Loop formulation for the supersymmetric non-linear $O(N)$ σ-model

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Overview

1. Discretisation of the non-linear SUSY $O(N)$ σ-model
   - Continuum model
   - Discretisation of the action
   - Constraints in the path integral
2. Loop formulation
   - Fermionic loop expansion
   - Graphical representation
   - Bosonic bond representation
3. Results for $O(2)$
   - Chiral point
   - Masses in the system
   - Mass degeneracy
   - Ward identity

Constructing a non-linear SUSY $O(N)$ σ-model

- Consider a superfield

$$\Phi = \phi + i\bar{\theta}\psi + \frac{i}{2}\bar{\theta}\theta f\,.$$

- ϕ and f are real N-tuples of scalar fields and ψ is an N-tuple of Majorana fields.
- Demand the constraint $\Phi\Phi = 1\,.$
- Constraints in component space

$$\phi^2 = 1\,, \quad \phi\psi = 0 \quad \text{and} \quad \phi f = \frac{i}{2}\overline{\psi}\psi\,.$$

Lagrangian density

- SUSY $O(N)$-invariant Lagrangian density

$$\mathcal{L} = \frac{1}{2g^2}\overline{D\Phi}D\Phi|_{\bar{\theta}\theta}\ ,$$

with $D_\alpha = \partial_{\bar{\theta}^\alpha} + i(\gamma^\mu\theta)_\alpha\partial_\mu$.

- EOM implies that f and ϕ must be parallel, so the on-shell Lagrangian density in component fields is

$$\mathcal{L} = \frac{1}{2g^2}\left(\partial_\mu\phi\partial^\mu\phi + i\overline{\psi}\partial\!\!\!/\psi + \frac{1}{4}(\overline{\psi}\psi)^2\right)\ ,$$

where $\phi = (\phi_1, \phi_2, .., \phi_N)$, and $\psi = (\psi_1, \psi_2, .., \psi_N)$ where ψ_i is a Majorana spinor with $\overline{\psi}_i = \psi_i^T\mathcal{C}$.

[Witten '77; Ferrara, Vecchia '77]

Symmetries

- Invariant under the $\mathcal{N} = 1$ supersymmetry transformations

$$\delta\phi = i\bar{\epsilon}\psi \quad \text{and} \quad \delta\psi = (\not{\partial} + \frac{i}{2}\bar{\psi}\psi)\phi\epsilon$$

- The action and the constraints satisfy the global $O(N)$-symmetry "flavour"-symmetry.
- Chiral $\mathbb{Z}_2$-symmetry: $\psi \to i\gamma_5\psi$ with $\gamma_5 = i\gamma_0\gamma_1$
- $\exists\ \mathcal{N} = 2$ extension for nonlinear $\sigma-$models with Kähler target manifold [Zumino '79], e.g. $O(3)$ [Flore, Körner, Wipf, Wozar '12]

Discretisation

- To solve the fermion doubling problem we use the Wilson derivative.

$$\hat{\partial}^{\mathcal{W}}_{\mu}(r) = \hat{\partial}^{S}_{\mu} - \frac{ra}{2}\Delta^{\mathcal{W}}$$

  - → Fine tuning of mass parameter.
  - → Explicit breaking of chiral-symmetry and SUSY.
  - → Also Wilson derivative for the boson.
  - → Yields nearest neighbour interaction and diagonal neighbour interaction for the bosonic fields.
- Good experience in our earlier work with the WZ-model. And by others [Golterman, Petcher '89; Catterall et al. '03; Wipf et al. since '07].
- Only surviving symmetry on the lattice: global "flavour"-rotation .

Constraints in the path integral

- Constraint $\phi\psi = 0$ and $\phi^2 = 1$ enter the partition function

$$Z = \int \mathcal{D}\phi\, \delta(\phi^2 - 1)\mathcal{D}\psi\, \delta(\phi\psi) e^{-S[\phi,\psi]}\,.$$

- With

$$\int d\eta\delta(\eta - \eta')f(\eta) = \int d\eta(\eta - \eta')f(\eta) = f(\eta')$$

where η is a Grassmann variable, one obtains for the fermionic constraint

$$Z = \int \mathcal{D}\phi\, \delta(\phi^2 - 1)\mathcal{D}\psi \left(\sum_{i,j=1}^{N} \phi_i \phi_j \overline{\psi}_i \psi_j \right) e^{-S[\phi,\overline{\psi},\psi]}\,,$$

Loop Expansion of the Majorana Wilson fermions

- Expand fermionic action using the nilpotency of Grassmann elements.

$$e^{\sum_{i,x} \overline{\psi}_{i,x}\psi_{i,x}} = \prod_{x,i}(1 - \overline{\psi}_{i,x}\psi_{i,x}) = \prod_{x,i} \sum_{m_{i,x}=0}^{1} \left(-\overline{\psi}_{i,x}\psi_{i,x}\right)^{m_{i,x}}$$

- Partition function becomes a sum over all:

$P_\mu = \frac{(1-\gamma_\mu)}{2}$

$\times$ Hopping terms $\quad \overline{\psi}_{i,x} P_\mu \psi_{i,x+\mu}$

$\times$ Monomer terms $\quad \overline{\psi}_{i,x}\psi_{i,x}$

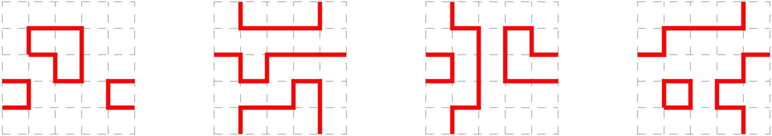
$\times$ Constraints $\quad \overline{\psi}_{i,x}\psi_{j,x}$

and

- Configurations with non-vanishing weight in Z need all $\overline{\psi}_i, \psi_i$ to appear exactly once (nilpotency) on every lattice site.
  $\rightarrow$ Only closed fermion loops survive the integration.

Loop expansion of the Majorana Wilson fermions

- The contribution of the fermionic strings to Z is only determined by their geometry (number of rotations and corners).
- If the loops are self-avoiding, the topologies



$\mathcal{L}_{00}$ $\mathcal{L}_{10}$ $\mathcal{L}_{01}$ $\mathcal{L}_{11}$

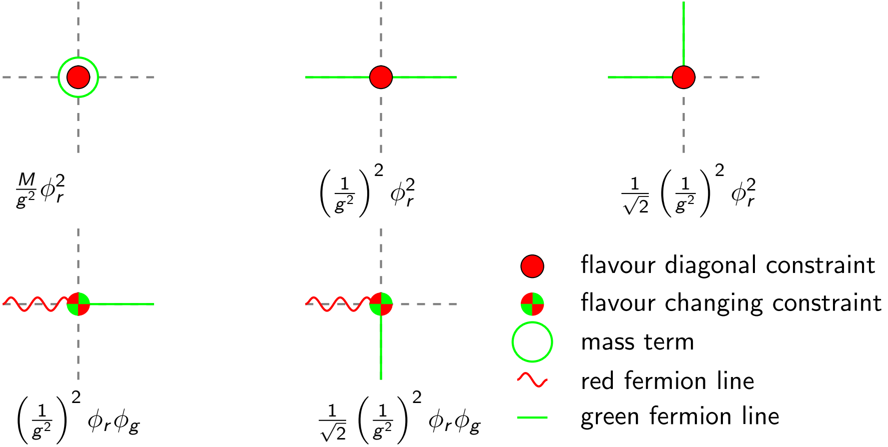
hold all information about the fermionic geometry.

$\rightarrow$ Sign of fermion loops under perfect control.

- For $N = 2$ the loops are self-avoiding and the four fermi term is irrelevant. This changes for $N > 2$.
- Simulate fermions by enlarging the configuration space by an open fermionic string. [Wenger '08]

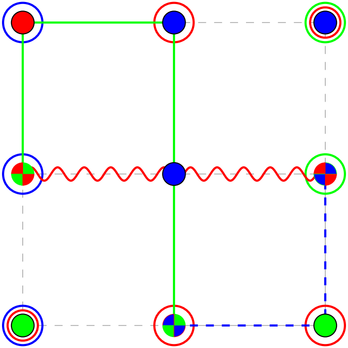
$N = 2$

Self-avoiding, non-backtracking and closed fermion loops



$N = 3$

Crucial difference for $N > 2$: loops are no longer self-avoiding.

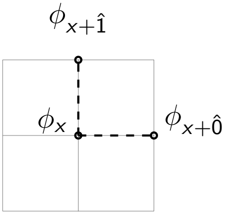


- Topology holds no information about the sign (geometry) of the configuration.

Bosonic bond representation

- Perform expansion of the bosonic action.
- Two types of bonds

nearest neighbour bonds and diagonal bonds



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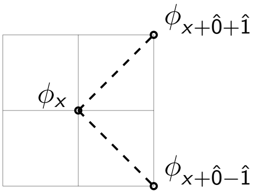
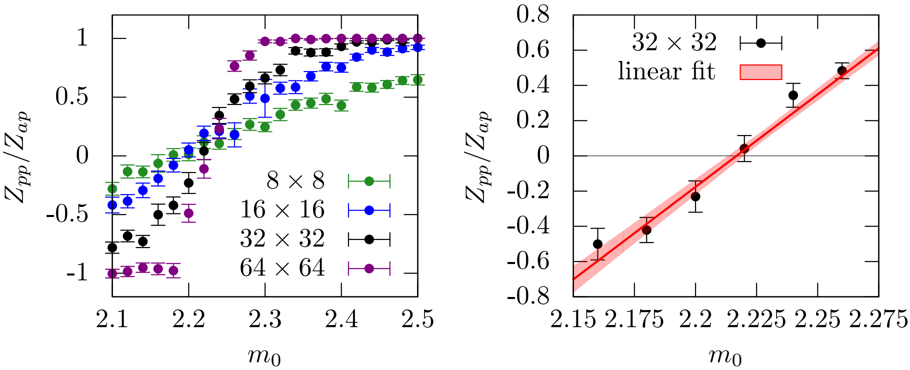


- $\exists$ an analytic form of the bosonic weight for each bond configuration for any $O(N)$.

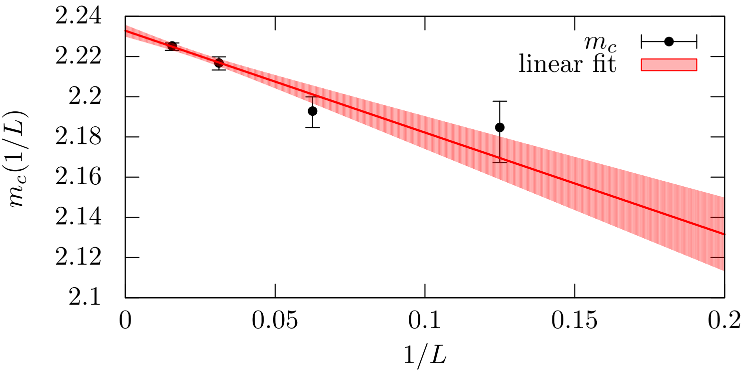
Massless point

Critical bare mass m_c, where Fermions are massless, i.e. $Z_{pp}/Z_{ap} = 0$



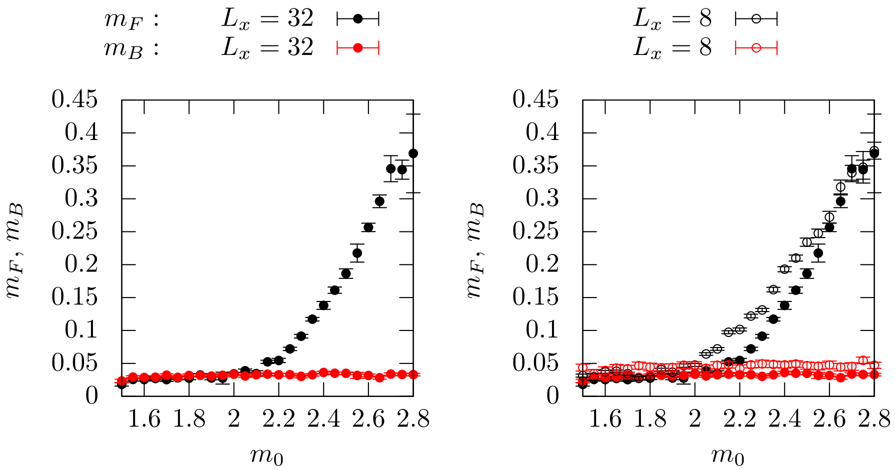
Massless point infinite volume extrapolation

Obtaining the critical mass in the infinite volume limit at fixed coupling g^2.



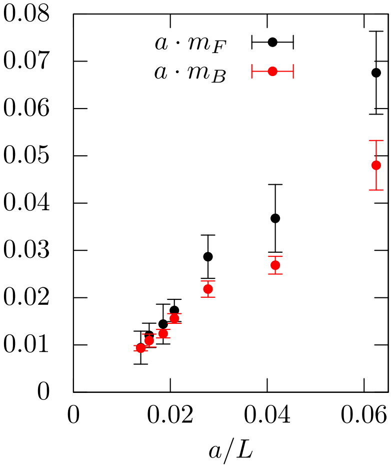
Fermion and boson mass at fixed coupling

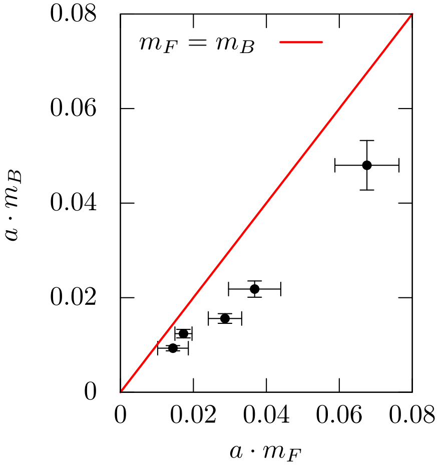
Measure correlators $\overline{\psi}(\tau)\psi(\tau+t)$ and $\phi(\tau)\phi(\tau+t)$ and fit $a\cosh(m(t-L_t/2))$ at fixed $g^2 = 0.25$ and $L_t = 32$.



Mass degeneracy

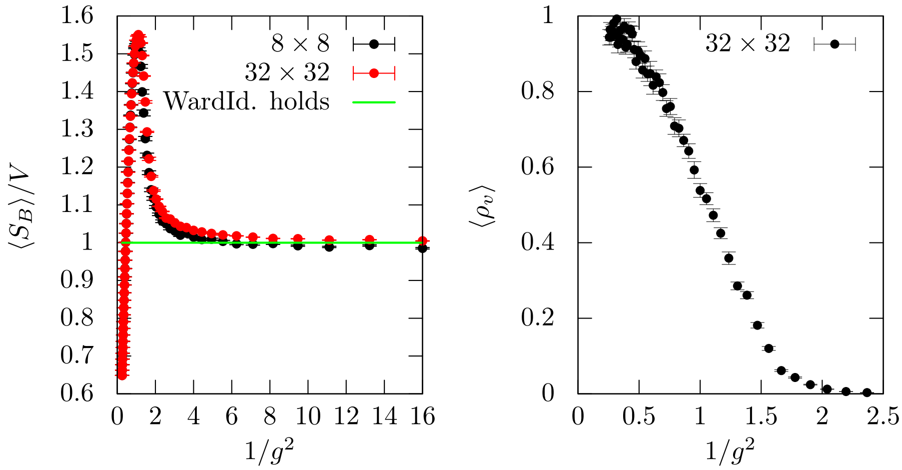
Simulations for various lattice sizes at the critical point m_c (determined via Z_{pp}/Z_{ap}) at fixed coupling $g^2 = 0.25$.





Ward identity

A Ward identiy can be constructed which depends only on the bosonic action $\langle S_B \rangle$. Non zero vortex-density $\langle \rho_v \rangle$ like in non-SUSY $O(2)$.



Conclusion and outlook

- We constructed a fermion loop expansion for the non-linear SUSY $O(N)$ σ-model (essentially implementable for any N).
- For $N = 2$ at fine tuned fermion mass we observe a mass degeneracy trend and an intact Ward identity. This gives us strong indications for a SUSY continuum limit.
- Proof of concept with $N = 2$. Use loop-expansion also for $N = 3$ where,
  - loops are no longer self avoiding. May lead to fluctuating sign.
  - data already exist to compare. [Flore, Körner, Wipf, Wozar '12]
- Implement description, where integration over all flavours is already performed.
- Check of SUSY continuum limit if only $\hat{\partial}^S$ is used for the boson.