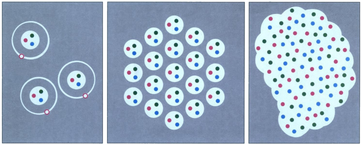

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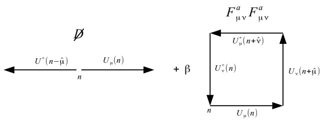
Why Strong Coupling Lattice QCD?

Look at Lattice QCD in a regime where the **sign problem** can be made **mild**:
Strong Coupling Limit: $\beta = \frac{2N_c}{g^2} \to 0$

- allows to integrate out the gauge fields completely, as **link integration factorizes**
  $\Rightarrow$ no fermion determinant
- drawback: strong coupling limit is converse to asymptotic freedom, lattice is maximally coarse



Strong coupling LQCD shares important features with QCD:

- exhibits **"confinement"**, only color singlet degrees of freedom survive:
  - **mesons** (represented by monomers and dimers)
  - **baryons** (represented by oriented self-avoiding loops)
- and **spontaneous chiral symmetry breaking/restoration:** (restored at T_c)
  $\Rightarrow$ SC-LQCD is a great laboratory to study the full (μ, T) **phase diagram**

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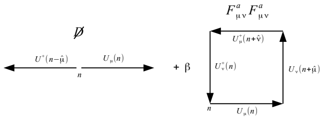

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SC-LQCD is a **1-parameter deformation of QCD in** β

Chiral transition and nuclear transition

This talk: focus on chiral transition and nuclear transition in the **chiral limit**

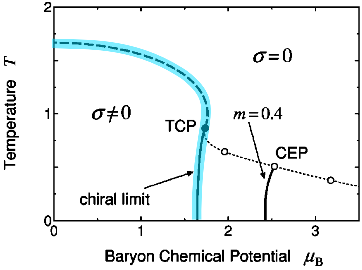
Chiral symmetry in SC-LQCD with **staggered fermions** for $N_f = 1$:

$$U(1)_V \times U(1)_{55}: \qquad \psi(x) \mapsto e^{i\epsilon(x)\theta_A + i\theta_V}\psi(x), \quad \epsilon(x) = (-1)^{x_1+x_2+x_3+x_4}$$

- $U(1)_V$ baryon number conserved
- $U(1)_{55}$ chiral symmetry spontaneously broken at low temperatures/densities
- expected to be O(2) 2nd order ($\mu = 0$)
- note: no chiral anomaly at $\beta = 0$

Nuclear Transition (T=0):

- baryon crystal forms
- chiral symmetry restored
- expected to be 1st order



Strong coupling phase diagram via Mean field: Nishida, PRD 69 (2004)

Long History of Staggered SC-LQCD

Mean field ($1/d$ expansion):

1983: development of the technique [Kluberg-Stern, Morel, Petersson]

1985: first **finite density analysis** [Damgaard, Hochberg & Kawamoto]

1992: $T_c(\mu = 0) = 5/3$, $\mu_c(T = 0) = 0.66$ [Bilic *et al.*]

1995: entropy per baryon [Bilic & Cleymans]

2004: full phase diagram and location of (tri)**crital point** [Nishida]

2009: include $\mathcal{O}(\beta)$ corrections [Ohnishi *et al.*]

Monte Carlo:

1984: formulation as a **dimer system** [Rossi & Wolff]

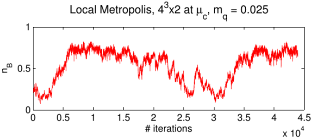
1989: first **finite density results** with MDP algorithm, $aT_c(\mu = 0) = 1.4$, $a\mu_c(T = 0) = 0.63$ [Karsch & Mütter]

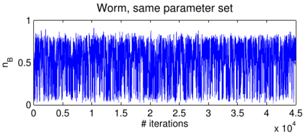
2003: first **Worm algorithm** applied to U(3): **fast**, easy to do **chiral limit** [Adams & Chandrasehkaran]

2010: full phase diagram and nuclear potential for SU(3) [de Forcrand & Fromm]

2011: continuous Euclidean time methods [de Forcrand & U.]

2011: include **$\mathcal{O}(\beta)$ corrections for U(3)** [Langelage, de Forcrand, Fromm, Miura, Philipsen, U.]





Strong Coupling Partition Function

After SU(N_c) gauge link integration only **hadronic d.o.f.** survive:

$$M_x = \bar{\chi}\chi(x), \quad B_x = \frac{1}{N_c!}\epsilon_{i_1\dots i_{N_c}}\chi_{i_1}\cdots\chi_{i_{N_c}}$$

Exact rewriting after Grassmann integration: Mapping to a MDP representation:

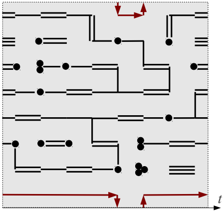
$$\mathcal{Z}(m_q,\mu,\gamma) = \sum_{\{k,n,\ell\}} \underbrace{\prod_{b=(x,\mu)} \frac{(N_c-k_b)!}{N_c!k_b!}\gamma^{2k_b\delta_{\mu 0}}}_{\text{meson hoppings } M_x M_y} \underbrace{\prod_x \frac{N_c!}{n_x!}(2am_q)^{n_x}}_{\text{chiral condensate } M_x} \underbrace{\prod_\ell w(\ell,\mu)}_{\text{baryon hoppings } \bar{B}_x B_y}$$

$$k_b \in \{0,\dots N_c\},\ n_x \in \{0,\dots N_c\},\ \ell_b \in \{0,\pm 1\}$$

- Grassmann constraint:

$$n_x + \sum_{\hat{\mu}=\pm\hat{0},\dots\pm\hat{d}} \left(k_{\hat{\mu}}(x) + \frac{N_c}{2}|\ell_{\hat{\mu}}(x)|\right) = N_c$$

- weight $w(\ell,\mu)$ and sign $\sigma(\ell) \in \{-1,+1\}$ for oriented baryonic loop ℓ depends on loop geometry



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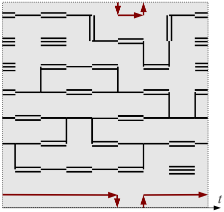


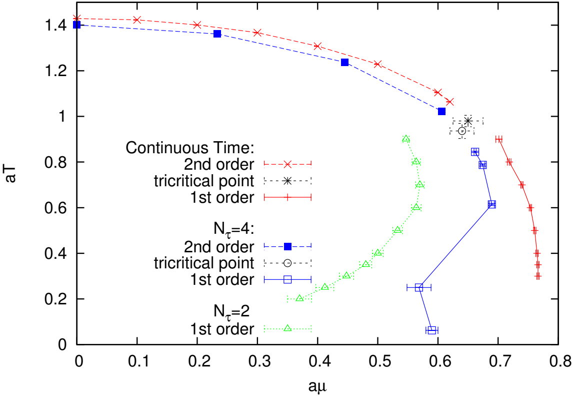
SC-LQCD Phase Diagram

Comparison of phase boundaries for $N_\tau = 2, 4$ and $N_\tau \to \infty$ (continuous time), studied with Worm algorithm [hep-lat/1111.1434]

identifications:

$$aT = \frac{\gamma^2}{N_\tau}$$

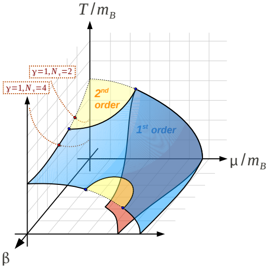
$$a\mu = \gamma^2 a_\tau \mu$$

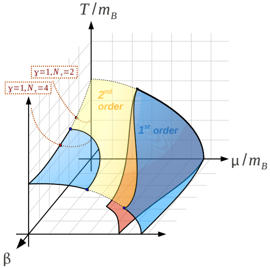


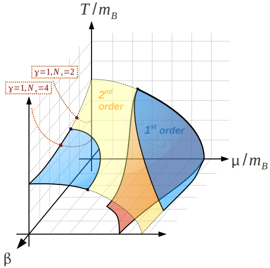
- behavior at low μ qualitatively the same, first order transition shifts to larger μ
- no re-entrance in continuous time (also seen by [Ohnishi *et al.* 2012] via auxilliary field Monte Carlo, see also $\to$ talk by T.Ichihara

Connection Between Strong Coupling and Continuum Limit?

Various possible scenarios for the extension to finite β:







- back plane: strong coupling phase diagram
- front plane: continuum phase diagram ($N_\mathrm{f} = 4$)

Questions we want to address:

- does **tricritical point** move to smaller or larger μ as β is increased?
- do the **nuclear and chiral transition split?**

Derivation of $\mathcal{O}(\beta)$ effective action

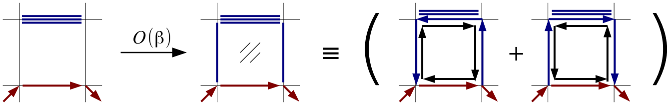
- Strong Coupling Partition function incorporating $\mathcal{O}(\beta)$ corrections:

$$Z = \int d\chi d\bar{\chi} dU e^{S_G + S_F} = \int d\chi d\bar{\chi} Z_F \left\langle e^{-S_G} \right\rangle_U$$

$$\langle O \rangle_U = \frac{1}{Z_F} \int dU O e^{-S_F}, \qquad Z_F = \int dU e^{-S_F} = \prod_{l=(x,\mu)} z(x,\mu)$$

- plaquette expectation value before Grassmann integration:

$$\left\langle \mathrm{tr}[U_P + U_P^\dagger] \right\rangle_U = \frac{1}{Z_F} \int dU \mathrm{tr}[U_P + U_P^\dagger] e^{-S_F} = \left(\prod_{l \in P} z_l \right)^{-1} \sum_{s=1}^{19} F_P^s(M, B, \bar{B})$$



Link Integrations for $\mathcal{O}(\beta)$ diagrams

One-Link integrals for links on the edge of an elementary plaquette (based on techniques from [Creutz 1978], [Azakov & Aliev 1988]):

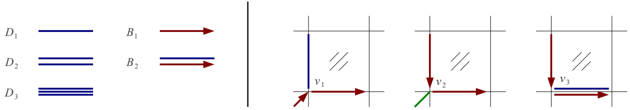
$$J_{ik} = \underbrace{\frac{1}{3}\bar{\chi}_k\varphi_i}_{D_1} - \underbrace{\frac{1}{6}M_\chi M_\varphi\bar{\chi}_k\varphi_i}_{D_2} + \underbrace{\frac{1}{12}M_\chi^2 M_\varphi^2\bar{\chi}_k\varphi_i}_{D_3} + \underbrace{\frac{1}{12}\epsilon_{ii_1i_2}\epsilon_{kk_1k_2}\bar{\varphi}_{i_1}\bar{\varphi}_{i_2}\chi_{k_1}\chi_{k_2}}_{B_1}$$

$$+\underbrace{\frac{1}{32}\epsilon_{ii_1i_2}\epsilon_{kk_1k_2}M_\chi M_\varphi\bar{\varphi}_{i_1}\bar{\varphi}_{i_2}\chi_{k_1}\chi_{k_2} + \frac{7}{24}\bar{B}_\varphi B_\chi\bar{\chi}_k\phi_i + \frac{1}{48}\epsilon_{ii_1i_2}M_\varphi B_\chi\bar{\varphi}_{i_1}\bar{\varphi}_{i_2}\bar{\chi}_k + \frac{1}{48}\epsilon_{kk_1k_2}M_\chi\bar{B}_\varphi\chi_{k_1}\chi_{k_2}\varphi_i}_{B2}$$

- determine plaquette link product $P = \mathrm{Tr}\, J_{ik}J_{kl}J_{lm}J_{mi}$
- result can be consistently re-expressed via

  link weights: $w(D_k) = \frac{(N_c-k)!}{N_c!(k-1)!}$, $\quad w(B_1) = \frac{1}{N_c!(N_c-1)!}$, $\quad w(B_2) = \frac{(N_c-1)!}{N_c!}$

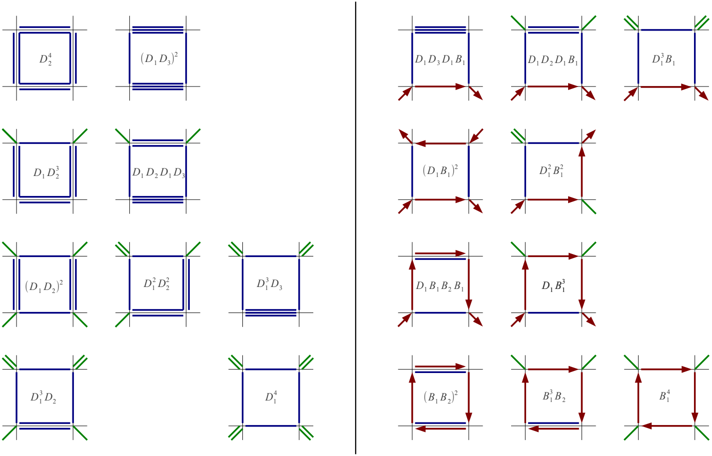
  and **site weights:** $v_1 = N_c!$, $\quad v_2 = (N_c-1)!$, $\quad v_3 = 1$



- Grassman constraint on sites touching a plaquette altered $N_c \to N_c + 1$

Classification of $\mathcal{O}(\beta)$ Diagrams

Diagrams classified by external legs (monomers or external dimers)



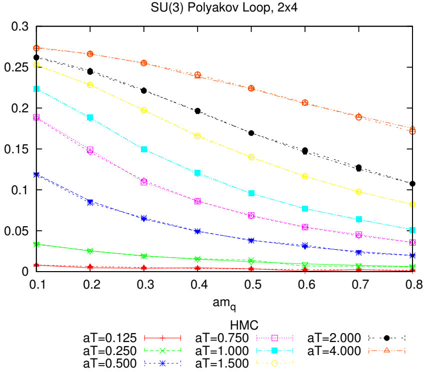
mesonic sector

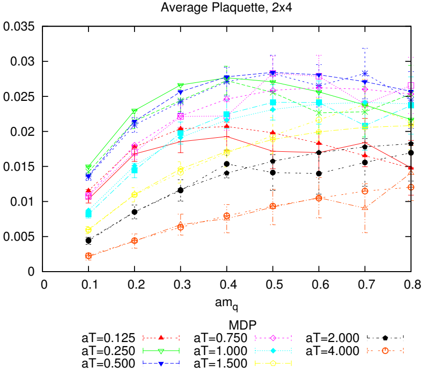
baryonic sector

Crosschecks at Finite Temperature

Croscheck on small lattices:

- comparison between **HMC** and **MDP** algorithms agrees well
- gauge observables are correctly obtained for **various** am_q, aT:



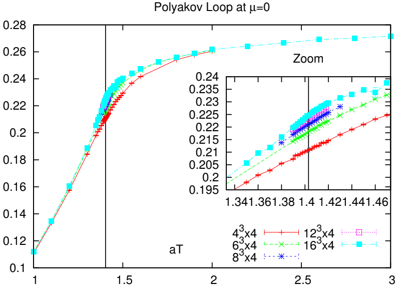


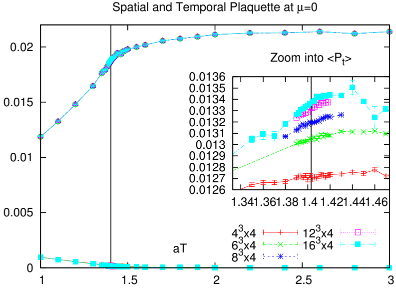
Gauge Observables at Zero Density

- Polyakov loop expectation value: ratio of partition function w/o static quark Q, measured via **reweighting from the SC-ensemble**:

$$\langle L \rangle = \frac{\int d\bar{\chi} d\chi \langle L \rangle_U Z_F}{\int d\bar{\chi} d\chi Z_F} \sim e^{-(F_Q - F_0)/T} = \frac{Z_Q}{Z}, \qquad L(\vec{x}) = \mathrm{Tr} J_{N_\tau,1}(\vec{x}) \prod_{t=1}^{N_\tau} J_{t,t+1}(\vec{x})$$

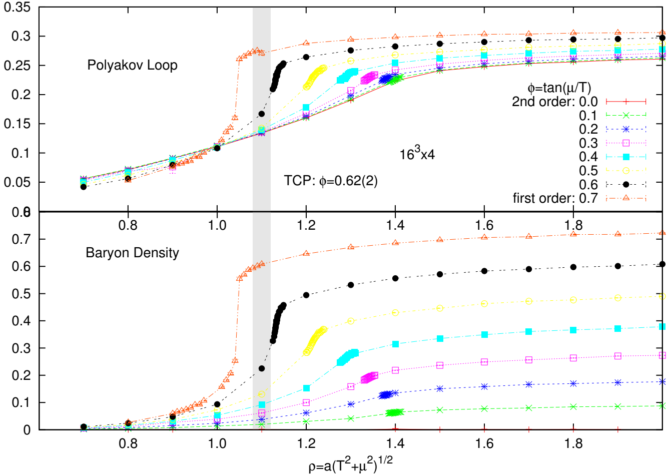
- $\langle L \rangle$ and $\langle P_t \rangle$ are sensitive to the chiral transition
- $\langle L \rangle$ rises significantly, indicating "deconfinement"





Gauge Observables at non-zero Density

- Scan at **finite density** in polar coordinates $(aT, a\mu) \mapsto (\rho, \phi)$
- Polyakov loop behaves similar to baryon number density, but also receives contributions from mesons



Chiral susceptibility in the chiral limit

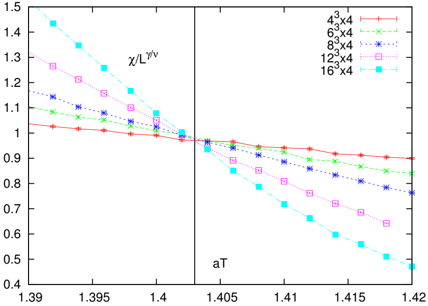
Full chiral susceptibility: $\chi = \frac{1}{V}\frac{\partial^2}{\partial(2am_q)^2}\log Z$ can be expressed in terms of monomers:

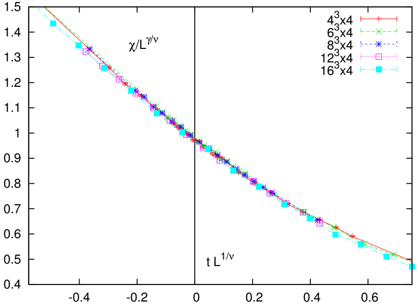
$$\chi = \frac{1}{(2am_q)^2 L^3 N_t}\left(\left\langle N_M^2\right\rangle - \langle N_M\rangle^2 - \langle N_M\rangle\right) = \frac{1}{L^3 N_t}\left(\sum_{x_1,x_2} G(x_1,x_2) - \frac{\langle N_M\rangle^2}{(2am_q)^2}\right)$$

In chiral limit:

- $\chi \sim \left\langle(\bar\psi\psi)^2\right\rangle$ is measured with high precision via Worm estimator $G(x_1,x_2)$
- χ has no peak, FSS via: $\chi_L/L^{\gamma/\nu}(t) = A + BtL^{1/\nu}, \quad t = \frac{T-T_c}{T_c}$

  with **3d O(2) critical exponents**





Taylor Expansion for the Susceptibility

For **fermionic observables**, the leading order β correction can be measured:

- obtain the slope of the transition temperature w.r.t. β from a Taylor coefficient:

$$\chi(\beta) = \chi_0 + \beta c_\chi^{(1)} + \mathcal{O}(\beta^2) \quad \text{with} \quad \chi_0 = \frac{Z_2}{Z},$$

$$c_\chi^{(1)} = \frac{\partial}{\partial\beta} \frac{Z_2(\beta)}{Z(\beta)}\bigg|_{\beta=0} = \left\langle (\bar{\psi}\psi)^2 P \right\rangle - \left\langle (\bar{\psi}\psi)^2 \right\rangle \langle P \rangle$$

- Z_2: 2-monomer sector sampled by $G(x_1, x_2)$ via Worm,
- necesssary condition: $c_\chi^{(1)}$ needs to obey **finite size scaling** to modify aT_c
- one can show that in the thermodynamic limit:

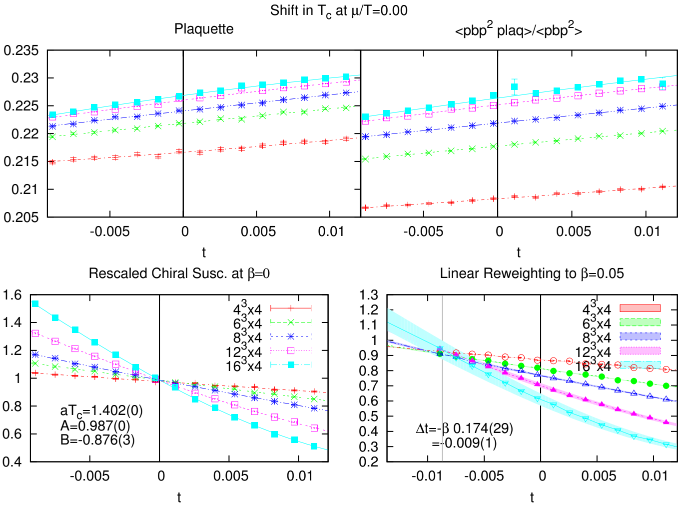
$$c_\chi^{(1)} \simeq (c_1 + c_2 L^{1/\nu} + c_3 t) \quad \text{in the vicinity of} \quad t = 0,$$

- the **shift in T_c** is then related to scaling function parameters A, B and c_2:

$$\Delta a T_c(\beta) \doteq -\beta a T_c \frac{A}{B} c_2$$

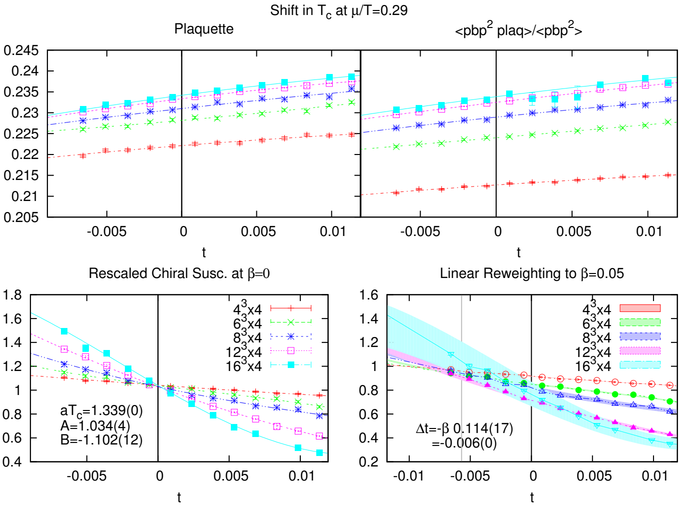
Results on the Slope at Zero and non-Zero Density

- We obtain for the **slope**: $\frac{\partial}{\partial\beta} aT_c(\beta) \simeq -0.24(3)$ at $\mu = 0$



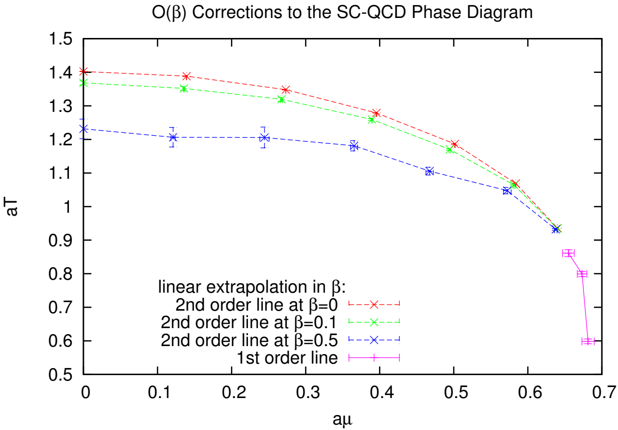
Results on the Slope at Zero and non-Zero Density

- We obtain for the **slope**: $\frac{\partial}{\partial\beta} aT_c(\beta) \simeq -0.15(2)$ at $\mu/T = 0.29$



Corrections to the SC-Phase diagram

- The slope vanishes at the tricritial point and along the first order line



ratio at strong coupling

$$\frac{T_c(\mu=0)}{\mu_c(T=0)} \approx \frac{1.403}{0.57} = 2.46$$

too large compared to $m_q = 0$ continuum result

$$\approx \frac{154\ \mathrm{MeV}}{0.93\ \mathrm{GeV}} = 0.165$$

but

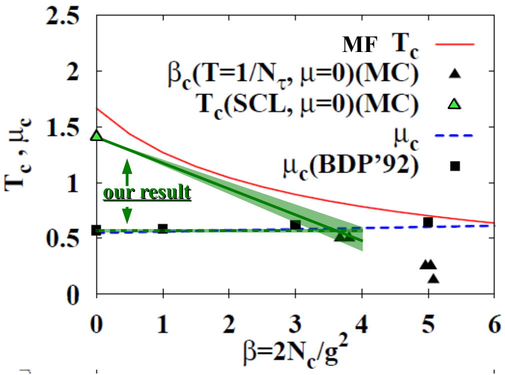
$$\frac{T_c(\mu=0)}{\mu_c(T=0)} \searrow \quad (\beta \nearrow)$$

at leading $\mathcal{O}(\beta)$

Conclusions

Achievements:

- correct average plaquette and Polyakov loop reproduced at $\beta = 0$ (checked with HMC)
- all measurements **extended to finite μ**
- $\langle L \rangle$ and $\langle P_t \rangle$ are sensitive to the chiral transition
- **slope of aT_c** determined at finite density up to the tricritical point



Comparison with mean field results by Miura *et. al*, Phys. Rev. D **80** (2009) 074034 (2009): **good agreement**

Further Goals:

- $\mathcal{O}(\beta^2)$ **corrections** needed
- determine whether the chiral and nuclear transition split at finite β

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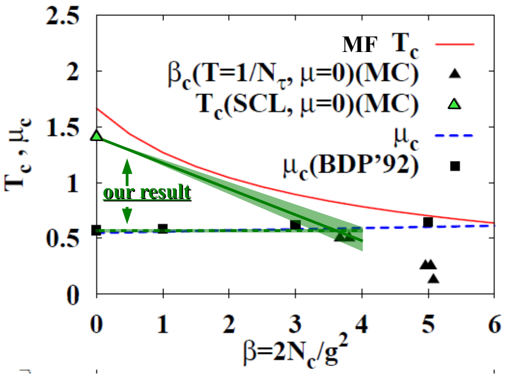

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Thank you for your attention!

Backup: SC-LQCD at finite temperature

How to vary the temperature?

- $aT = 1/N_\tau$ is discrete with N_τ even
- $aT_c \simeq 1.5$, i.e. $N_\tau{}^c < 2$ $\Rightarrow$ we cannot address the phase transition!

Solution: introduce an **anisotropy** γ in the Dirac couplings:

$$\mathcal{Z}(m_q, \mu, \gamma, N_\tau) = \sum_{\{k,n,l\}} \prod_{b=(x,\mu)} \frac{(3-k_b)!}{3!k_b!} \gamma^{2k_b\delta_{\mu 0}} \prod_x \frac{3!}{n_x!} (2am_q)^{n_x} \prod_l w(\ell, \mu)$$

Should we expect $a/a_\tau = \gamma$, as suggested at weak coupling?

- **No**: meanfield predicts $a/a_\tau = \gamma^2$, since $\gamma_c^2 = N_\tau \frac{(d-1)(N_c+1)(N_c+2)}{6(N_c+3)}$

  $\Rightarrow$ sensible, N_τ-independent definition of the temperature: $aT \simeq \frac{\gamma^2}{N_\tau}$

- Moreover, SC-LQCD partition function is a function of γ^2

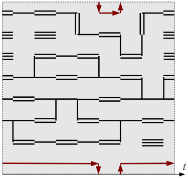
However: **precise correspondence between a/a_τ and γ^2 not known**

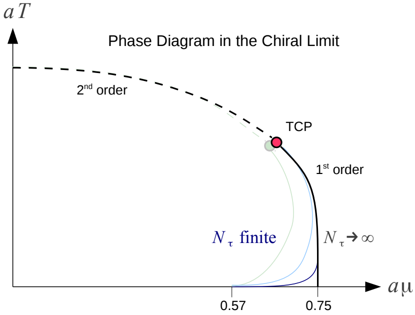
Backup: The Fate of the Nuclear and Chiral Transition

Strong Coupling Limit:

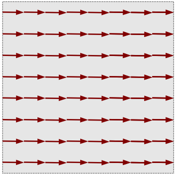
- finite temperature chiral transition takes place when spatial dimers vanish
- nuclear and chiral transition coincide: $\langle\bar{\chi}\chi\rangle$ vanishes as baryonic crystal forms

$\mu = 0$, $T = 0$





$T = 0$, $\mu > \mu_c$, $\beta = 0$



Backup: The Fate of the Nuclear and Chiral Transition

Strong Coupling Limit:

- finite temperature chiral transition takes place when spatial dimers vanish
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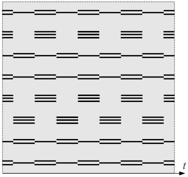

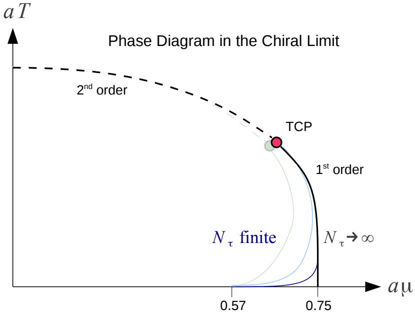

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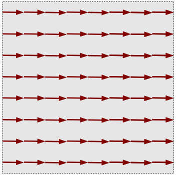

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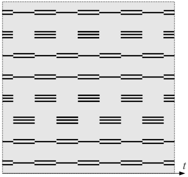
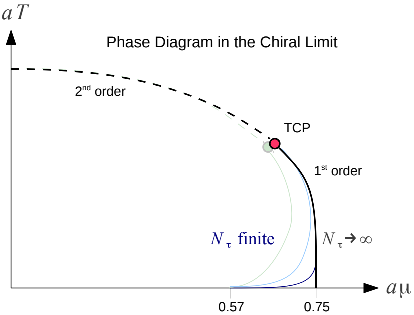
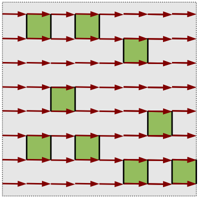




$T = 0$, $\mu > \mu_c$, $\beta > 0$



Possibility for $\beta > 0$:

- chiral transition takes place at larger μ_c than nuclear transition, as chiral condensate can be non-zero even though baryonic crystal has formed

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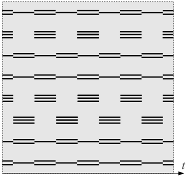
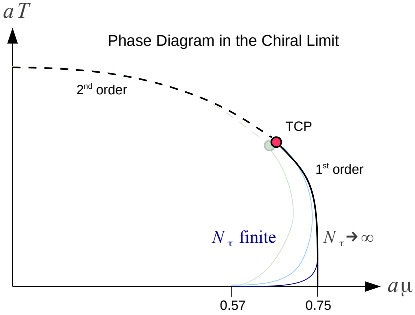
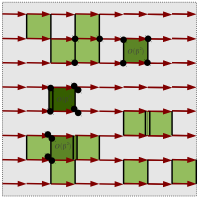




$T = 0$, $\mu > \mu$, $\beta \gg 0$



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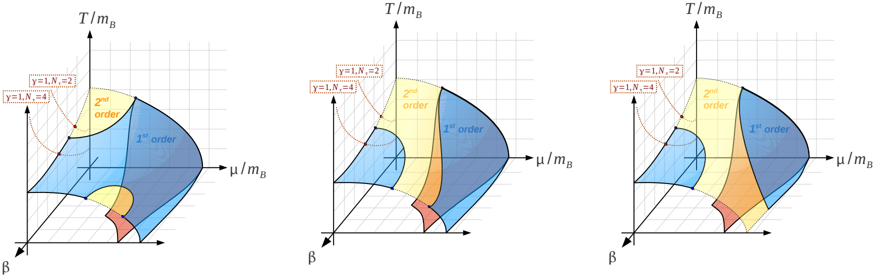

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Backup Slide: SC + Plaquette Partition Function at $\mathcal{O}(\beta)$

partition function can be expanded up to $\mathcal{O}(1/g^{2N_c})$ as Grassmann integration terminates at this order:

$$Z = \int d\chi d\bar{\chi} Z_F \prod_P \left(1 + \frac{1}{g^2}\left(\prod_{l\in P} z_l\right)^{-1} \sum_{s=1}^{19} F_P^s + \dots\right)$$

- new set of **plaquette variables** $q_P \in \{0, \dots N_c\}$ and auxiliary variables

$$q_x = \sum_P^{x\in P} q_P \in \{0, \dots N_c\}, \quad q_b = \sum_P^{b\in P} q_P \in \{0, \dots N_c\}$$

- help to write down Z after Grassmann integration:

$$Z = \sum_{\{k,n,\ell,q\}} \prod_{b=(x,\mu)} w_b \prod_x w_x \prod_\ell w_\ell \prod_P w_P,$$

$$w_x = \frac{N_c!}{n_x!}(2am_q)^{n_x} v_i(x), \qquad w_b = \frac{(N_c - k_b)!}{N_c!(k_b - q_b)!}, \qquad w_P = g^{-2q_P}$$

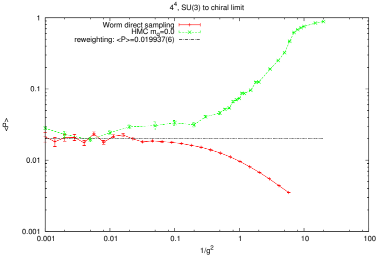
$$n_x + \sum_{\hat{\mu}=\pm\hat{0},\dots\pm\hat{d}} \left(k_{\hat{\mu}}(x) + \frac{N_c}{2}|\ell_{\hat{\mu}}(x)|\right) = N_c + q_x$$

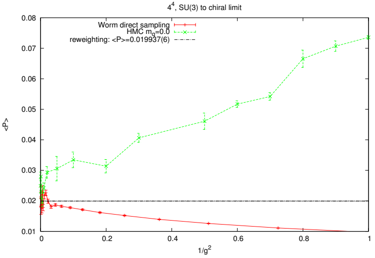
Backup: Crosschecks at $\mu = 0$, $T = 0$

- Sampling average plaquette at finite β:

$$\langle P \rangle = \frac{2}{Vd(d-1)} \frac{\partial}{\partial \beta} \log(Z) = \frac{1}{\beta} \langle n_P \rangle, \quad n_P = \frac{2}{Vd(d-1)} \sum_P q_P$$

- **saturation expected:** $\langle n_p \rangle \leq \frac{N_c}{2d(d-1)}$
  (at most N_c plaquettes can join at a bond or site)
- numerical results show indeed saturation of $\langle n_p \rangle$, $\quad \Rightarrow \quad \langle P \rangle \to 0$ for $\beta \to \infty$





- reweighting from the SC-ensemble, $\langle P \rangle = Z_P / Z$, gives very precise results

Backup: FSS Scaling of Taylor Coefficient

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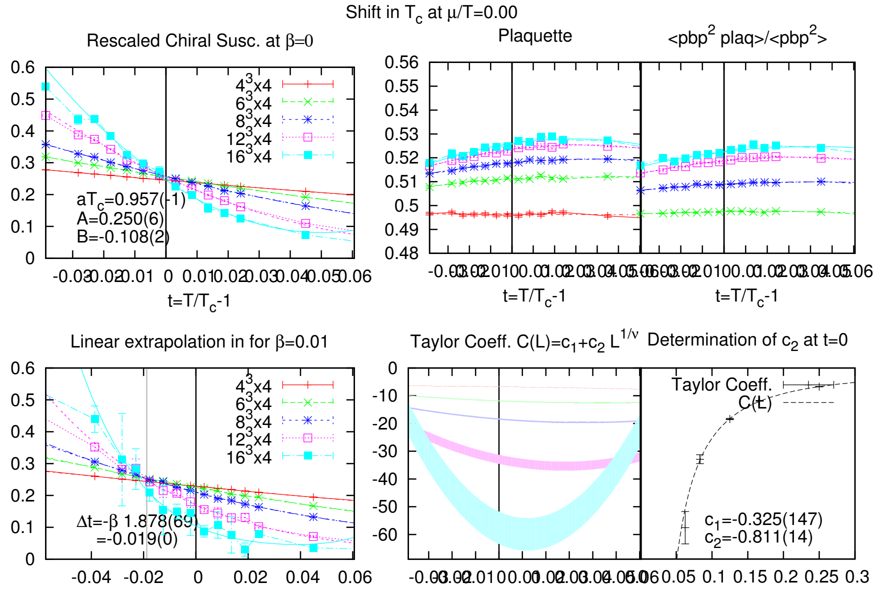

Backup: Results on the Slope at Zero and non-Zero Density

The slope gets smaller for increasing μ