ETH
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Swiss Federal Institute of Technology Zurich

Euclidean Dynamical Triangulation revisited: is the phase transition really 1st order?

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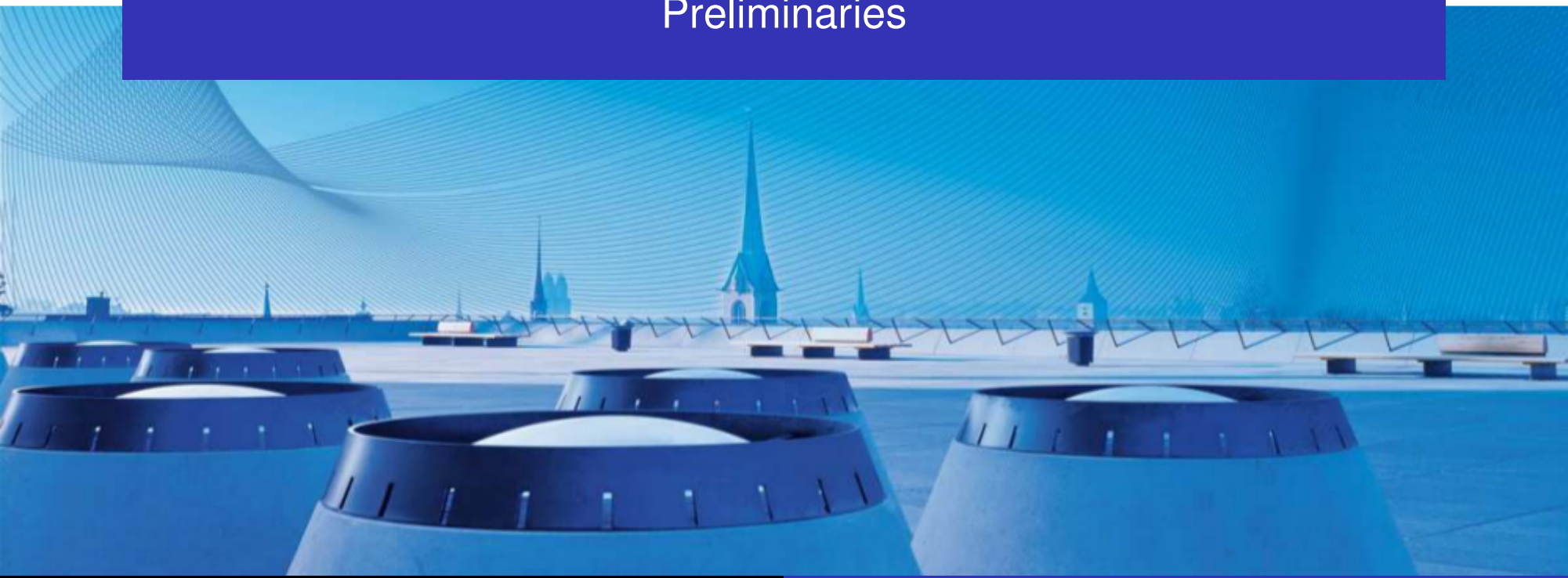
Section 1

Preliminaries



Motivation

Why we reconsidered the phase transition in 4D EDT?

- numerical part of my Master's thesis: write own program to cross-check results of D. Coumbe & J. Laiho: "'Exploring the Phase Diagram of Lattice Quantum Gravity"' (arXiv:1201.2864 [hep-lat])

- not convinced that phase transition in standard EDT is 1st order:
  - simulation methods correct?
  - scaling behavior of $B_4 = \dfrac{\left\langle (N_2 - \langle N_2 \rangle)^4 \right\rangle}{\left\langle (N_2 - \langle N_2 \rangle)^2 \right\rangle^2}$?

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Dynamical Triangulation

- Formal path integral for Euclidean quantum gravity in four dimensions,

$$Z = \int \mathcal{D}\left[g_{\mu\nu}\right] \mathrm{e}^{-S_{EH}\left[g_{\mu\nu}\right]},$$

is regularized by approximating configuration space with space of piecewise linear (PL) manifolds consisting of equilateral 4-simplices with fixed edge length a, s.t.:

- $S_{EH} = -\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\,(R-2\Lambda) \quad\longrightarrow\quad S_{ER} = -\kappa_2 N_2 + \kappa_4 N_4,$

$$\text{with } \kappa_2 = \frac{V_2}{4G},\ \kappa_4 = \frac{10\arccos(1/4)\,V_2 + \Lambda V_4}{8\pi G},\ V_n = a^n\frac{\sqrt{n+1}}{n!\sqrt{2^n}}$$

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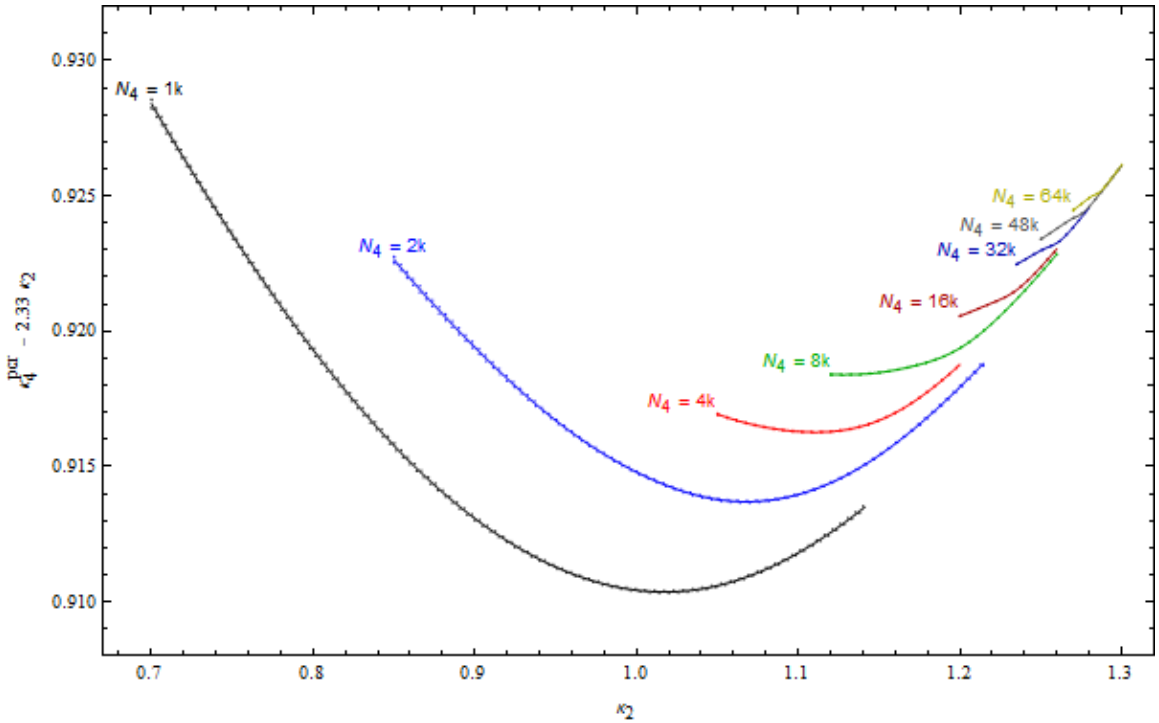

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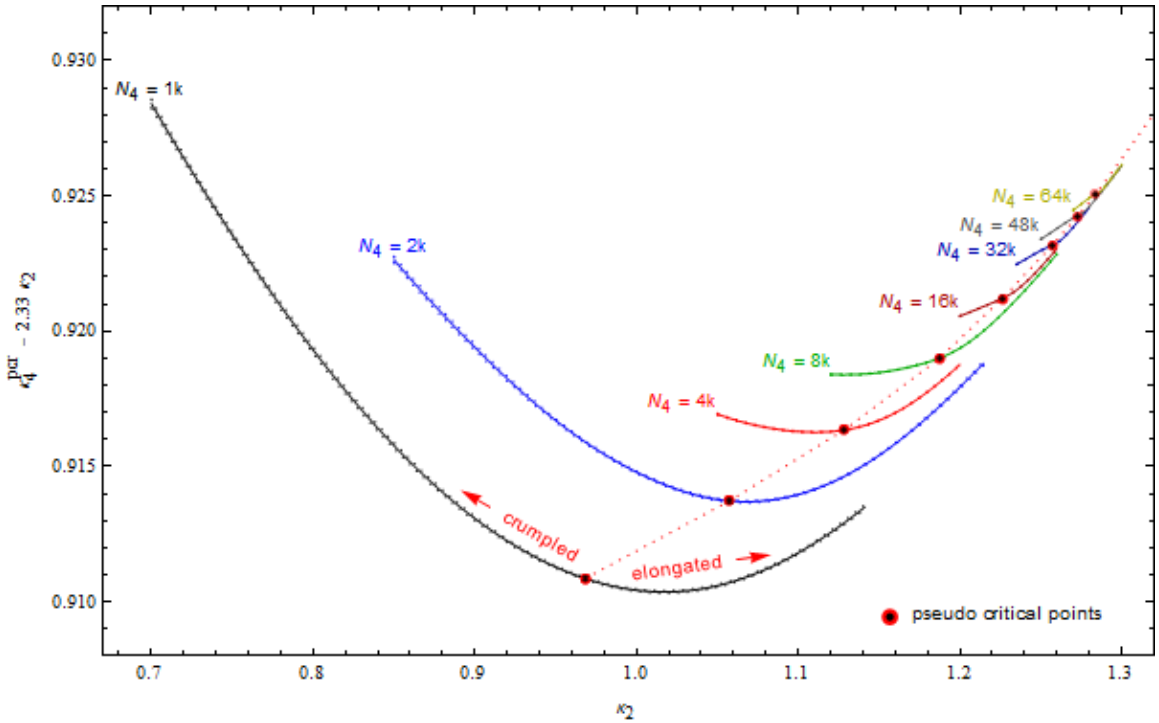

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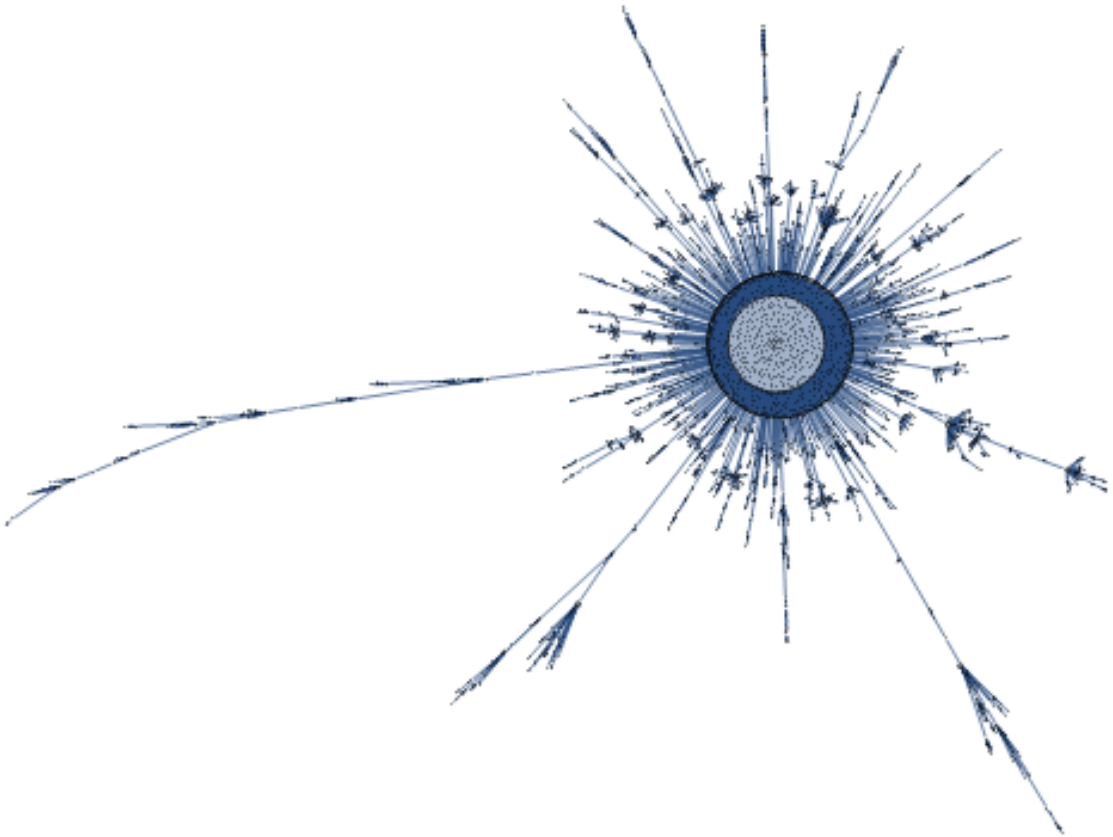

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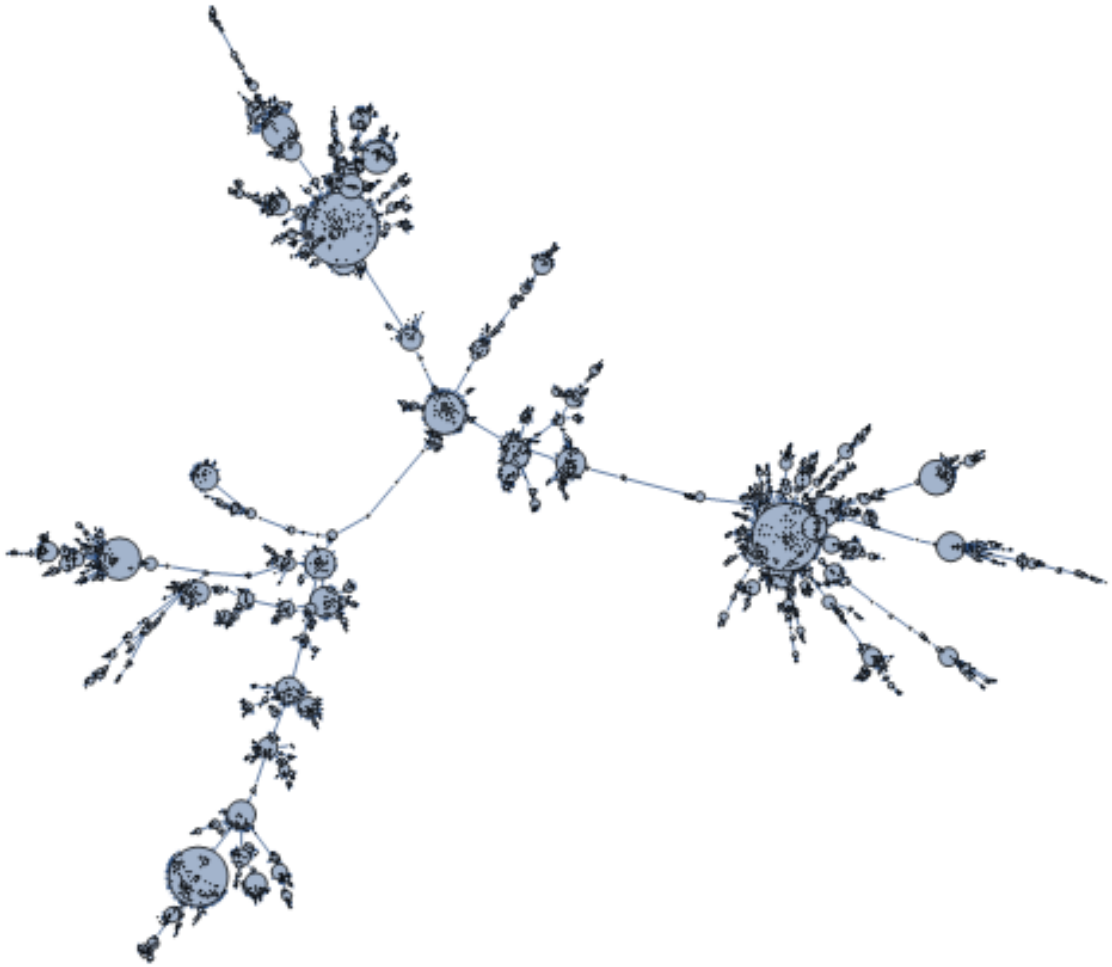

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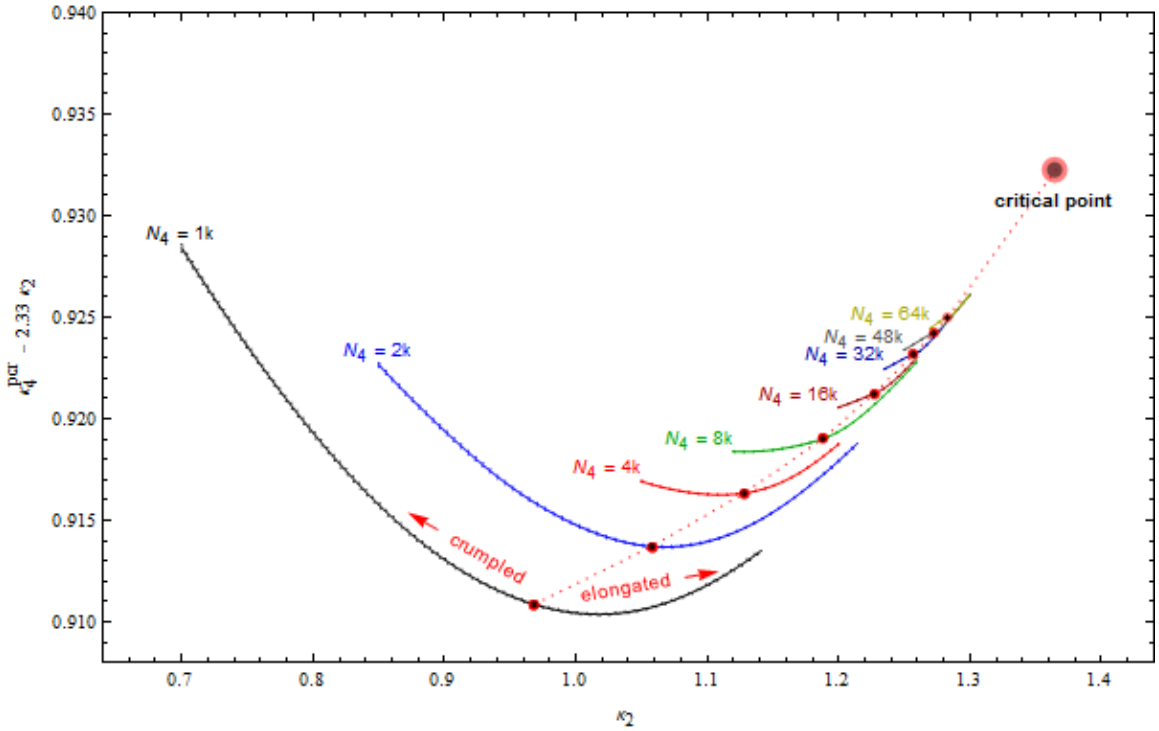

- critical point: $(\kappa_2^{cr}, \kappa_4^{cr}(\kappa_2^{cr})) = \lim_{N_4 \to \infty} (\kappa_2^{pcr}(N_4), \kappa_4^{pcr}(\kappa_2^{pcr}(N_4), N_4))$

Order of Phase Transition

- phase transition was found to be of 1^{st} order:
  1. P. Bialas, Z. Burda, A. Krzywicki, B. Petersson: arXiv:hep-lat/9601024 (1996) ($N_4 = 32$k)
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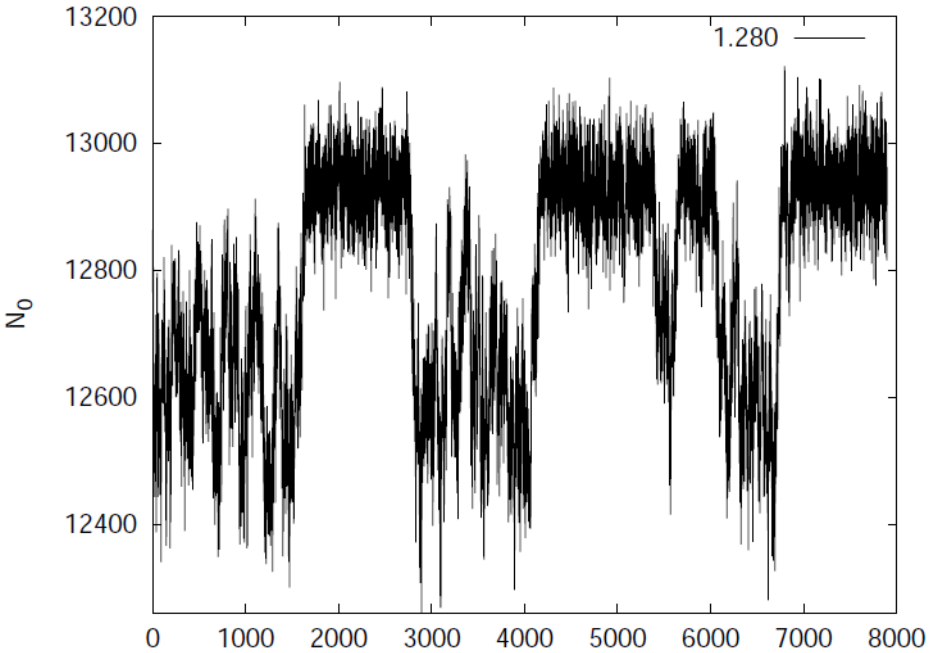
N_0-time history shows jumps



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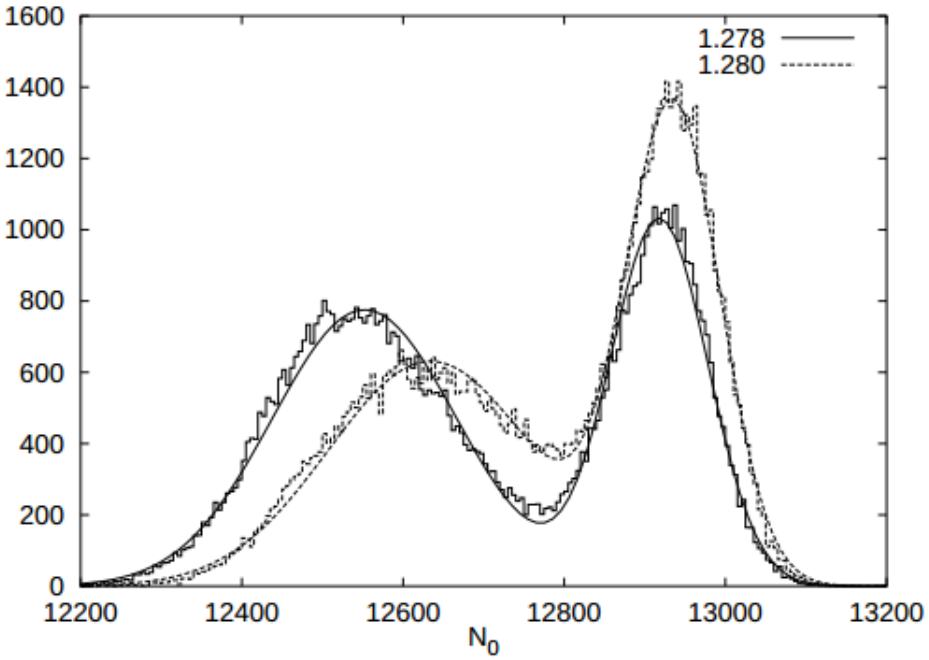
N_0-histogram shows double-gaussian structure



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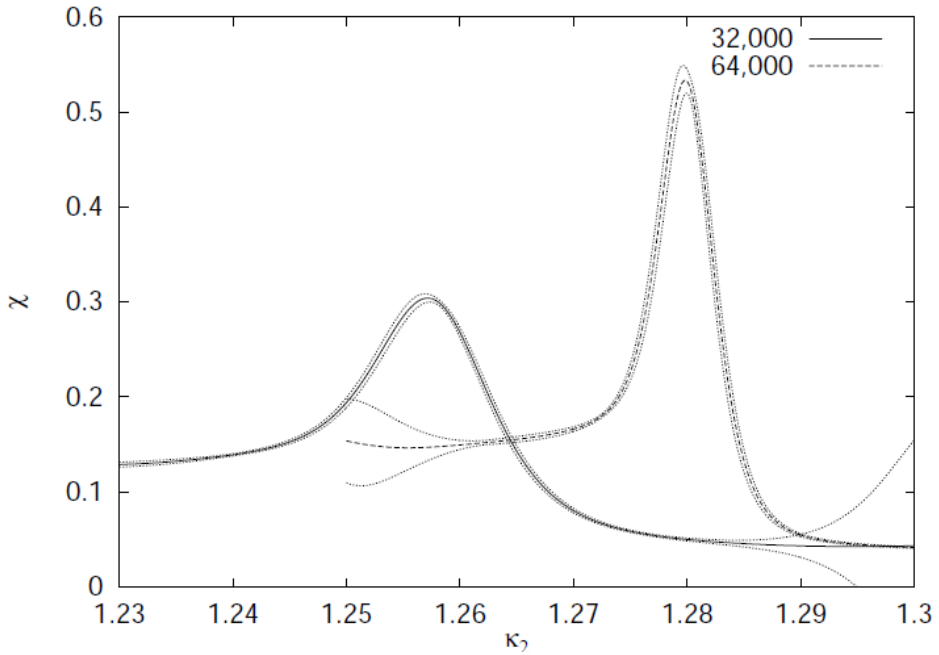
scaling of N_0-susceptibility compatible with 1st order transition



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Section 2

Possible Problems & Modifications

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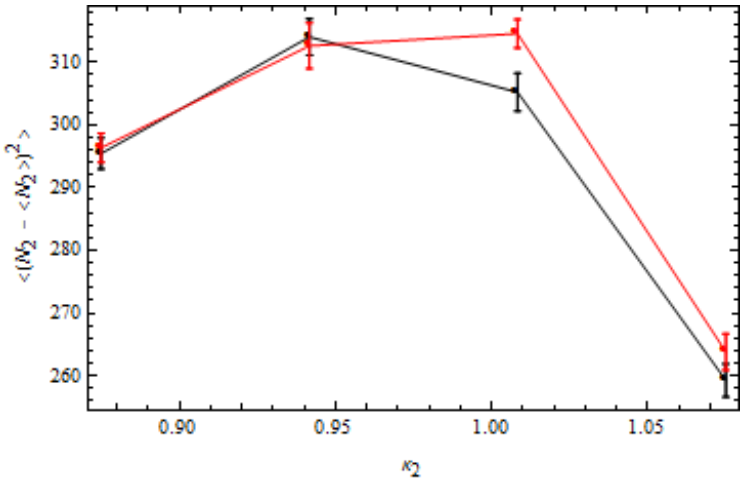

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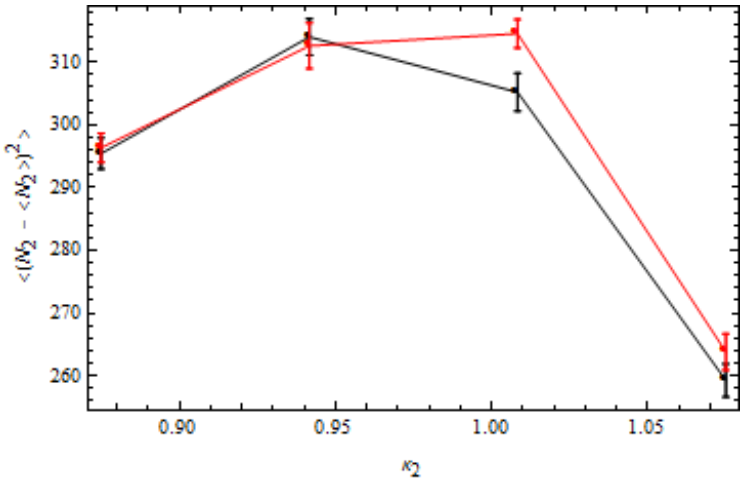

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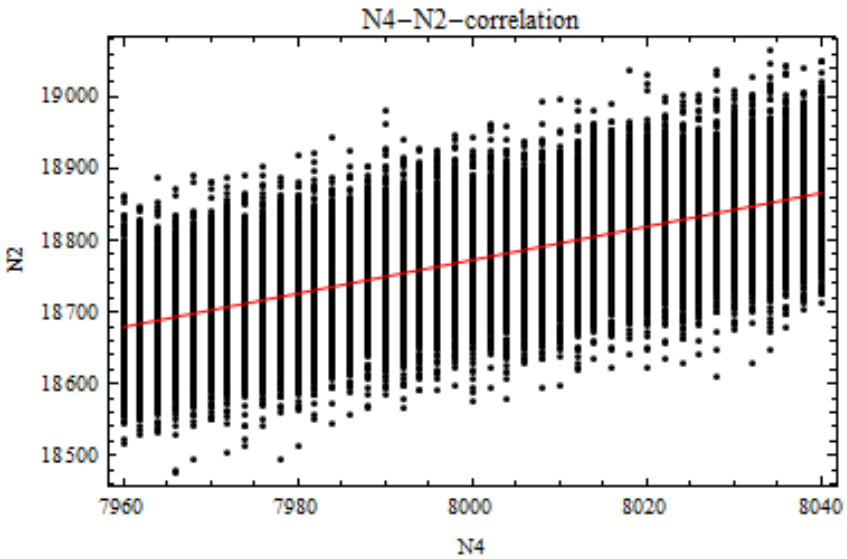

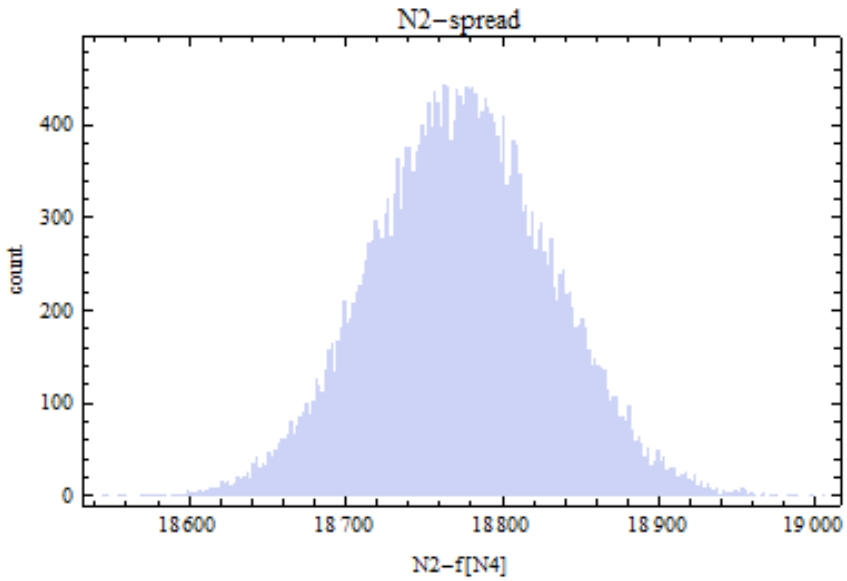

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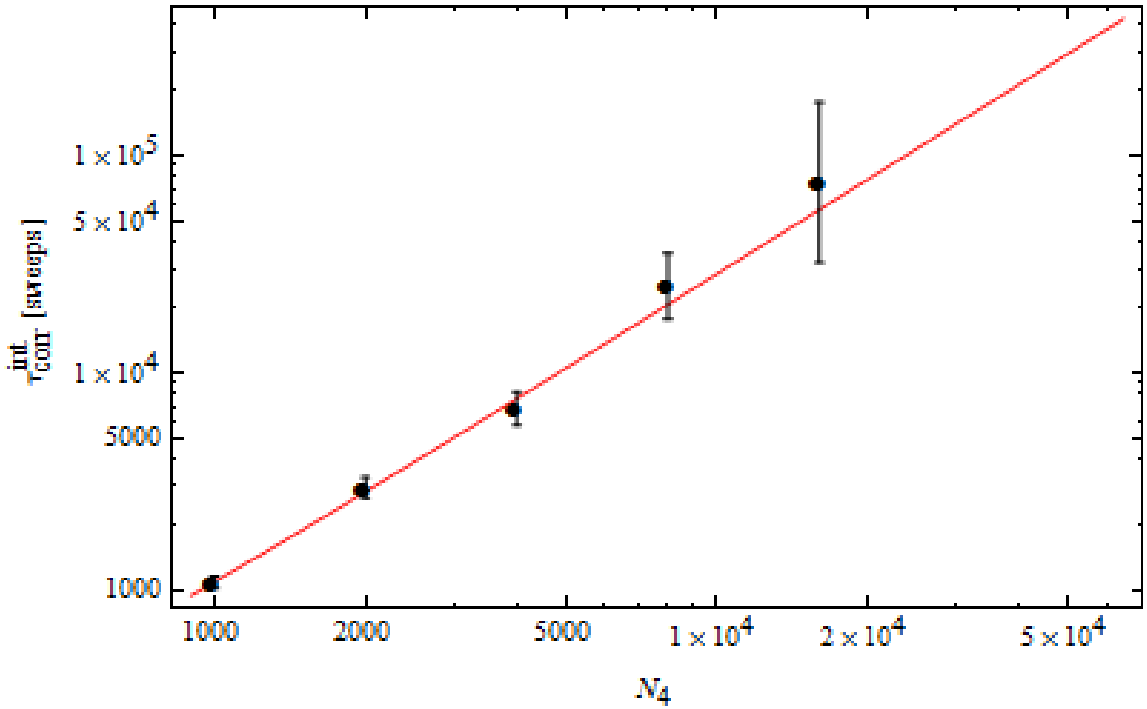

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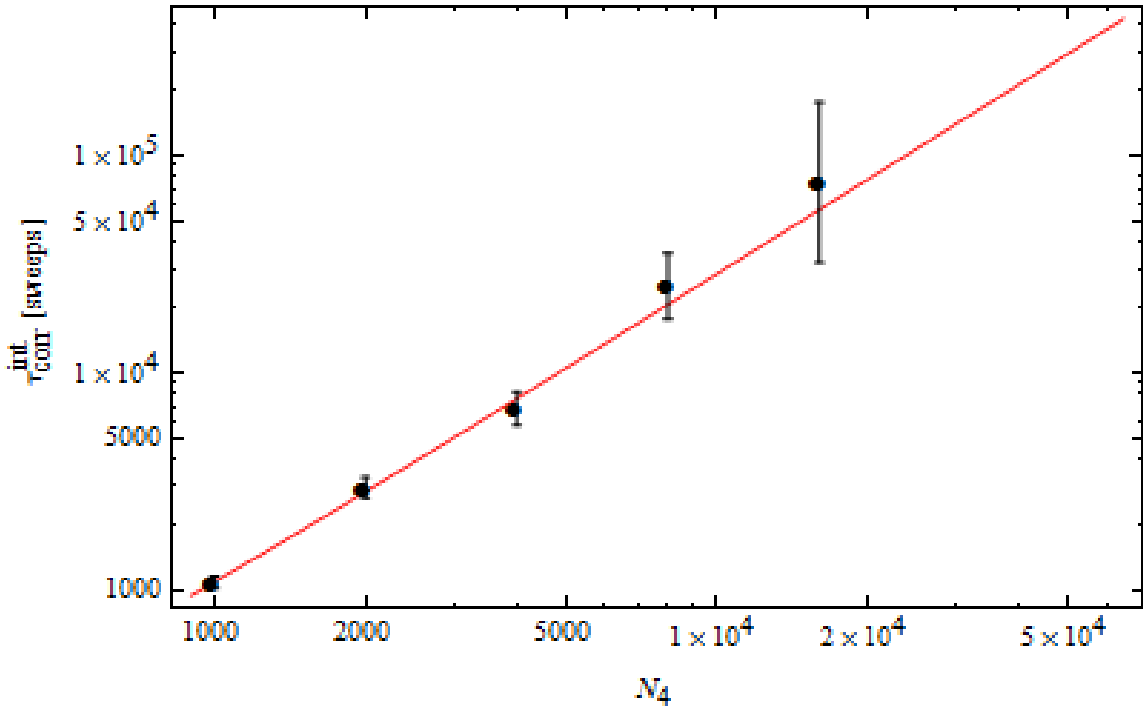

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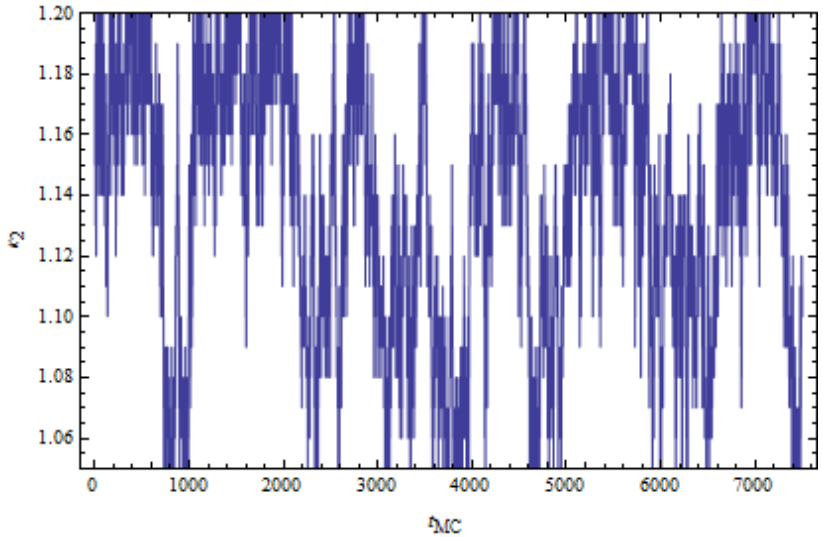

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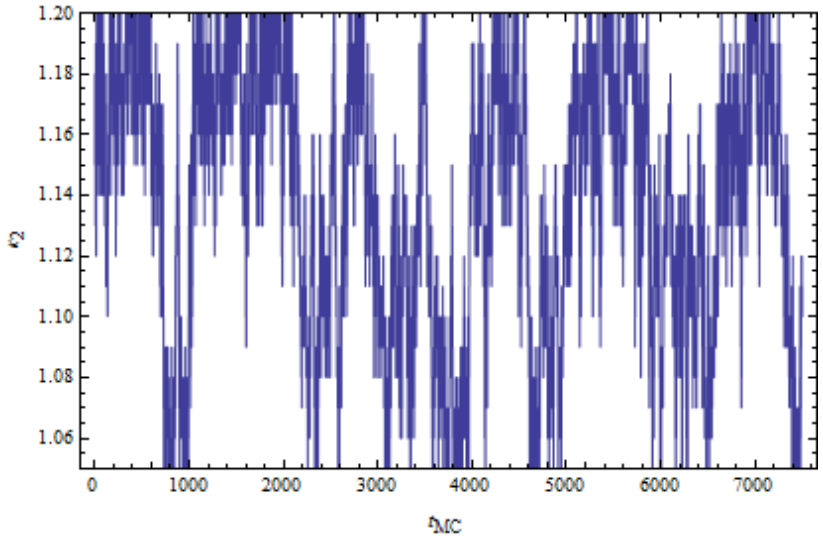
- after some number of sweeps, swap configurations between neighboring runs with probability $p_{\text{swap}} = \min\left\{1, e^{-\Delta\kappa_2 \Delta N_2 + \Delta\kappa_4 \Delta N_4}\right\}$
- configurations wander through κ_2-space:
  - configurations reach frequently regions where relaxation is fast
  - all n Markov chains contribute to statistical ensemble at each κ_2



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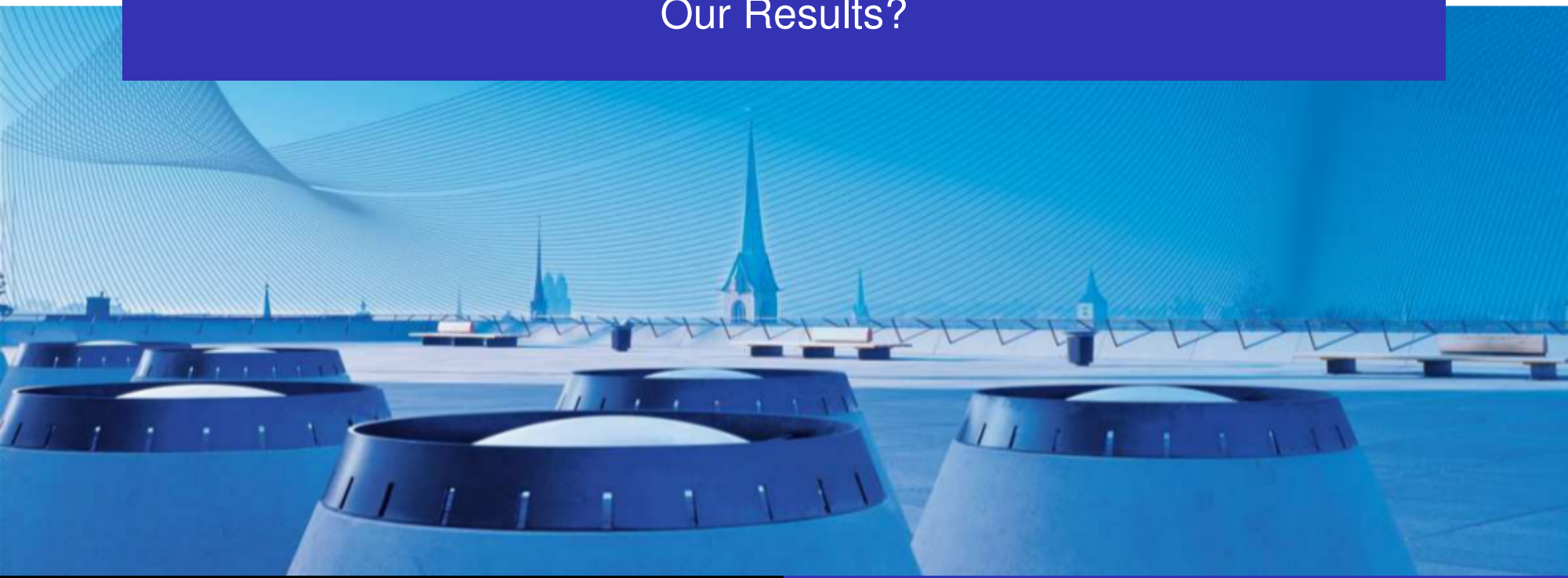
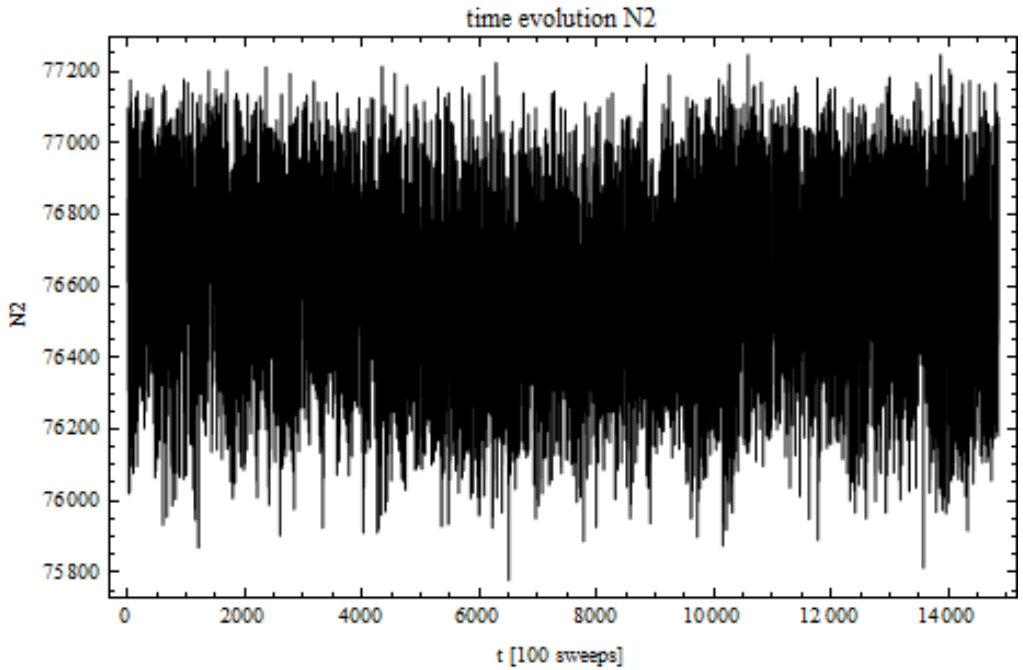
Section 3

Our Results?



Results: N_2-Distribution

- thanks to parallel tempering N_2-time history looks much better
- still observe double peak structure



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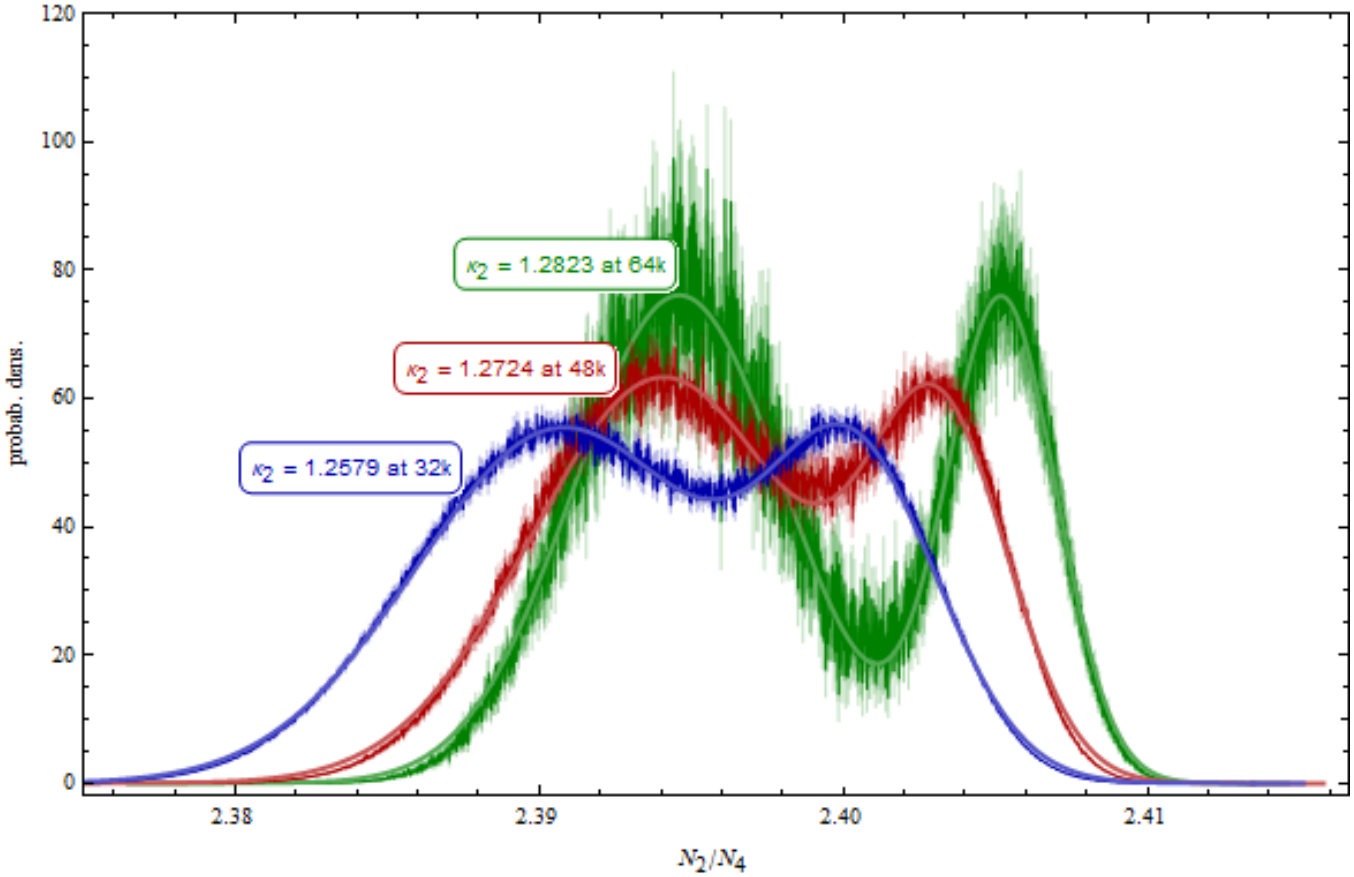
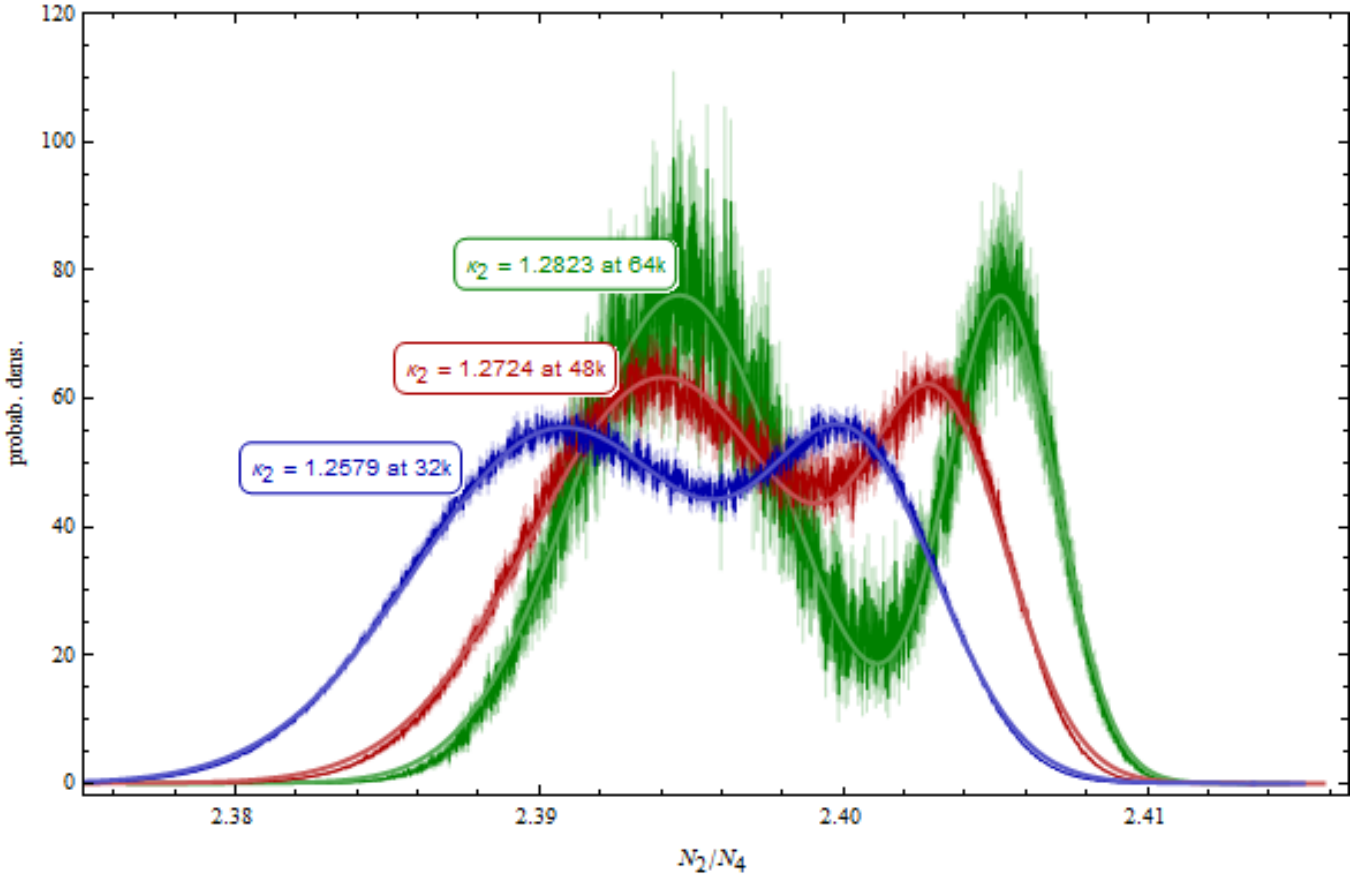


Results: N_2-Distribution

- thanks to parallel tempering N_2-time history looks much better
- still observe double peak structure $\Rightarrow$ **characteristic for 1st order transition**



Results: Scaling of B_4

- 4^{th} Binder cumulant (Kurtosis): $$B_4\left[N_2\right] = \frac{\left\langle \left(N_2 - \langle N_2 \rangle\right)^4 \right\rangle}{\left\langle \left(N_2 - \langle N_2 \rangle\right)^2 \right\rangle^2}$$

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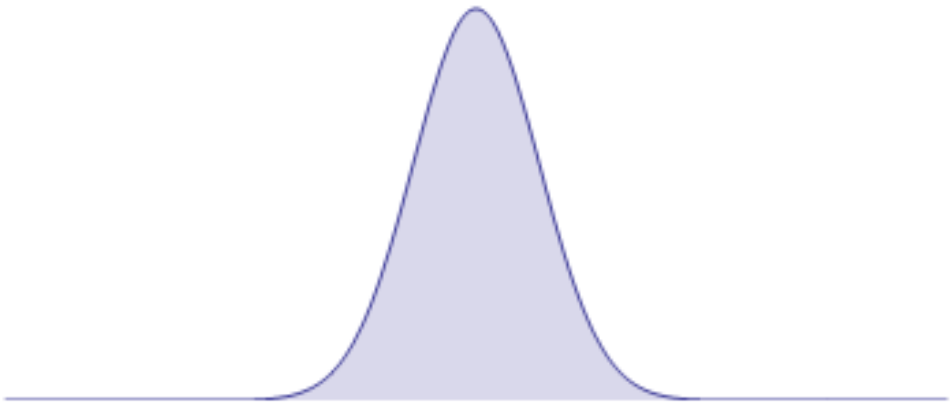
$B_4 = 3.00000$



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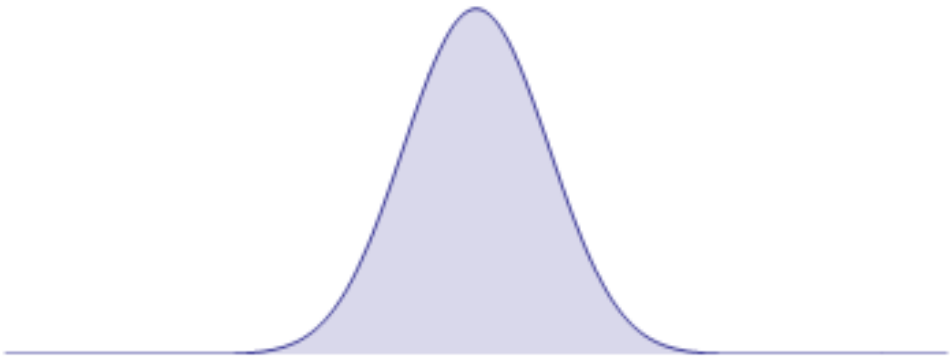
$B_4 = 2.92000$



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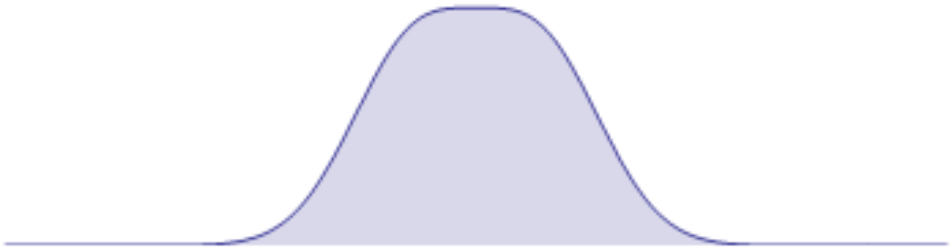
$B_4 = 2.50000$



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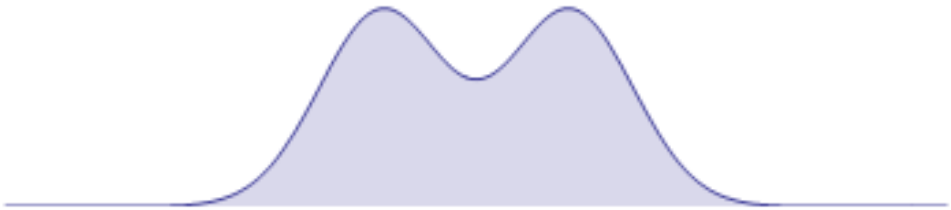
$$B_4 = 2.04140$$



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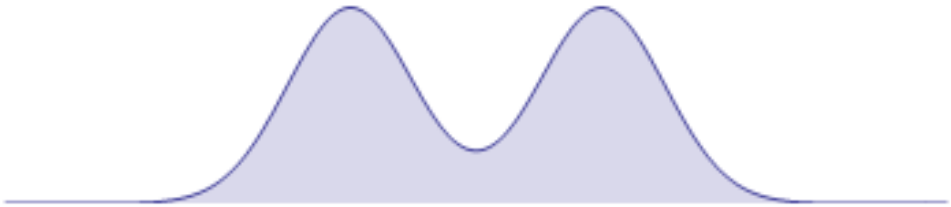
$B_4 = 1.72000$



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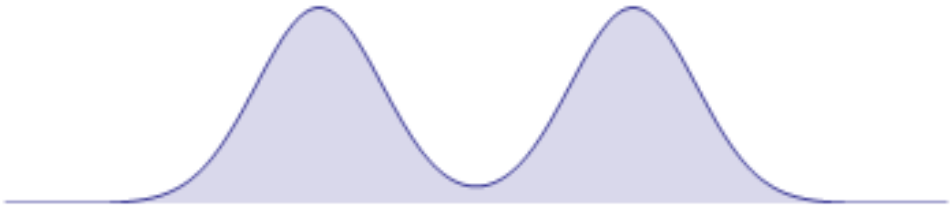
$$B_4 = 1.51370$$



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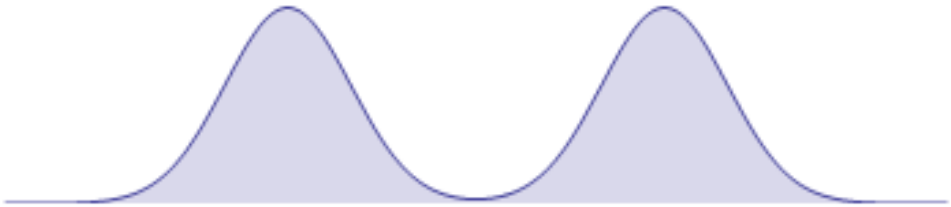

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- Gaussian distribution $\Rightarrow$ $B_4 = 3$.

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Results: Scaling of B_4

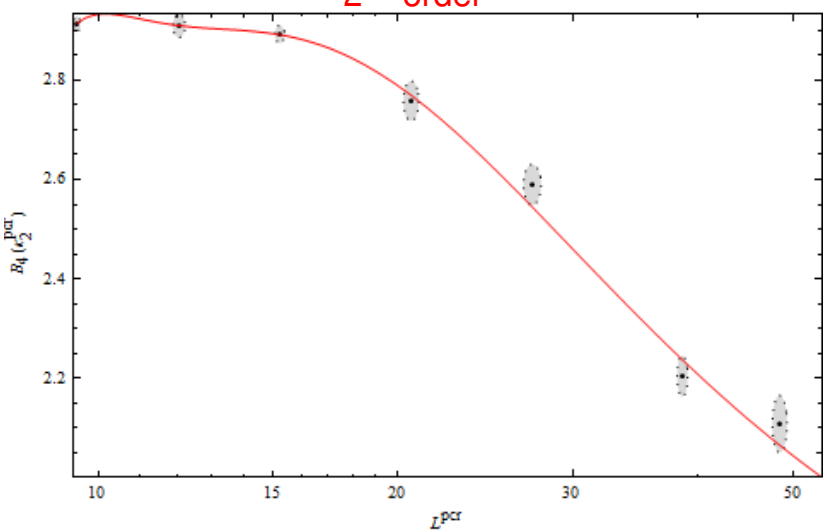
- finite size scaling of $B_4^{pcr}[N_2](N_4) = B_4[N_2](\kappa_2^{pcr}(N_4), N_4)$:
  - assuming 2nd order transition:

$$B_4^{pcr}[N_2](L(N_4)) = B_4^{cr}[N] + c_1 L^{-\omega}(N_4) + c_2 c_1^2 L^{-2\omega}(N_4) + \ldots$$

    where $L(N_4) = L(\kappa_2^{pcr}, N_4)$: average **linear** system size at pseudo-critical point

  - assuming 1th order transition (largest 3 volumes): $B_4^{pcr}[N_2](N_4) = B_4^{cr} + c_1 N_4^{-1}$

2nd order



$B_4^{cr}[N_2] = 1.53 \pm 1.00, \quad \omega = 1.57 \pm 1.25$

Results: Scaling of B_4

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$$B_4^{pcr}\left[N_2\right]\left(L\left(N_4\right)\right) = B_4^{cr}\left[N\right] + c_1\, L^{-\omega}\left(N_4\right) + c_2\, c_1^2\, L^{-2\,\omega}\left(N_4\right) + \ldots$$

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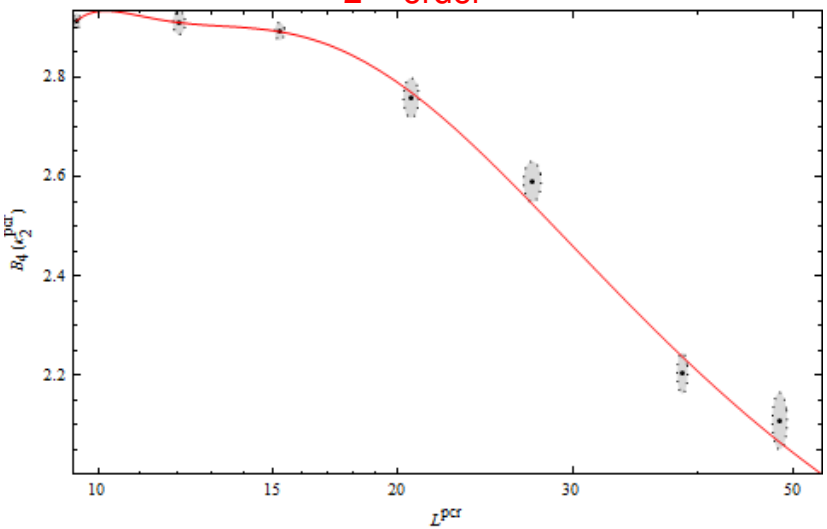

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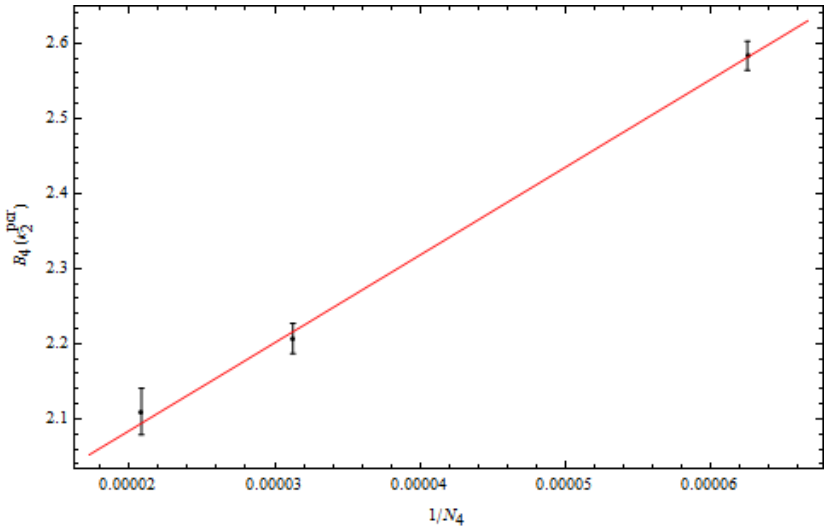
1st order



$B_4^{cr}\left[N_2\right] = 1.85 \pm 0.02$

Results: Scaling of B_4

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  - too small volumes?
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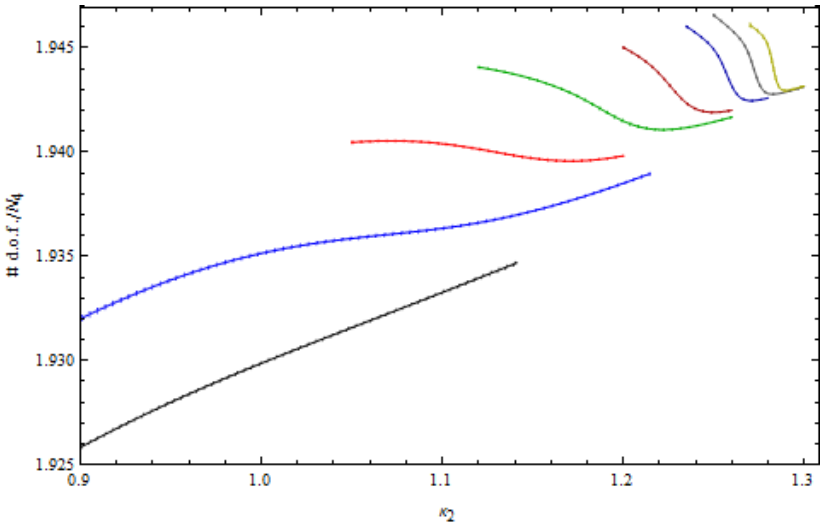
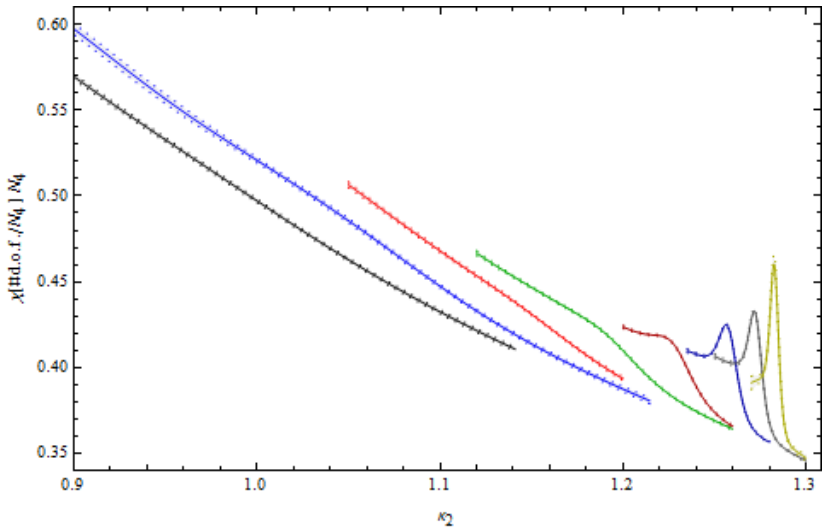
$\left\langle \frac{\text{d.o.f.}}{N_4} \right\rangle$



$N_4 \times \chi\left[\frac{\text{d.o.f.}}{N_4}\right]$



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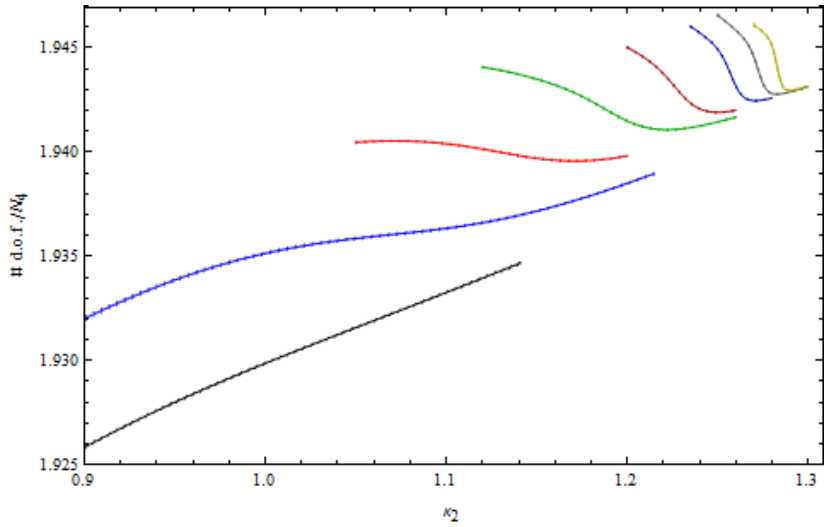

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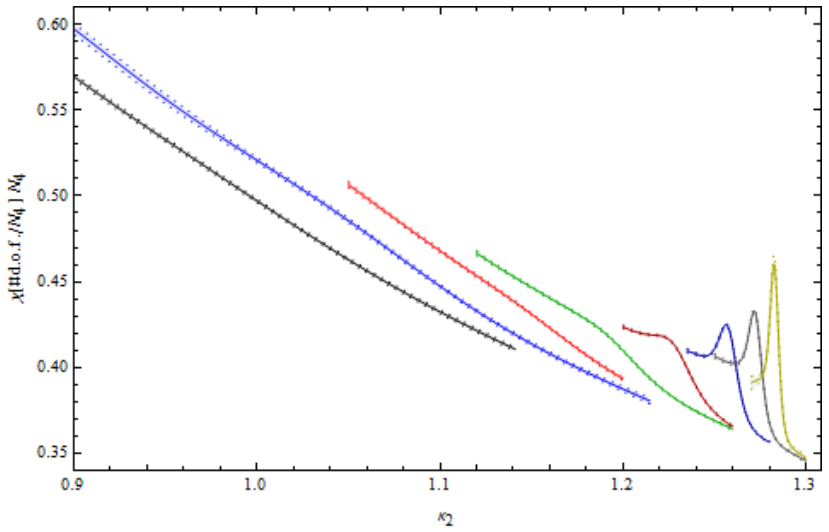

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Conclusion

- numerical methods used in [de Bakker, arXiv:hep-lat/9603024 (1996)] were wrong
  - measurements taken after every n^{th} *accepted* move
  - harmonic potential to control N_4

  $\Big\}$ $\Longrightarrow$ systematic error

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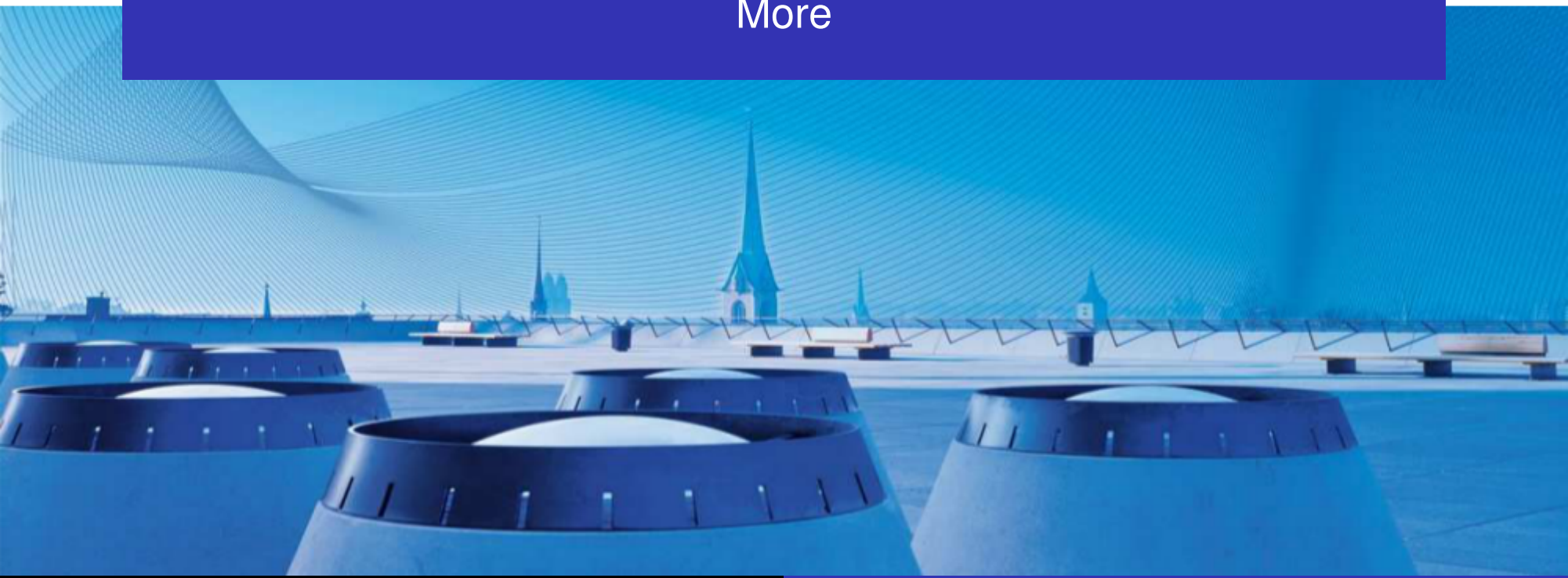
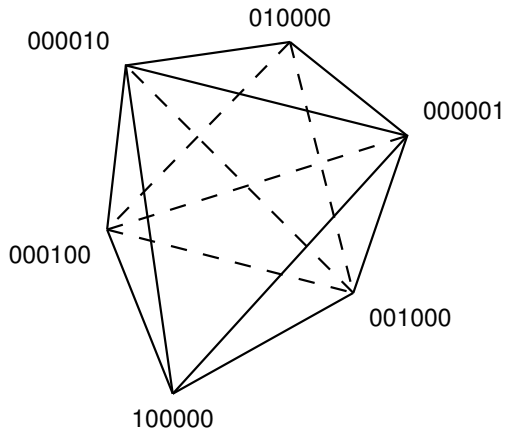
Section 4

More



Pachner Moves

- *n*-move:
  - select *n*-simplex, contained in $(5-n)$ 4-simplices



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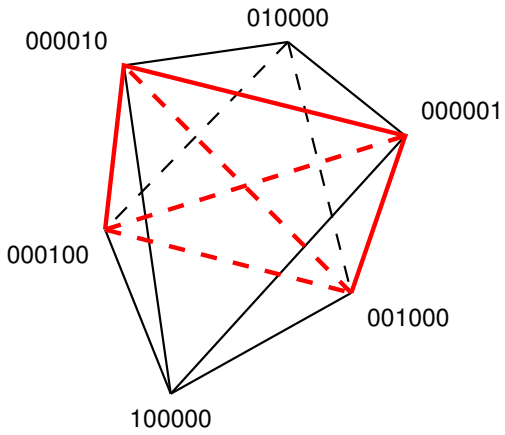

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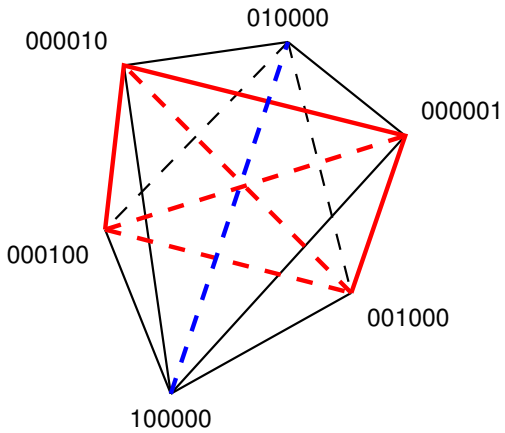

e.g. 3-move:

$$\begin{matrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{matrix} \xrightarrow{3-\text{move}} \begin{matrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{matrix}$$

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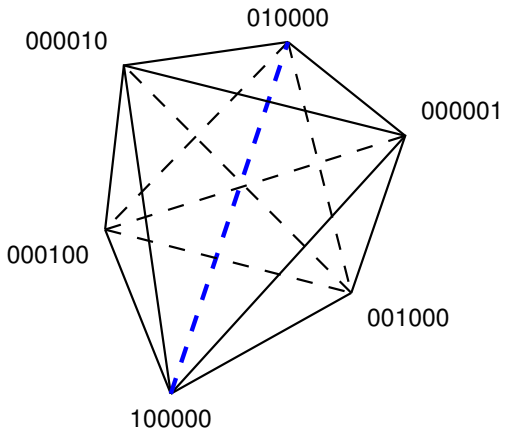

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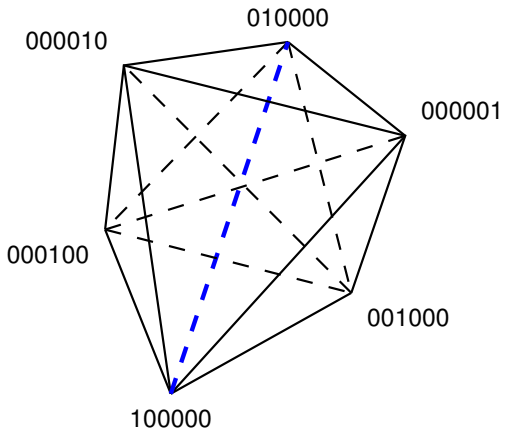

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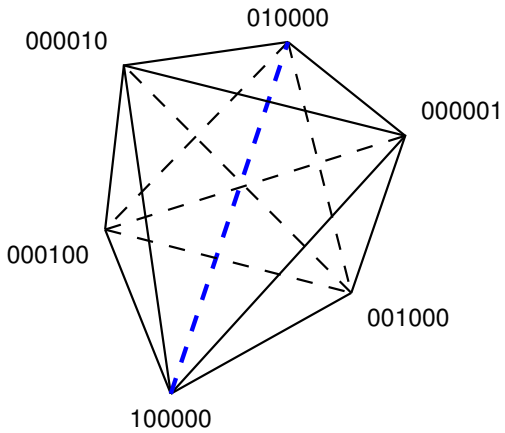

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- *manifold constraint*: dual $(4-n)$-simplex must not already exist in the triangulation.

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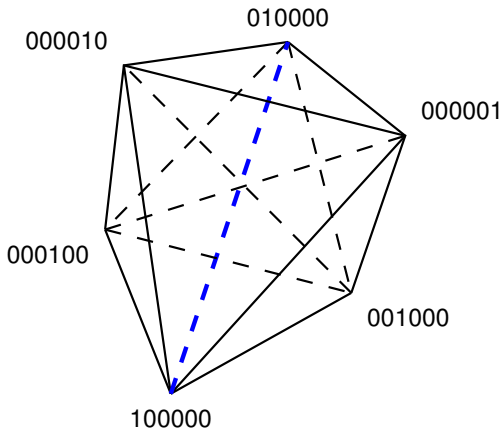

e.g. 3-move:

$$\begin{array}{cccccc} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{array} \xrightarrow{3-\text{move}} \begin{array}{cccccc} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

back

Metropolis Update Scheme

- detailed balance:

$$\rho(T)\, P\left(T \overset{n}{\to} T'\right) = \rho\left(T'\right) P\left(T' \overset{4-n}{\to} T\right)$$

for valid n-move (relating configurations T, T'):

- $\rho(T) = e^{\kappa_2 N_2(T) - \kappa_4 N_4(T)}$
- $P\left(T \overset{n}{\to} T'\right) = \dfrac{5-n}{5\, N_4(T) \binom{5}{n+1}}\, p_n(T)$

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probability for choosing the correct n-simplex

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randomly choose move type $n \in [0, \ldots, 4]$

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randomly chose a 4-simplex

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randomly chose a sub-n-simplex

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n-simplex accessible through $(5-n)$ different 4-simplices

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reduced transition probability:

$$p_n(T) = \min\left\{1, \frac{N_4(T)}{N_4(T) + \Delta N_4(n)}\, e^{\kappa_2 \Delta N_2(n) - \kappa_4 \Delta N_4(n)}\right\}$$

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for invalid n-move:

- $\rho\left(T'\right) = 0$
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