Discretization Effects in the ϵ Domain of QCD

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Acknowledgments

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Relevant Papers

P.H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Microscopic Spectrum of the Wilson Dirac Operator, Phys. Rev. Lett. 105 (2010)

G. Akemann, P.H. Damgaard, K. Splittorff and J. J. M. Verbaarschot, Spectrum of the Wilson Dirac Operator at Finite Lattice Spacing, Phys. Rev. D83 (2011).

K. Splittorff and J. J. M. Verbaarschot, The Wilson Dirac Spectrum for QCD with Dynamical Quarks, Phys. Rev. **D** 84 (2011 [arXiv:1104.6229 [hep-lat]].

K. Splittorff and J. J. M. Verbaarschot, The Microscopic Twisted Mass Dirac Spectrum, Phys. Rev. **D8** 5 (2012) 105008, arXiv:1201.1361 [hep-lat].

M. Kieburg, J. J. M. Verbaarschot, S. Zafeiropoulos Eigenvalue Density of the Non-Hermitian Wilson Dirac Operator,, Phys. Rev. Lett. **108** (2012) 022001 arXiv:1109.0656 [hep-lat].

M. Kieburg, K. Splittorff and J. J. M. Verbaarschot, The Realization of the Sharpe-Singleton Scenario, Phys. Rev. **D85**(2012) 094011 [arXiv:1202.0620 [hep-lat]].

M. Kieburg, S. Saveiropoulos and J. J. M. Verbaarschot, Spectral Properties of the Wilson Dirac Operator and RMT, arXiv:1307.7251

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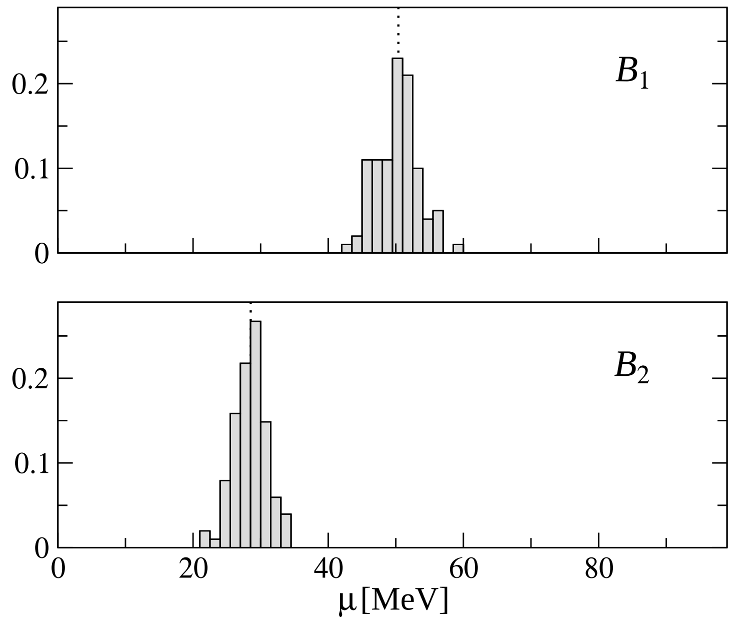
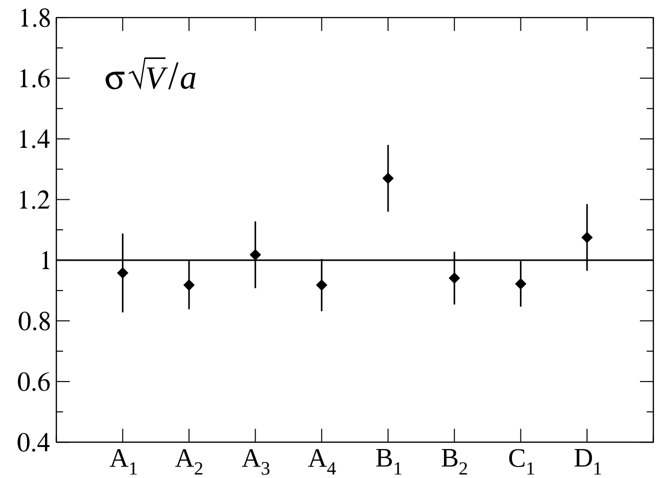
I. Motivation



Distribution of the smallest eigenvalue of the Hermitian Wilson Dirac operator on a 64×32^3 lattice for two different values of the quark mass.

Del Debbio-Giusti-Lüscher-Petronzio-Tantalo-2005



Scaling of the width of the distribution.

Del Debbio-Giusti-Lüscher-Petronzio-Tantalo-2005

Can we understand this scaling behavior?

I. Discretization Effects of the Wilson Dirac Operator

Chiral Lagrangian

Random Matrix Theory

Constraints on Low Energy Constants

First Order Scenario

Chiral Lagrangian in the ϵ Domain

Chiral Lagrangian for Wilson Fermions

$$
\begin{aligned}
-\mathcal{L} &= \frac{1}{2}mV\Sigma\mathrm{Tr}(U+U^\dagger) - \frac{1}{2}zV\Sigma\mathrm{Tr}(U-U^\dagger) \\
&\quad -a^2VW_6[\mathrm{Tr}(U+U^\dagger)]^2 - a^2VW_7[\mathrm{Tr}(U-U^\dagger)]^2 - a^2VW_8\mathrm{Tr}(U^2+U^{-2}).
\end{aligned}
$$

Sharpe-Singleton-1998, Rupak-Shoresh-2002, Bär-Rupak-Shoresh-2004,

Damgaard-Splittorff-JV- 2010

- Partition function for fixed index

$$
Z_\nu = \int_{U\in U(N_f)} dU \det{}^\nu U e^{-\int d^4x\mathcal{L}}.
$$

- For twisted mass fermions the mass term is replaced by

$$
\frac{i}{2}\mu V\Sigma\mathrm{Tr}\tau_3(U-U^\dagger).
$$

Random Matrix Theory for the Wilson Dirac Operator

Since the chiral Lagrangian is determined uniquely by symmetries, in the microscopic domain it also can be obtained from a random matrix theory with the same symmetries. In the sector of index ν the random matrix partition function is given by

$$Z^{\nu}_{N_f} = \int dAdBdW \det{}^{N_f}(D_W + m + z\gamma_5)\, P(D_W),$$

with

$$D_W = \begin{pmatrix} aA & C + aD \\ -C^\dagger + aD^\dagger & aB \end{pmatrix} \qquad \text{and} \quad A^\dagger = A, \qquad B^\dagger = B.$$

A is a square matrix of size $n \times n$, and B is a square matrix of size $(n+\nu)\times(n+\nu)$. The matrices C and D are complex $n \times (n+\nu)$ matrices. Damgaard-Splittorff-JV-2010

Eigenvalue Fluctuations and Low Energy Constants

- Since the QCD partition function is the average of a determinant, eigenvalue fluctuations determine the low-energy constants.
- Which spectral fluctuations are responsible for W_6 and W_7 and W_8 ?
- We will find that the interpretation of the low energy constants in terms of eigenvalue fluctuations will impose constraints on the value of the constants.

Distribution of "Topological" Eigenvalues

For $a = 0$ the eigenvalue density of D_5 can be decomposed as

$$\rho_5^\nu(\lambda) = \nu\delta(\lambda - m) + \rho_{\lambda > m}(\lambda).$$

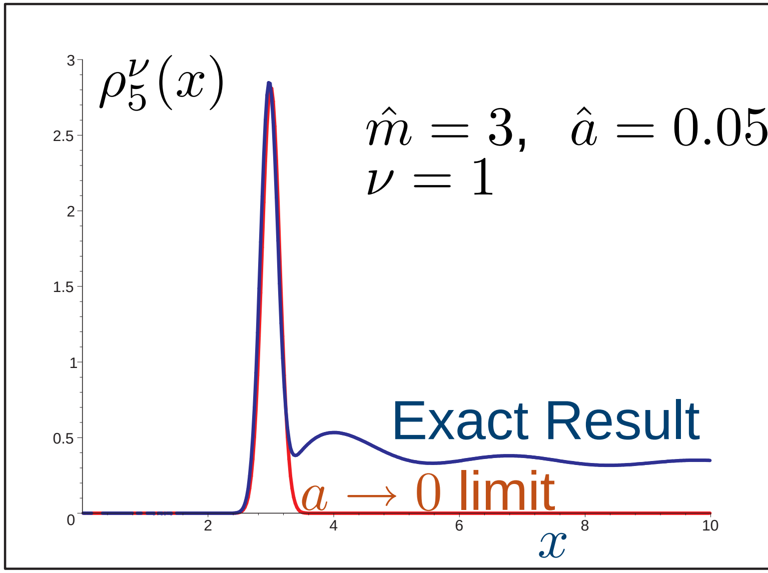
For $a \neq 0$ the width of the peak at $\lambda = m$ becomes finite.

- For $\nu = 1$ and small a the result is given by (see red curve in figure)

$$\rho_{5,\text{topo}}^{\nu=1}(x) = \frac{e^{-\frac{V\Sigma^2(x-m)^2}{16a^2(W_8-W_6-W_7)}}}{4a\sqrt{\pi V(W_8 - W_6 - W_7)}}.$$

Akemann-Damgaard-Splittorff-JV-2010,Kieburg-Zafeiropoulos-JV-2013

- $W_8 - W_6 - W_7 > 0$. This also follows from the positivity of the two-flavor partition function at fixed ν .



Akemann-Damgaard-Splittorff-JV-2010

Random Mass and Trace Squared Terms

For $W_6 < 0$ we have

$$e^{-a^2 V W_6 \mathrm{Tr}^2(U+U^{-1})} \sim \int dy e^{-y^2/(16V|W_6|a^2) - \frac{1}{2} y \mathrm{Tr}(U+U^{-1})}.$$

For $W_7 < 0$ we have

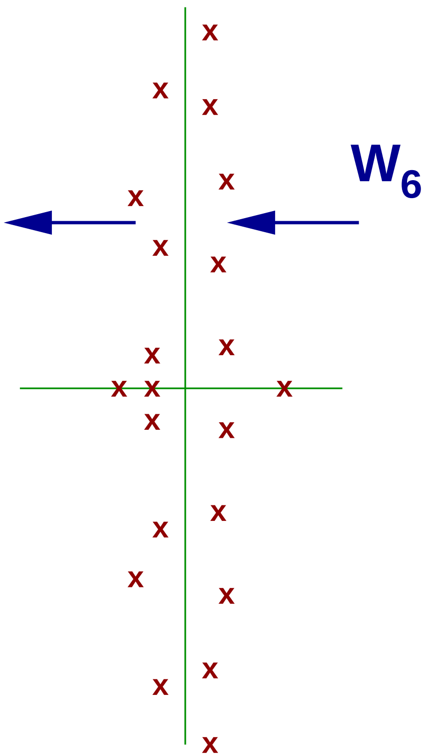
$$e^{-a^2 V W_7 \mathrm{Tr}^2(U+U^{-1})} \sim \int dy e^{-y^2/(16V|W_7|a^2) - \frac{1}{2} y \mathrm{Tr}(U-U^{-1})}.$$

Akemann-Damgaard-Spittorff-JV-2010

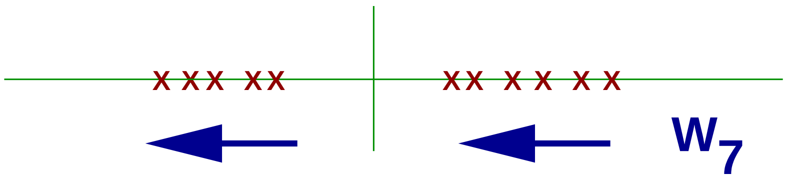
The trace square terms are generated by a random mass or the real part of the eigenvalues of D_W (for W_6) or a chiral random mass or the eigenvalues of $D_5 \equiv \gamma_5 D_W$ (for W_7).

Kieburg-Spittorff-JV-2012

Collective Spectral Fluctuations of D_W



Collective fluctuations of the real part of the eigenvalues of D_W generate a nonzero negative value of W_6 .



Collective fluctuations of the eigenvalues of D_5 generate a nonzero negative value of W_7 .

Kieburg-Splittorff-JV-2012

Caveats

- In lattice simulations one does not expect global collective fluctuations. At most I would expect collective fluctuations on the microscopic scale.
- Are there other eigenvalue fluctuations that can give rise to a nonzero W_6 and W_7 ?

Signs of Low-Energy Constants

- $W_8 > 0$ independent of the value of W_6 and W_7 .

  Akemann-Damgaard-Splittorf-JV-2010,Hansen-Sharpe-2011

- Positivity of the QCD partition function requires that
  $W_8 - W_6 - W_7 > 0$.

- Interpretation in terms of eigenvalue fluctuations requires that
  $W_6 < 0$, $W_7 < 0$.

- Twisted mass Wilson fermions lattice simulations find $m_0^{\mathrm{PS}} < m_\pm^{\mathrm{PS}}$

$$m_0^{\mathrm{PS}\,2} - m_\pm^{\mathrm{PS}\,2} = \frac{16a^2(W_8 + 2W_6)}{F_\pi^2}.$$

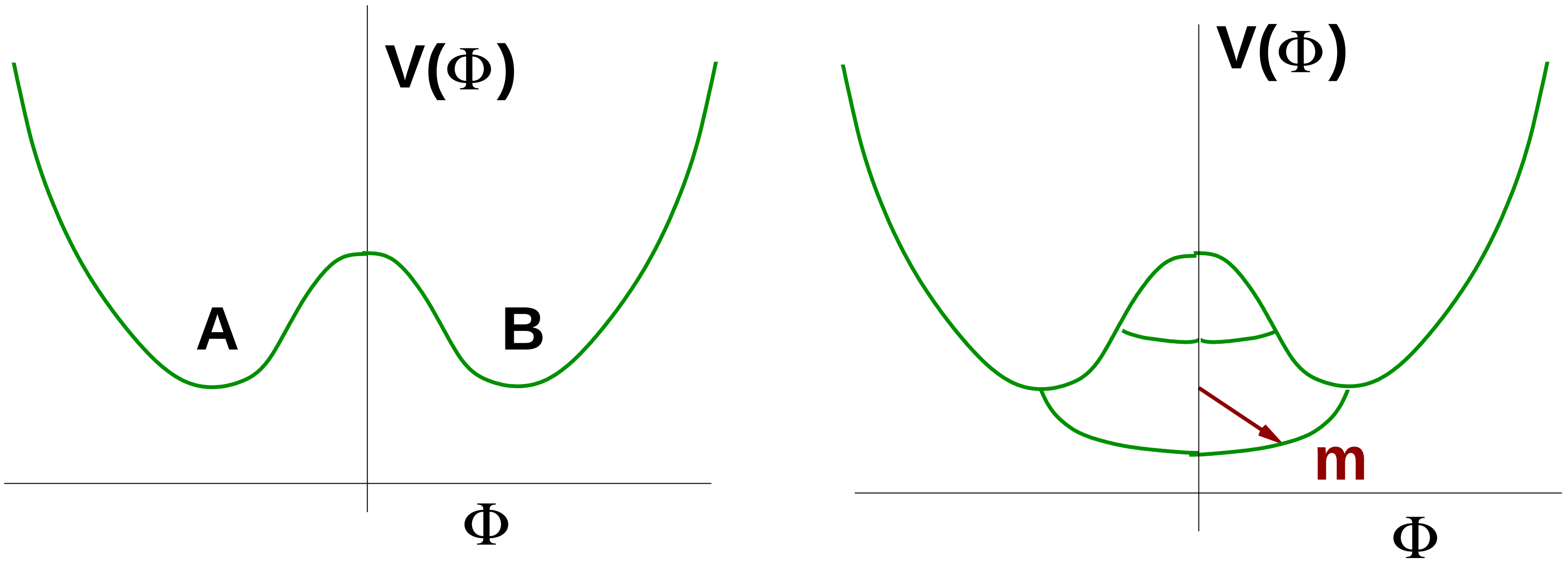
Münster-2004, Sharpe-Wu-2004

	W_6'	W_8'
Iwasaki	0.0049(38)	–0.0119(17)
tlSym	0.0082(34)	–0.0138(22)

Herdoiza-etal-2013

Extrapolation to the chiral limit.

First Order Scenario



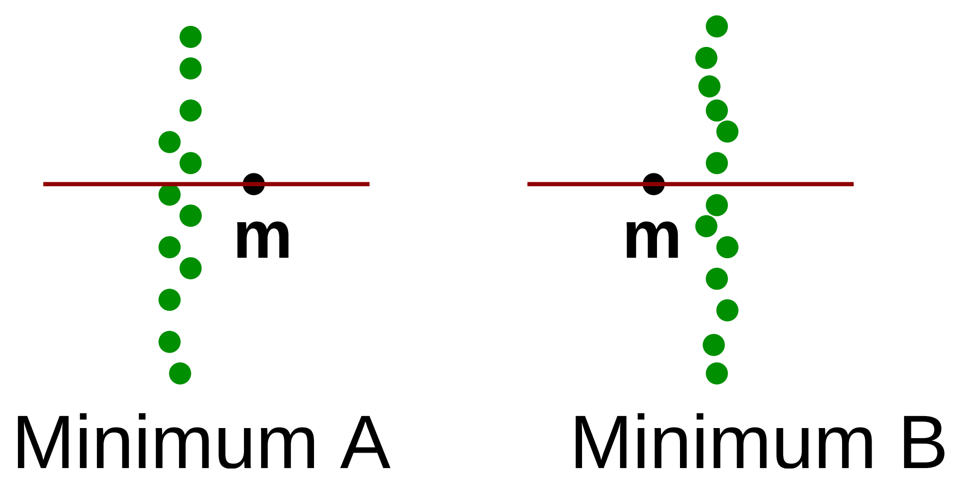
Effective potential for the order parameter. In the first order scenario (left) there is an effective potential potential between the two minima while in the usual case the effective potential is only slightly tilted by the quark mass.

In terms of the chiral Lagrangian a first order scenario takes place if $2W_6 + W_8 < 0$ when there is a potential barrier between the minima.

Sharpe-Singleton-2004

First Order Scenario and Dirac Spectra

- Because the Wilson Dirac operator in neither Hermitian nor anti-Hermitian its eigenvalues can move in the complex plane.
- Because of the fermion determinant they will be repelled from the quark mass.
- The finite jump of the Dirac spectrum results in a first order phase transition.



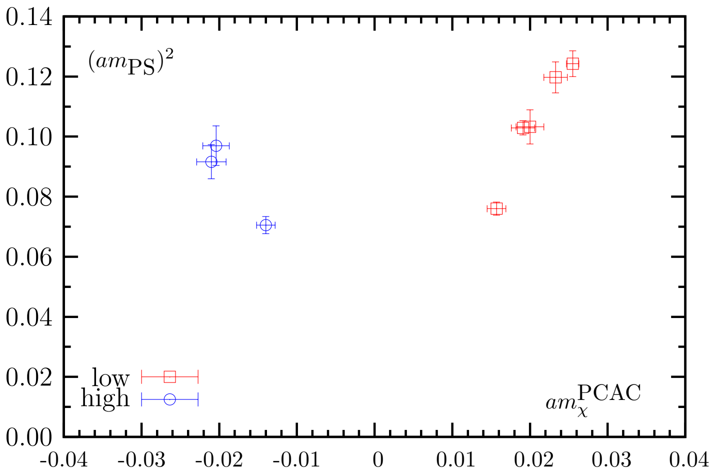
The fuzzy string of eigenvalues is repelled from the mass, m , which results in a first order phase transition.

Predictions

- Pion mass Sharpe-Singleton-2004, Münster-2004

$$m_\pi^2 = \frac{2|m|\Sigma - 16(W_8 + 2W_6)a^2}{F_\pi^2}.$$

When $W_8 + 2W_6 < 0$ we have a minimum pion mass. This has been observed in lattice simulations with twisted mass fermions. Jansen-etal-2005



The minimum pion mass is $O(a)$.

- The first order scenario has only been observed for dynamical Wilson quarks, whereas the Aoki phase has been found both in the quenched case and in the case with dynamical Wilson quarks.

III. Discretization Effects for the Overlap Dirac Operator

Overlap Dirac Operator at $\mu = 0$

Overlap Dirac Operator at $\mu \neq 0$

Overlap Dirac Operator at $a = 0$

The overlap Dirac operator

$$D_{\text{ov}} = 1 + \gamma_5 U \text{sign}(D_5) U^{-1}, \qquad D_5 = D_W + m\gamma_5.$$

Narayanan-Neuberger-1994,1995, Neuberger-1998, Edwards-Heller-Narayanan-1999

- Looks drastic to replace the eigenvalues by their sign, but at zero lattice spacing this is actually exact.
- The eigenvectors contain the information on the eigenvalues.

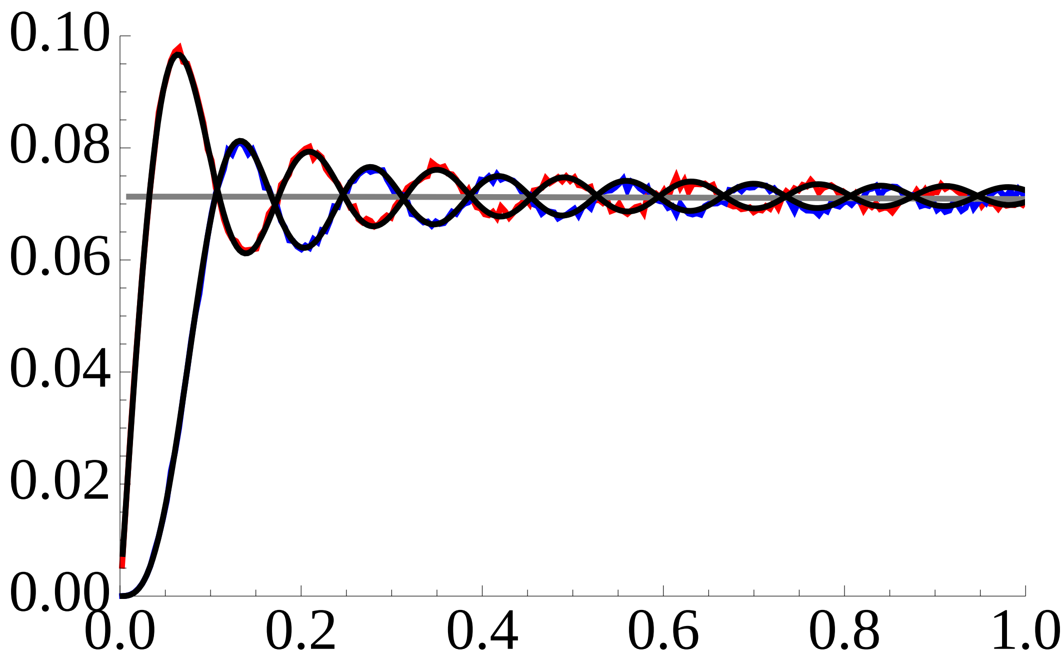
$$D_5 = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} m & \lambda_k \\ \lambda_k & -m \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix}.$$

- Complete diagonalization by an additional rotation with $\tan 2\phi_k = \lambda_k / m$.

Overlap Dirac Operator at $a = 0$

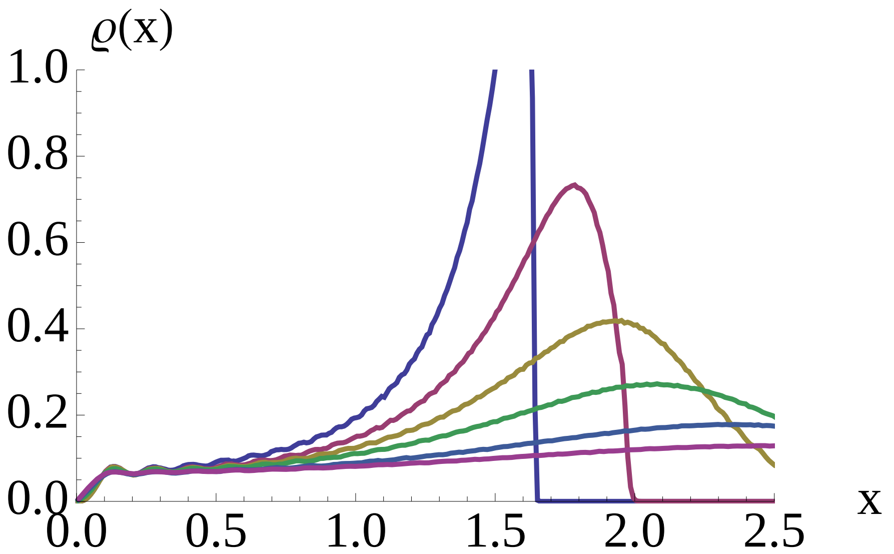
- The projected eigenvalues are given by a smooth function of λ_k/m .
- At nonzero lattice spacing overlap eigenvalues are expected to have correlations that differ by $O(a)$ or $O(a^2)$ terms from the continuum limit.
- What happens to eigenvalue correlation of when the Wilson Dirac operator is in the Aoki phase?

Large Mass RMT Overlap Dirac Operator at $a \neq 0$



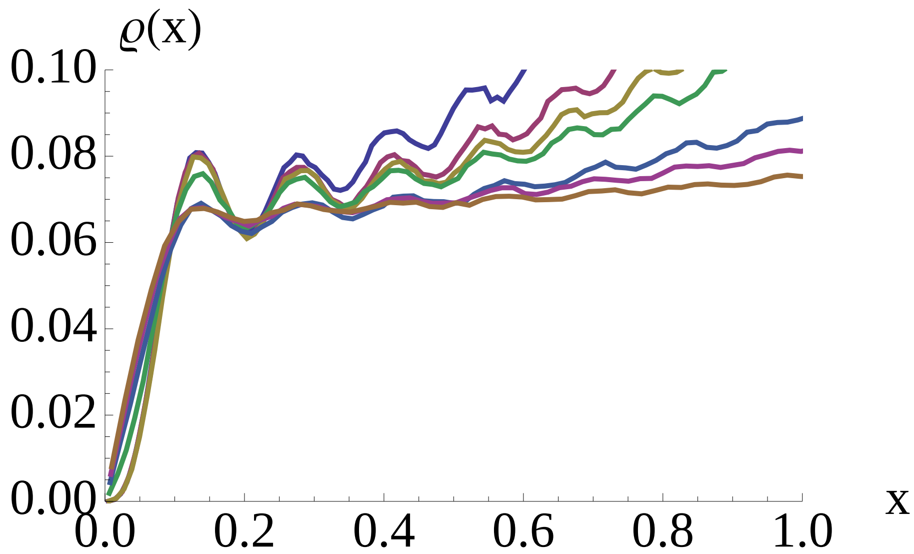
The spectral density of the projected overlap Dirac operator for $a = 0.3$, $m = 100$ and index $\nu = 0$ and $\nu = 1$. The black curve shows the analytical result and the red and blue curve the result from the computed eigenvalues.The grey curve is the mean field result for the spectral density.

Global Spectral Density of the Overlap Dirac Operator at $a \neq 0$



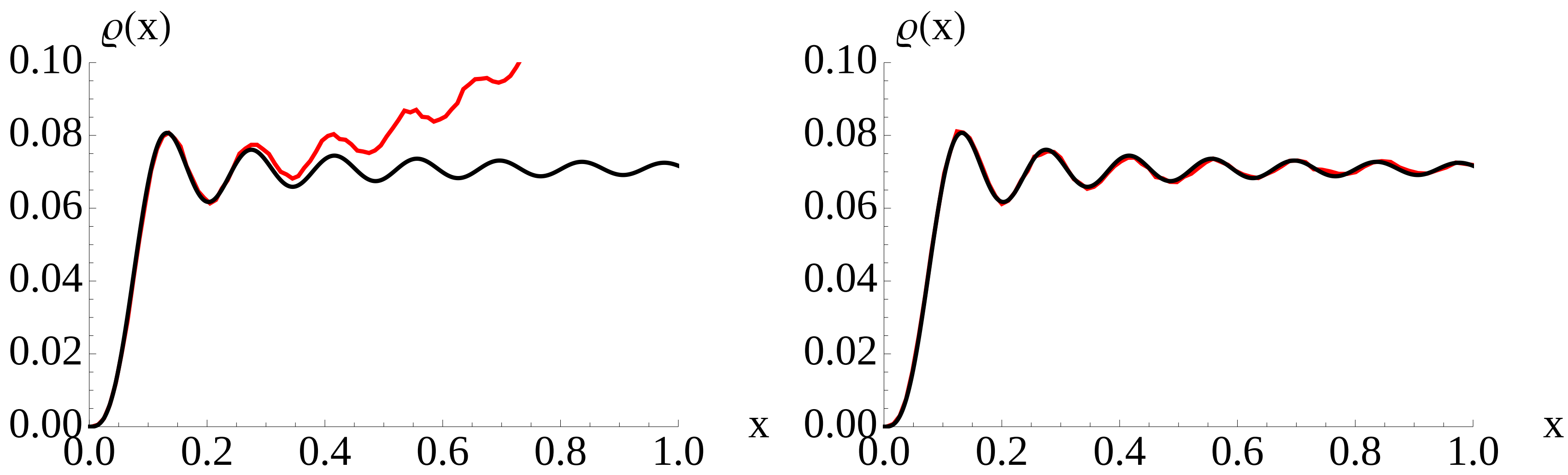
Global behavior of the spectral density of the projected overlap Dirac operator for $m = 0.2$ and $a = 0.05,\ a = 0.15,\ a = 0.24,\ a = 0.35,\ a = 0.45,\ a = 0.55$ (from top to bottom). The critical value for the transition to the Aoki phase is $a = 0.35$ (green curve). The curves have been rescaled to have the same small x behavior, separately for the normal phase and the Aoki phase.

Microscopic Spectral Density Overlap Dirac Operator at $a \neq 0$



The spectral density of the projected overlap Dirac spectra for $m = 0.2$ and $a = 0.05,\ a = 0.15,\ a = 0.24,\ a = 0.35,\ a = 0.45,\ a = 0.55$ (from top to bottom). The critical value for the transition to the Aoki phase is $a = 0.35$ (green curve). The curves have been rescaled to have the same small x behavior, separately for the normal phase and the Aoki phase.

Unfolded Spectral Density of the Overlap Dirac Operator at $a \neq 0$



Comparison of the spectral density of the overlap Dirac operator for $a = 0.15$ and $m = 0.2$ and the analytical chiral random matrix theory result.The unfolded eigenvalues are shown right.

Overlap Dirac Operator at Nonzero μ

Dirac operator

$$D_W(\mu, a) = \begin{pmatrix} m + aA & W + \mu \\ -W^\dagger + \mu & m + aB \end{pmatrix}.$$

Bloch-Wettig Overlap Dirac operator

Bloch-Wettig-2006

$$D_{\rm ov}(\mu, a) = 1 + \gamma_5 V \mathrm{sign}(\gamma_5 D_W(a, mu) V^{-1}.$$

- Sign is determined by the imaginary part of the eigenvalues
- V is not unitary.

Large Mass Limit of Overlap Dirac Operator

For $a = 0$, the Dirac operator $\gamma_5 D_W$ can be written in the form

$$\gamma_5 D_W = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} m & X \\ Y & -m \end{pmatrix} \begin{pmatrix} U_1^{-1} & 0 \\ 0 & U_2^{-1} \end{pmatrix}$$

with X and Y triangular matrices, and U and V unitary. We diagonalize the matrix inbetween

$$\begin{pmatrix} m & X \\ Y & -m \end{pmatrix} = V^{-1} \begin{pmatrix} \sqrt{m^2 + xy} & 0 \\ 0 & -\sqrt{m^2 + xy} \end{pmatrix} V.$$

For large m

$$V = \begin{pmatrix} 1 & -X/2m \\ Y/2m & 1 \end{pmatrix}.$$

Large Mass Limit of Overlap Dirac Operator Nonzero μ and a

Overlap Dirac operator (up to unitary transformation)

$$D_{\rm ov} = 1 - \gamma_5 V^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V = \begin{pmatrix} 0 & -X/m \\ Y/m & 0 \end{pmatrix},$$

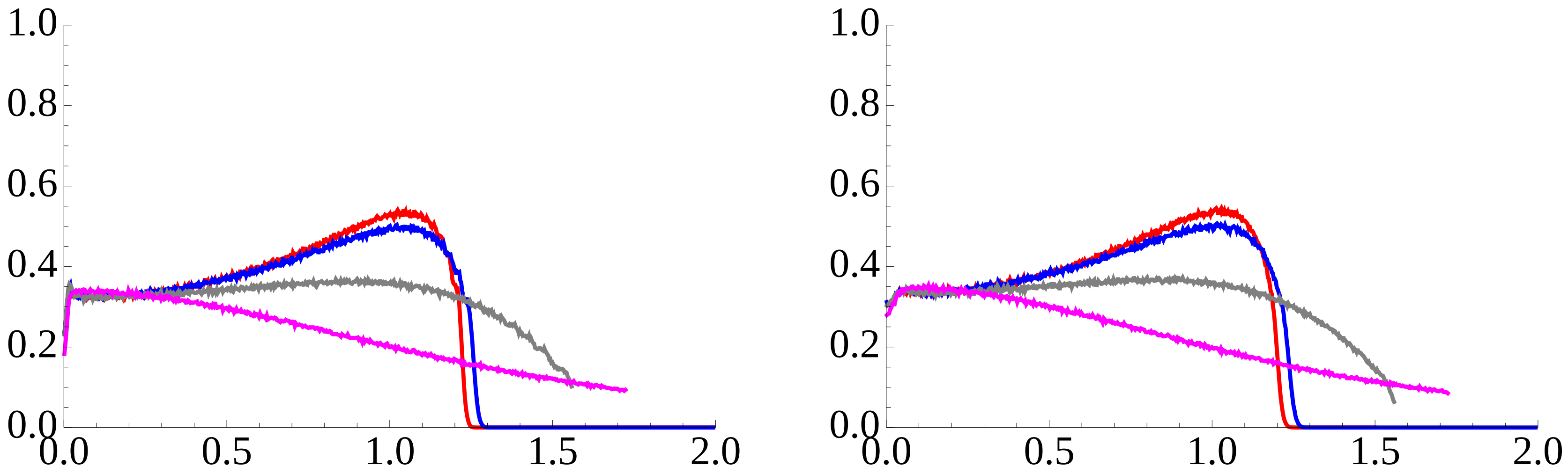
Eigenvalues of the overlap Dirac operator

$$\lambda_k = \pm i \frac{\sqrt{x_k y_k}}{m},$$

which differ by a factor $1/m$ from the original Dirac operator.

At finite a we again expect $O(a)$ or $O(a^2)$ corrections.

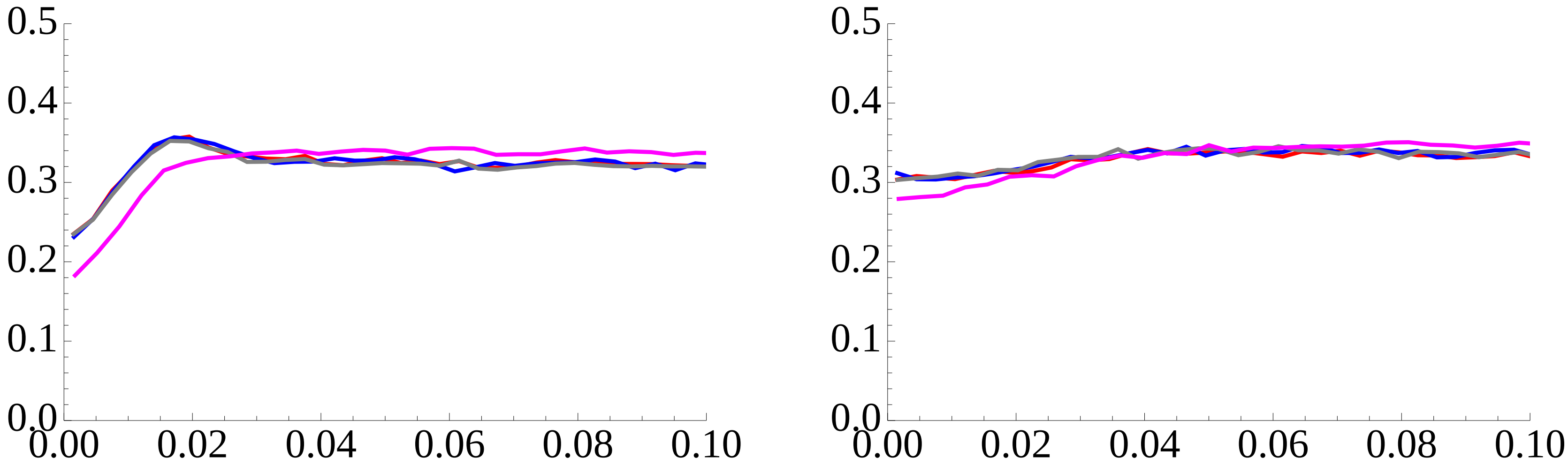
Global Spectral Density of the Overlap Dirac Operator at Nonzero μ and a



Spectral density $\int dx \rho(x, y)$ as a function of y of the overlap Dirac operator for $\mu = 0.1$ (left) and $\mu = 0.3$ (right) and lattice spacing equal to $a = 0$ (red), $a = 0.2$ (blue), $a = 0.5$ (grey) and $a = 1$ (magenta), and $m = 1$.

- Contrary to the lattice spacing, the chemical potential does not have a large effect on the ultraviolet part of the Dirac spectrum.

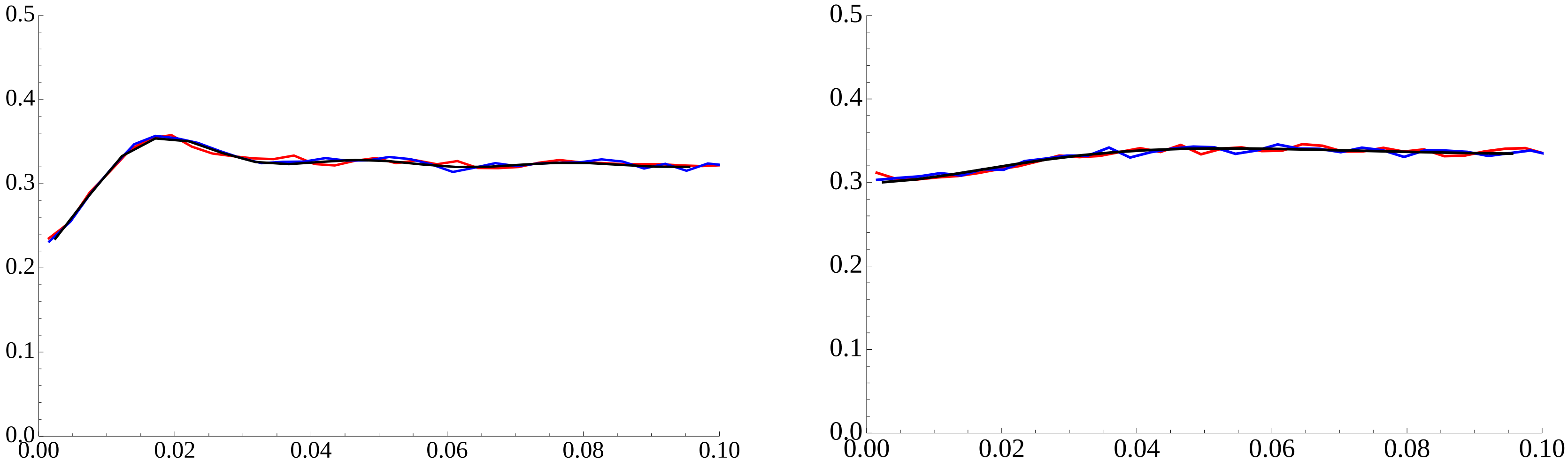
Infrared Part of the Dirac Spectrum



Spectral density $\int dx \rho(x, y)$ as a function of y of the overlap Dirac operator for $\mu = 0.2$ (left) and $\mu = 0.5$ (right) and lattice spacing equal to $a = 0$ (red), $a = 0.2$ (blue), $a = 0.5$ (grey) and $a = 1$ (magenta).

- Eigenvalue fluctuations do not depend on the lattice spacing (after rescaling) if the quark mass is outside of the Aoki domain.
- The scale factors do not depend on μ, and the deformed Dirac spectrum at nonzero a only gives rise to μ-independent overall factor.

Microscopic Spectral Density of Overlap Dirac Operator at $\mu \neq 0$



Microscopic spectral density for $a = 0.2$ (red) and $a = 0.5$ (blue) compared to the analytical result for $a = 0$ for $\mu = 0.1$ (left) and $\mu = 0.3$ (right).

IV. Conclusions

- We have seen that the interpretation of low-energy constants of Wilson Chiral perturbation theory in terms of eigenvalue fluctuations imposes constraints on these constants.

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- The first order scenario has been understood in terms of collective eigenvalue fluctuations of D_W .

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- We have seen that the interpretation of low-energy constants of Wilson Chiral perturbation theory in terms of eigenvalue fluctuations imposes constraints on these constants.
- The first order scenario has been understood in terms of collective eigenvalue fluctuations of D_W .
- The overlap operator is very robust against discretization effects due to the Wilson term as long as the quark mass is outside the Aoki phase. This is also the case for the overlap Dirac operator and nonzero chemical potential.