Finite-volume effects in the evaluation of Δm_K

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(Based on work with Norman Christ and Guido Martinelli)

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- In the previous two talks by N.H.Christ and J.Yu we have heard about the RBC-UKQCD programme to evaluate the long-distance contributions to $\Delta m_K = m_{K_L} - m_{K_S}$. This builds on the exploratory work reported in

  Long-distance contributions to the K_L - K_S mass difference

  N.H.Christ, T.Izubuchi, C.T.Sachrajda, A.Soni, J.Yu arXiv:1212.5931.

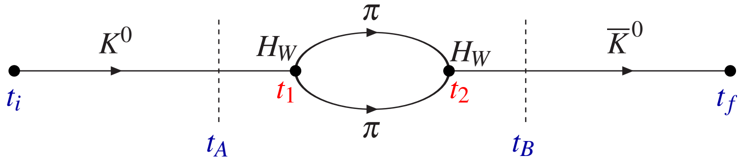
- We need to compute the amplitude

$$\mathscr{A} = \frac{1}{2}\int_{-\infty}^{\infty} dt_1\,dt_2\; T\,\langle \bar{K}^0\,|\,H_W(t_2)\,H_W(t_1)\,|\,K^0\rangle$$

and to determine the K_L-K_S mass difference:

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = \frac{1}{2m_K}\,2\mathscr{P}\sum_\alpha \frac{\langle \bar{K}^0\,|\,H_W\,|\,\alpha\rangle\,\langle\alpha\,|\,H_W\,|\,K^0\rangle}{m_K - E_\alpha} = 3.483(6)\times 10^{-12}\,\mathrm{MeV}.$$

where the sum over $|\alpha\rangle$ includes an energy-momentum integral.



- The above correlation function gives ($T = t_B - t_A + 1$)

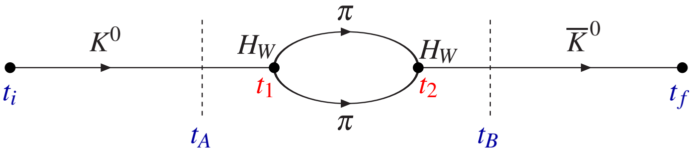
$$C_4(t_A, t_B; t_i, t_f) = |Z_K|^2 e^{-m_K(t_f - t_i)} \sum_n \frac{\langle \bar{K}^0 \,|\, H_W \,|\, n\rangle \, \langle n \,|\, H_W \,|\, K^0\rangle}{(m_K - E_n)^2} \times \left\{ e^{(m_K - E_n)T} - (m_K - E_n)T - 1 \right\}.$$

- From the coefficient of T we can therefore obtain

$$\Delta m_K^{\mathrm{FV}} \equiv 2 \sum_n \frac{\langle \bar{K}^0 \,|\, H_W \,|\, n\rangle \, \langle n \,|\, H_W \,|\, K^0\rangle}{(m_K - E_n)}\,.$$

- In this talk I discuss the evaluation of FV effects necessary to relate Δm_K^{FV} to the physical mass difference.
- Note that the correlation function itself does not have a singularity as $m_K \to E_n$!

FV Effects in ΔM_K



$$C_4(t_A,t_B;t_i,t_f) = |Z_K|^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0\,|\,H_W\,|\,n\rangle\,\langle n\,|\,H_W\,|\,K^0\rangle}{(m_K-E_n)^2} \left\{ e^{(m_K-E_n)T} - (m_K-E_n)T - 1 \right\}$$

- From the time dependence we obtain

$$2\sum_n \frac{\langle \bar{K}\,|\,H_W\,|\,n\rangle\,\langle n\,|\,H_W\,|\,K\rangle}{(M_K-E_n)}\,.$$

- If the volume is tuned so that one state has $E_{n_0} = m_K$ then we obtain

$$2\sum_{n\neq n_0} \frac{\langle \bar{K}\,|\,H_W\,|\,n\rangle\,\langle n\,|\,H_W\,|\,K\rangle}{(M_K-E_n)}\,.$$

Relating FV sums and IV integrals

- The issue we consider is how to relate the FV sums

$$2\sum_n \frac{\langle \bar{K}\,|\,H_W\,|\,n\rangle\,\langle n\,|\,H_W\,|\,K\rangle}{(M_K-E_n)} \quad \text{or} \quad 2\sum_{n\neq n_0} \frac{\langle \bar{K}\,|\,H_W\,|\,n\rangle\,\langle n\,|\,H_W\,|\,K\rangle}{(M_K-E_n)}$$

  to the infinite volume integral

$$\Delta M_K = 2\sum_\alpha \mathscr{P}\int dE\,\frac{\langle \bar{K}\,|\,H_W\,|\,\alpha(E)\rangle\,\langle \alpha(E)\,|\,H_W\,|\,K\rangle}{M_K-E}\,,$$

  and hence to obtain ΔM_K from a realistic lattice simulation.

- Using degenerate perturbation theory $\Rightarrow$ N.H.Christ, arXiv:1012.6034

$$\begin{aligned}\Delta M_K \;=\;& 2\sum_{n\neq n_0} \frac{\langle \bar{K}\,|\,H_W\,|\,n\rangle\,\langle n\,|\,H_W\,|\,K\rangle}{(M_K-E_n)} + \frac{1}{\frac{\partial(\phi+\delta)}{\partial E}}\left[\frac{1}{2}\,\frac{\partial^2(\phi+\delta)}{\partial E^2}\,|\langle n_0\,|\,H_W\,|\,K_S\rangle|^2\right.\\ &\left. -\frac{\partial}{\partial E_{n_0}}\left\{\left.\frac{\partial(\phi+\delta)}{\partial E}\right|_{E=E_{n_0}}|\langle n_0\,|\,H_W\,|\,K_S\rangle|^2\right\}_{E_{n_0}=M_K}\right]\end{aligned}$$

- For s-wave two-pion states, Lüscher's quantization condition is $h(E,L)\pi \equiv \phi(q)+\delta(k) = n\pi$, where $q = kL/2\pi$, ϕ is a kinematical function and δ is the physical s-wave $\pi\pi$ phase shift for the appropriate isospin state.

References

1. C. J. D. Lin, G. Martinelli, CTS and M. Testa,
"$K \to \pi\pi$ decays in a finite volume,"
Nucl. Phys. B **619** (2001) 467 [hep-lat/0104006].
2. C. J. D. Lin, G. Martinelli, E. Pallante, CTS and G. Villadoro
"$K^+ \to \pi^+\pi^0$ decays on finite volumes and at next-to-leading order in the chiral expansion,"
Nucl. Phys. B **650** (2003) 301 [hep-lat/0208007].
3. C. h. Kim, CTS and S. R. Sharpe,
"Finite-volume effects for two-hadron states in moving frames,"
Nucl. Phys. B **727** (2005) 218 [hep-lat/0507006].

C. J. D. Lin, G. Martinelli, CTS and M. Testa, hep-lat/0104006

- Consider the correlation function

$$C(t)=\int_V\langle 0|J(\vec{x},t)J(0)|0\rangle \quad = \quad V\sum_n|\langle 0|J(0)|\pi\pi,n\rangle_V|^2\,e^{-E_nt}$$

- The Lüscher quantization condition (for s-wave dominance) is

$$h(E,L)\equiv\phi(q)+\delta(k)=n\pi \quad \text{with} \quad E^2=4(m^2+k^2) \text{ and } q=kL/2\pi\,,$$

  ϕ is a kinematic function and δ is the s-wave phase shift.

- As $V\to\infty$, the excitation level at fixed physics becomes large, i.e. $h(E,L)\to\infty$.
- Poisson-Summation formula $\Rightarrow$

$$\sum_n f(E_n)=\int dE\,\rho_V(E)f(E)+\sum_{l\neq 0}\int dE\,\rho_V(E)f(E)\,e^{i2\pi l\,h(E,L)}\,,$$

  where

$$\rho_V(E)=\frac{dn}{dE}=\frac{q\phi'(q)+k\delta'(k)}{4\pi k^2}\,E\,,$$

- Applying this formula to $C(t)$ we obtain:

$$C(t)\to V\int dE\,\rho_V(E)\,|\langle 0|J(0)|\pi\pi,E\rangle_V|^2\,e^{-Et}+\text{exponentially small corrections.}$$

$$C(t) = V\int dE\,\rho_V(E)\,|\langle 0\,|\,J(0)\,|\,\pi\pi,E\,\rangle_V|^2\,e^{-Et}$$

- On the other hand, clustering implies that the finite-volume correlation function $\to$ the infinite-volume one up to exponentially small corrections:

$$C(t) = \frac{\pi}{2(2\pi)^3}\int \frac{dE}{E}\,e^{-Et}|\langle 0\,|\,J(0)\,|\,\pi\pi,E\,\rangle|^2\,k(E)$$

  where $E^2 = 4(m_\pi^2 + k^2(E))$.

- Comparing the two expressions, we obtain

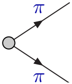
$$|\pi\pi,E\rangle = 4\pi\sqrt{\frac{VE\rho_V(E)}{k(E)}}\,|\pi\pi,E\rangle_V \quad \text{where} \quad k(E) = \sqrt{\frac{E^2}{4} - m^2}\,,$$

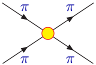
  the key ingredient of the Lellouch-Lüscher formula,

  L.Lellouch&M.Lüscher, hep-lat/0003023

  (note also the relation between the normalisations of single-particle states: $|K(\vec{p}=0) = \sqrt{2m_K V}\,|K(\vec{p}=0\,\rangle_V)$).

- Passing remark: $C(t)$ has no non-exponential FV corrections, but the energies and the matrix elements do.

- It is instructive to confirm the above in perturbation theory with weak vertex 

and strong vertex 

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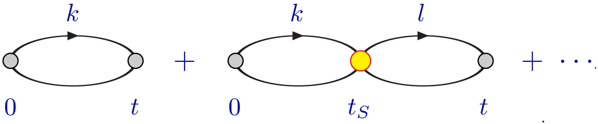
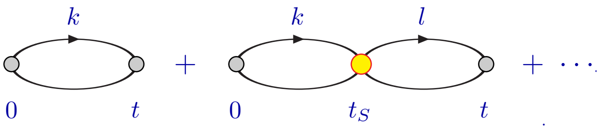
- Expand the correlation function $\langle \pi\pi(t)\ \pi\pi(0)\rangle$ in perturbation theory:



- The power corrections arise from the above two-loop diagram with $0 \le t_S \le t$.
- (The full calculation for $K \to \pi\pi$ decays in chiral perturbation theory was presented in C. D. Lin, G. Martinelli, E. Pallante, CTS and G. Villadoro, hep-lat/0208007.)

Perturbation theory for two-pion states



- The correlation function, with total momentum zero, is given by

$$C(t)=\frac{1}{V}\sum_{\vec{k}}\frac{1}{(2\omega_k)^2}e^{-2\omega_k t}+\frac{\hat{\lambda}}{V^2}\sum_{\vec{k},\vec{l}}\frac{1}{(2\omega_k)^2(2\omega_l)^2}\,\frac{1}{2(\omega_k-\omega_l)}\left\{e^{-2\omega_k t}-e^{-2\omega_l t}\right\}$$

  where $\omega_k^2=\vec{k}^2+m_\pi^2$ $(\omega_l^2=\vec{l}^2+m_\pi^2)$.

- There is no singularity at $\omega_k=\omega_t$! $\Rightarrow$ confirms LMST procedure, which assumes no FV corrections in the correlation function.
- The terms with $|\vec{l}|=|\vec{k}|$ combined with LO term gives:

$$\frac{1}{V}\sum_{\vec{k}}\frac{1}{(2\omega_k)^2}e^{-2\omega_k t}+\frac{\hat{\lambda}}{V^2}\sum_{\vec{k}}\frac{\nu_k t}{(2\omega_k)^4}\simeq\frac{1}{V}\sum_{\vec{k}}\frac{1}{(2\omega_k)^2}e^{-(2\omega_k+\Delta E(k))t}\,,$$

  where $\Delta E(k)$ is the finite-volume energy shift as given by Lüscher's formula.

- The terms with $|\vec{l}|\neq|\vec{k}|$ correctly give the Lellouch-Lüscher formula.
- Sum of the two corrections is zero.

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When there is a pole the summation formula has a correction term. For example:

$$\frac{1}{L}\sum_n \frac{f(p_n^2)}{k^2-p_n^2} = \mathscr{P}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\frac{f(p^2)}{k^2-p^2} + \frac{f(k^2)\cot(kL/2)}{2k}\,.$$

(This is the one-dimensional version of the key ingredient in the derivation of the Lüscher quantisation formula.)

- The optimal choice of volume appears to be $L=\pi/k$, i.e. to take the infinite-volume limit by keeping $kL=(2n+1)\pi$, so that the cotangent term vanishes.
- Another is to choose volumes such that $k\to p_{n_0}$ say, for some integers n_0.

$$\frac{1}{L}\sum_n{}' \frac{f(p_n^2)}{k^2-p_n^2} = \mathscr{P}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\frac{f(p^2)}{k^2-p^2} + \frac{2f'(k^2)}{L} - \frac{f(k^2)}{2Lk^2}\,.$$

  - $f'(k^2)$ indicates the derivative of f w.r.t. k^2 .
  - The $'$ on the sum denotes that the two terms with $p_n^2=p_{n_0}^2$ are omitted.

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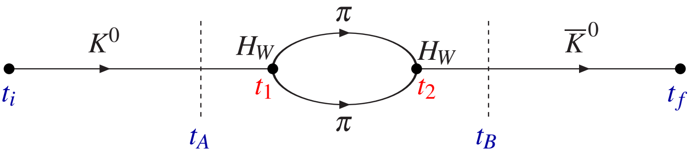
Return to discussion of FV Effects in ΔM_K



$$C_4(t_A,t_B;t_i,t_f) = |Z_K|^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0 \,|\, H_W \,|\, n\rangle \, \langle n \,|\, H_W \,|\, K^0\rangle}{(m_K-E_n)^2} \left\{ e^{(m_K-E_n)T} - (m_K-E_n)T - 1 \right\}$$

- We now introduce a pole as $E_n \to m_K$, and hence power-like FV corrections, by replacing C_4 by

$$-T\,|Z_K|^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0 \,|\, H_W \,|\, n\rangle \, \langle n \,|\, H_W \,|\, K^0\rangle}{(m_K-E_n)}$$

- It is convenient to replace

$$\frac{1}{m_K-E_n} \to \frac{m_K+E_n}{4(k^2-p_n^2)}$$

where $m_K^2 \equiv 4(k^2+m_\pi^2)$ and $E_n^2 \equiv 4(p_n^2+m_\pi^2)$.

- Starting point for the derivation of

$$\frac{1}{L}\sum_n \frac{f(p_n^2)}{k^2-p_n^2} = \mathscr{P}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\frac{f(p^2)}{k^2-p^2} + \frac{f(k^2)\,\cot(kL/2)}{2k}\,.$$

is

$$\frac{1}{L}\sum_n \frac{f(p_n^2)-f(k^2)}{k^2-p_n^2} = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\frac{f(p^2)-f(k^2)}{k^2-p^2}$$

- Using the same procedure we obtain:

$$\sum_n \frac{f(E_n)}{m_K-E_n} = \mathscr{P}\int dE\,\rho_V(E)\,\frac{f(E)}{m_K-E} + f(m_K)\pi\left(\cot(\pi h)\,\frac{dh}{dE}\right)_{m_K}$$

where $f(E_n) = 2\,{}_V\langle\bar{K}^0\,|\,H_W\,|\,n\rangle_V\,{}_V\langle n\,|\,H_W\,|\,K^0\rangle_V$.

- Now we can imagine taking the large L limit by taking a sequence of volumes such that there is a two-pion state with $E_{\pi\pi} = m_K$ in which case:

$$\sum_n{}' \frac{f(E_n)}{m_K-E_n} = \mathscr{P}\int dE\,\rho_V(E)\,\frac{f(E)}{m_K-E} + f'(m_K) + \frac{1}{2}f(m_K)\,\frac{h''}{h'}\,.$$

This is N.Christ's result from Lattice 2010.

- Alternatively, it seems attractive to choose the volumes such that $\cot(\pi h)\simeq 0$, with no non-exponential FV corrections.

- In addition to improved precision for standard quantities, it is important to continue extending the range of physical processes which can be studied.
  - We are hearing much progress in this direction at Lattice 2013, including contributions from RBC-UKQCD.
- For Δm_K, among the theoretical issues is the control of finite-volume effects.

$$\Delta m_K = \Delta m_K^{\mathrm{FV}} - 2\pi\, {}_V\langle \bar{K}^0 \,|\, H \,|\, n_0 \rangle_V \, {}_V\langle n_0 \,|\, H \,|\, K^0 \rangle_V \left[\cot \pi h \, \frac{dh}{dE}\right]_{m_K} .$$

  - This formula reproduces the result for the special case when the volume is such that there is a two-pion state with energy $= m_K$.

    N.H.Christ, arXiv:1012.6034
  - Increasing the volumes keeping $h = n/2$ and thus avoiding the power corrections is an intriguing possibility.
- The practical implementation of the above relations needs to be developed.

RBC

Ziyuan Bai, Thomas Blum, Norman Christ, Tomomi Ishikawa, Taku Izubuchi, Luchang Jin, Chulwoo Jung, Taichi Kawanai, Chris Kelly, Hyung-Jin Kim, Christoph Lehner, Jasper Lin, Meifeng Lin, Robert Mawhinney, Greg McGlynn, David Murphy, Shigemi Ohta, Eigo Shintani, Amarjit Soni, Oliver Witzel, Hantao Yin, Jianglei Yu, Daiqian Zhang

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