Chiral condensate from the Banks–Casher relation

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Banks-Casher relate condensate Σ to spectral density ρ of Dirac operator

$$\Sigma \equiv -\langle\bar{\psi}\psi\rangle = \pi \lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty}\rho(\lambda, m), \qquad \rho(\lambda, m) = \frac{1}{V}\sum_{k=1}^{\infty}\langle\delta(\lambda-\lambda_k)\rangle \quad (1)$$

Calculated on the lattice: mode number $\nu \equiv$ integrated density

The number of modes of the massive hermitian Dirac operator $D^\dagger D - m^2$, with eigenvalues $\alpha \leq \Lambda_{\mathrm{R}}^2 + m_{\mathrm{R}}^2$, is renormalization-group invariant[1]

$$\nu_{\mathrm{R}}(\Lambda_{\mathrm{R}}, m_{\mathrm{R}}) = V\int_{-\Lambda_{\mathrm{R}}}^{\Lambda_{\mathrm{R}}} d\lambda\,\rho(\lambda, m_{\mathrm{R}}) \quad (2)$$

To extract Σ: define effective condensate (removing threshold effects)

$$\widetilde{\Sigma}(\Lambda_1, \Lambda_2, m) = \frac{\pi}{2V}\frac{\nu(\Lambda_2)-\nu(\Lambda_1)}{\Lambda_2-\Lambda_1} \xrightarrow{V\to\infty;\,m,\Lambda_i\to 0} \Sigma \quad (3)$$

[1] L. Giusti and M. Lüscher, JHEP903(2008)13

NLO ChPT ($n_f = 2$) in the continuum theory[1,2]

$$\widetilde{\Sigma}^{\mathrm{NLO}} = \Sigma\left\{1 + \frac{m\Sigma}{(4\pi)^2 F^4}\left[3\bar{l}_6 + 1 - \ln(2) - 3\ln\left(\frac{\Sigma m}{F^2 M^2}\right) + \tilde{g}_\nu\left(\frac{\Lambda_1}{m}, \frac{\Lambda_2}{m}\right)\right]\right\} \tag{4}$$

with

$$\tilde{g}_\nu(x_1, x_2) = \frac{f_\nu(x_1) + f_\nu(x_2)}{2} + \frac{1}{2}\frac{x_1 + x_2}{x_2 - x_1}\left[f_\nu(x_2) - f_\nu(x_1)\right]$$

$$f_\nu(x) = \left(x - \frac{1}{x}\right)\arctan(x) - \frac{\pi}{2}x - \ln(x + x^3)$$

- No chiral logs for fixed Λ
- $m \neq 0$: $\widetilde{\Sigma}^{\mathrm{NLO}}$ is decreasing with Λ
- $m = 0$: $\widetilde{\Sigma}^{\mathrm{NLO}}$ is independent of Λ

[1] L. Giusti and M. Lüscher, JHEP903(2008)13
[2] S. Necco and A. Shindler, JHEP1104(2010)31

NLO WChPT ($n_f = 2$) in the $\mathcal{O}(a)$ improved lattice theory[2]

$$\widetilde{\Sigma}^{\rm NLO} = \Sigma\left\{1 + \frac{m\Sigma}{(4\pi)^2 F^4}\left[3\,\bar{l}_6 + 1 - \ln(2) - 3\ln\left(\frac{\Sigma m}{F^2 M^2}\right) + \tilde{g}_\nu\left(\frac{\Lambda_1}{m}, \frac{\Lambda_2}{m}\right)\right]\right\}$$
$$-32(W_0 a)^2 \frac{W_8' m}{\Lambda_1 \Lambda_2} \qquad (4)$$

- No chiral logs for fixed Λ
- $m \neq 0$: $\widetilde{\Sigma}^{\rm NLO}$ is decreasing with Λ
- $m = 0$: $\widetilde{\Sigma}^{\rm NLO}$ is independent of Λ

- Lattice: 2 further NLO LECs; W_8' expected negative[3]
- No discretization effects in the chiral limit at NLO

[2] S. Necco and A. Shindler, JHEP1104(2010)31
[3] M.T. Hansen and S.R. Sharpe, PRD85(2012)14593; K. Splittorff and J. Verbaarschot, PRD85(2012)105008

Parameters of the simulation: CLS-lattices[4]

id	L/a	m_π [MeV]	$m_\pi L$	a [fm]	$R\tau_{\rm exp}$ [MDU]	$R\tau_{\rm int}(m_\pi)$ [MDU]	$R\tau_{\rm int}(\nu)$ [MDU]	$\Delta_{\rm cnfg}$ [MDU]	$N_{\rm cnfg}$
A3	32	490	6.0	0.0755(9)(7)	25	7	2.01(0.18)(1.1)	128	55
A4		380	4.7			5		144	55
A5		330	4.0			5		36	55
B6	48	280	5.2			6		24	50
E5	32	440	4.7	0.0658(7)(7)	50	9	3.7(0.5)(1.5)	96	92
F6	48	310	5.0			8		80	40
F7		270	4.3			7		72	50
G8	64	190	4.1			8		48	22

- 2 flavors of $\mathcal{O}(a)$ improved Wilson quarks
- Autocorrelation of mode number ν negligible
- Finite volume effects tiny
- 9 values of cutoff Λ for each ensemble, 20 MeV $\leq \Lambda \leq$ 115 MeV
- R...ratio of active links in DD–HMC, MDU...molecular dynamics units

[4] P. Fritzsch et al, NPB865(2012)397; M. Marinkovic et al, PoS Lat(2011)232

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O7	64	270	4.2						50

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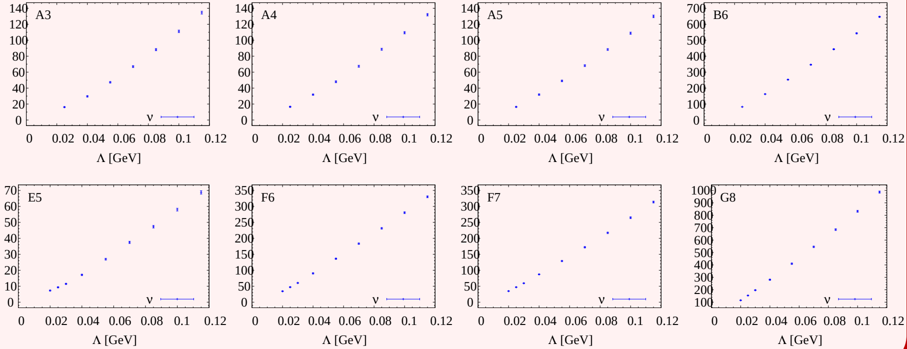
Stochastic evaluation of ν through VEV of projector to low modes

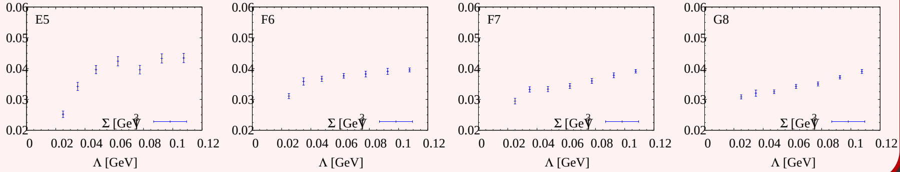
$$\nu = \langle \mathrm{tr}[\mathbb{P}_M] \rangle \,, \qquad M = \sqrt{\Lambda^2 + m^2} \tag{5}$$

$$= \frac{1}{N} \sum_{k=1}^{N} \left(\eta_k, \mathbb{P}_M \eta_k \right) \,, \qquad \eta_k \text{... pseudo-fermion fields} \tag{6}$$

Collection of data on the mode number: 64 data points



Indications for NNLO effects



- $\widetilde{\Sigma}$ should decrease with Λ according to NLO WChPT
- $\widetilde{\Sigma}$ increases with Λ in all ensembles: first indication for NNLO effects
- Fit function resting on NLO:

$$\widetilde{\Sigma}(\Lambda_i, m) = c_0 + c_1 m + c_2 g[\Lambda_i, m] \quad (7)$$

$$\text{with} \quad g[.] = m(\tilde{g}_\nu[.] - 3\ln[m]) \quad (8)$$

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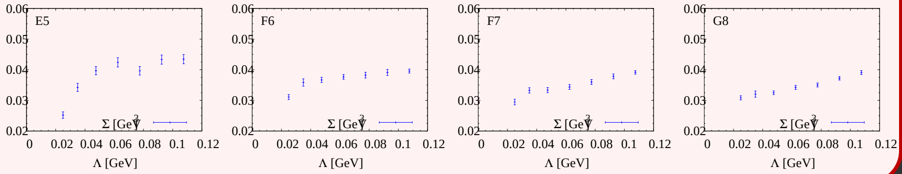


- $\widetilde{\Sigma}$ should decrease with Λ according to NLO WChPT
- $\widetilde{\Sigma}$ increases with Λ in all ensembles: first indication for NNLO effects
- Fit function resting on NLO but capable of higher order effects:

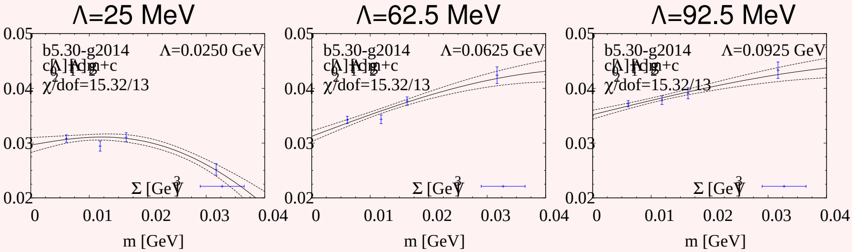
$$\widetilde{\Sigma}(\Lambda_i, m) = c_0[\Lambda] + c_1[\Lambda]m + c_2 g[\Lambda_i, m] \quad (7)$$

$$\text{with } g[.] = m(\tilde{g}_\nu[.] - 3\ln[m]) \quad (8)$$

- $c_0[\Lambda]$ and $c_1[\Lambda]$ allow for most general functions of Λ, including:
  - NLO discretization effects: $a^2 m/\Lambda^2$
  - NNLO effects: Λ^2, $m\Lambda$, ...

Fitting strategy II

Fit to $\beta = 5.3$ following Eq. (7): $\Sigma(m)$



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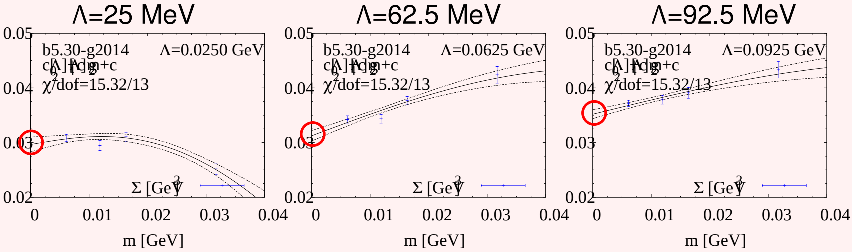
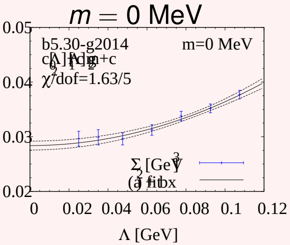
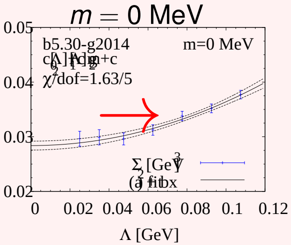


Fit to $\beta = 5.3$ following Eq. (7): $\Sigma(\Lambda)$



Fit to $\beta = 5.3$ following Eq. (7): $\Sigma(\Lambda)$



Second indication for NNLO effects

- $c_0[\Lambda] \equiv \Sigma(\Lambda, m{=}0)$ flat within NLO
- NNLO Λ^2 effects significant for $\Lambda \gtrsim 50$ MeV at $m = 0$ MeV
- Results for $c_0[\Lambda]$ and $c_1[\Lambda]$ agree with ChPT expectation

Recall generic fit function Eq. (7) capable of higher order effects:

$$\widetilde{\Sigma}(\Lambda_i, m) = c_0[\Lambda] + c_1[\Lambda]m + c_2 g[\Lambda_i, m]$$

Specify particular functions for $c_0[\Lambda]$ and $c_1[\Lambda]$ → 3 main fitting strategies

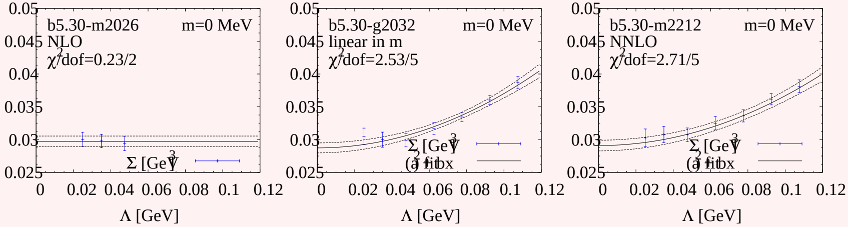
name	Λ[MeV]	m[MeV]	functional form
NLO	<50	<25	$c_0 + c_1 m + c_2 g[.] + c_3 a^2 m/\Lambda^2$
lin. in *m*	<110	<25	$c_0 + c_1 m + c_3 a^2 m/\Lambda^2 + c_4 \Lambda^2 + c_5 m\Lambda$
"NNLO"	<110	<40	$\underbrace{c_0 + c_1 m + c_2 g[.]}_{\text{NLO}} + \underbrace{c_3 a^2 m/\Lambda^2}_{\text{NLO discret.}} + \underbrace{c_4 \Lambda^2 + c_5 m\Lambda + c_6 m^2}_{\text{NNLO}}$

Further consistency checks:

- Allow for generic functions $c_0[\Lambda]$ and/or $c_1[\Lambda]$
- Use ChPT/LQCD input for $c_2 = \Sigma^2/(16\pi^2 F^4) \stackrel{\text{GMOR}}{=} m_\pi^4/(64\pi^2 m^2)$
- Use denser set of light Λ, relax m^2, exclude data points from fit...

Fit to $\beta = 5.3$: 3 main fits, plot Σ(Λ,m=0)

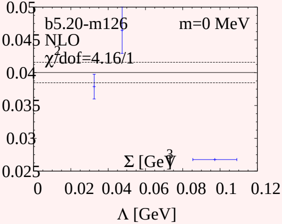
NLO | linear in *m* | "NNLO"



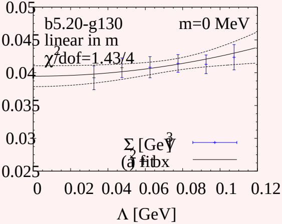
- Different strategies agree in their result for Σ
- NLO fit works for light Λ, *m* ($\Lambda < 50$ MeV, $m < 20$ MeV)

Fit to $\beta = 5.2$

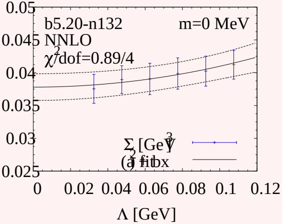
NLO



linear in *m*



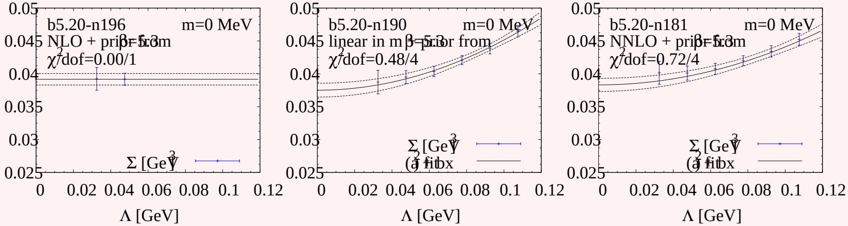
"NNLO"



- Different strategies agree in their result for Σ
- NLO fit needs more data for light $\Lambda < 50$ MeV (work in progress)

Fit to $\beta = 5.2$ using knowledge from $\beta = 5.3$

NLO | linear in m | "NNLO"

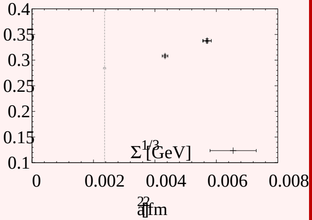


- Fit parameters consistent for different lattice spacings
- Combined fit improves accuracy

Conclusion

Summary

- Ab-initio determination of the chiral condensate from Banks-Casher
- Head at high precision study:
  - 3 lattice spacings: $0.048 < a < 0.076$ fm
  - 4 pion masses: $190 < m_\pi < 490$ MeV
  - 9 cutoffs: $20 < \Lambda < 115$ MeV

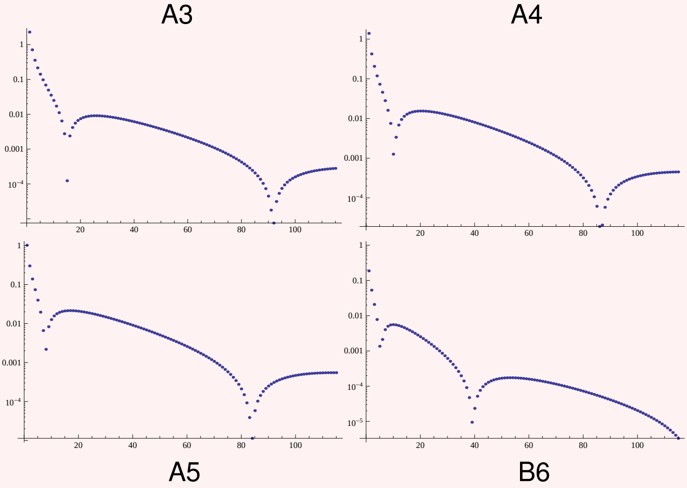


- NNLO ChPT effects observed:
  - $\widetilde{\Sigma}$ increases with Λ
  - Λ^2 effects in the chiral limit
- Different fit strategies agree (NLO, lin. in m, “NNLO”)
- Different β: agree; significant discretization effects

Outlook

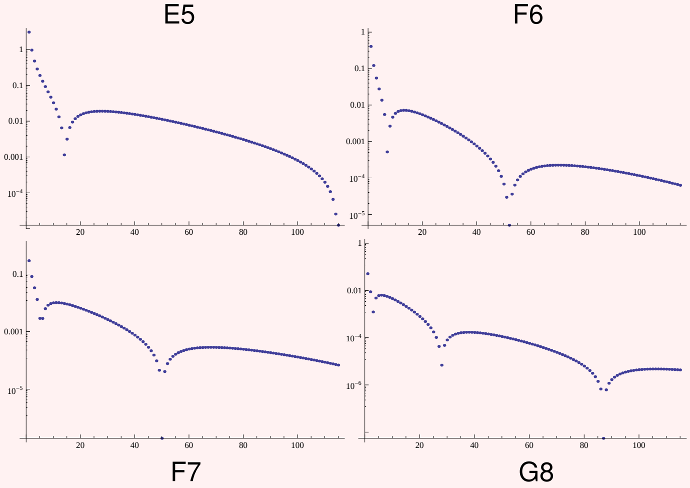
- Complete $\beta = 5.2$ and 5.3; analyze $\beta = 5.5$, continuum limit

$|\Delta\widetilde{\Sigma}^{\mathrm{FV}}|/\Sigma$ vs. Λ for $\beta = 5.2$



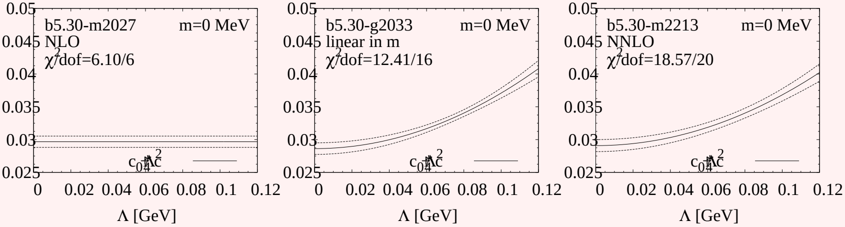
Backup: Finite volume effects from ChPT II

$|\Delta\widetilde{\Sigma}^{\mathrm{FV}}|/\Sigma$ vs. Λ for $\beta = 5.3$



Fit to $\beta = 5.3$: 3 main fits, plot $\Sigma(\Lambda, m=0)$

NLO | linear in *m* | "NNLO"



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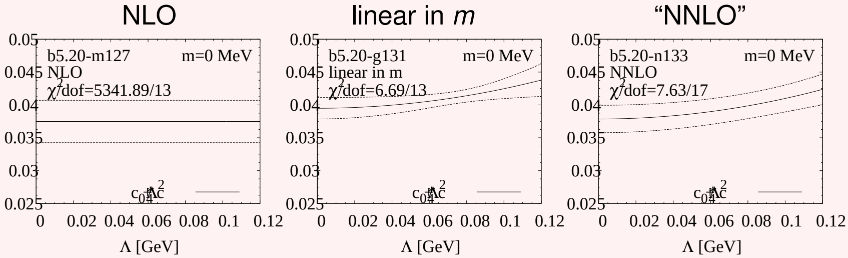
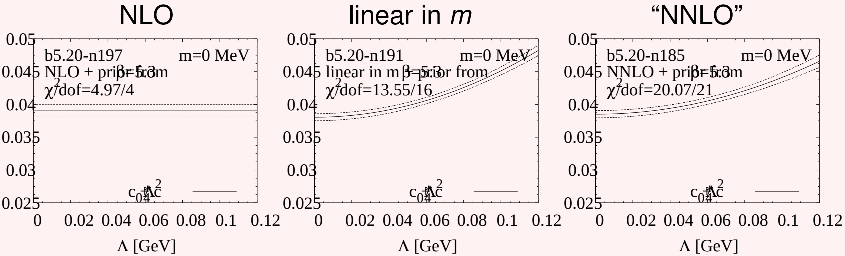
- NLO fit works for light Λ, m ($\Lambda < 50$ MeV, $m < 20$ MeV)

Fit to $\beta = 5.2$



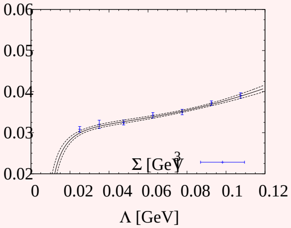
- Different strategies agree in their result for Σ
- NLO fit needs more data for light $\Lambda < 50$ MeV (work in progress)

Fit to $\beta = 5.2$ using knowledge from $\beta = 5.3$



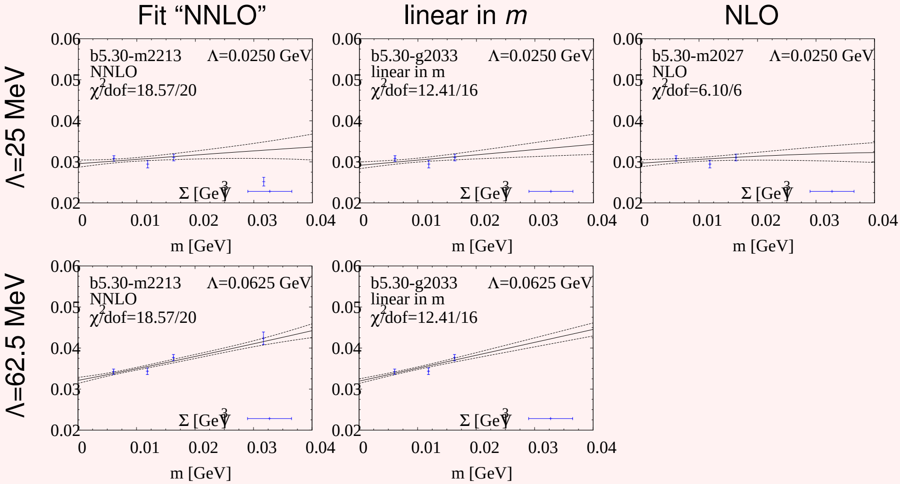
- Fit parameters consistent for different lattice spacings
- Combined fit improves accuracy

$\widetilde{\Sigma}$ vs. Λ for G8: ChPT fit g2033 (linear in m)



Backup: Fit function shown vs. m for $\beta = 5.3$

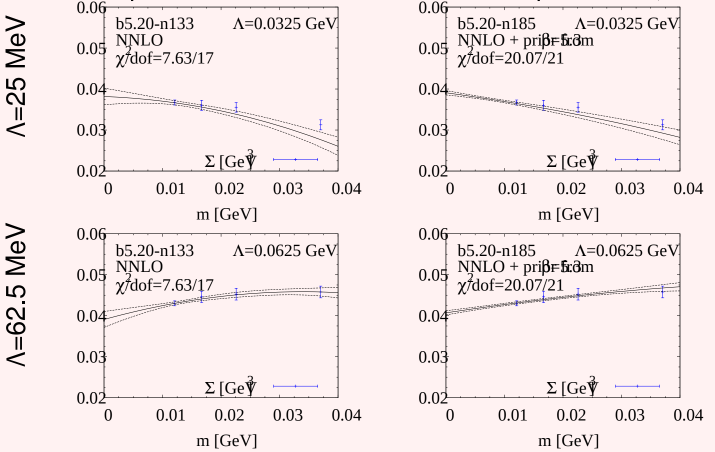
$\widetilde{\Sigma}$ vs. m for different Λ at $\beta = 5.3$ ($a = 0.066$ fm)



- Strategies agree well towards the chiral limit, all $0.75 < \chi^2/\text{dof} < 1.1$

$\widetilde{\Sigma}$ vs. m for different Λ at $\beta = 5.2$ ($a = 0.076$ fm)

plain "NNLO" | "NNLO" with prior from $\beta = 5.3$



$\widetilde{\Sigma}$ vs. Λ in the chiral limit at $\beta = 5.2$ ($a = 0.076$ fm)

plain "NNLO" | "NNLO" with prior from $\beta = 5.3$

generic $c_0[\Lambda]$

modelling $c_0 + c_4\Lambda^2$